ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

in cotutela con Universitat de Barcelona

DOTTORATO DI RICERCA IN

MATEMATICA

Ciclo 36

**Settore Concorsuale:** 01/A3 - ANALISI MATEMATICA, PROBABILITÀ' E STATISTICA MATEMATICA

**Settore Scientifico Disciplinare:** MAT/05 - ANALISI MATEMATICA

# CONGESTED OPTIMAL TRANSPORT IN THE HEISENBERG GROUP

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Esame finale anno 2024

# Introduction

The aim of this thesis is to study the problem of continuous congested optimal transport in the Heisenberg group, equipped with a sub-Riemannian metric. We will introduce three equivalent problems: one of them will give rise to a quasilinear PDE in divergence form.

The classical Monge-Kantorovich formulation of the optimal transport problem

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{X \times X} c(x,y) d\gamma(x,y), \tag{1}$$

has been widely studied in literature in general metric spaces $(X, d)$: here the set $\Pi(\mu, \nu)$ consists of probability measures $\gamma$ on the product space $X \times X$, with prescribed marginals $\mu, \nu$ and the cost function $c \geq 0$ is typically a power of the distance $d$. See [87] for an overview on this topic. This formulation depends only on the amount of mass $\gamma(x, y)$ transiting between two points $x, y$, but not on the paths followed by it. The problem of congested optimal transport is a variant of the problem (1), which take into account congestion over paths. The first contribution in this direction was given by Wardrop in [94], where he introduced a concept of equilibrium for a traffic problem on a network. Let us consider two discrete probability measures $\mu$ and $\nu$ on it, representing the distributions of agents and destinations, respectively. For any pair of nodes $x$ and $y$, $\gamma(x, y)$ represents the amount of agents that have to go from the node $x$ to the node $y$. Wardrop supposed that each path, connecting $x$ to $y$, has a cost that depends on the portion of $\gamma(x, y)$ transiting through it. Of course, on the network there is an equilibrium if and only if all the actually used paths, connecting the nodes $x$ and $y$, provide a cost that is less than, or equal to, the one provided by any other path between the same nodes. Later in [14], Beckmann et al. showed that an equilibrium is a solution to a convex minimization problem, taking into account the total congestion over the whole network. See for instance [87, Section 4.4] and [44] for an overview on this topic.

In [43] Carlier et al. proposed a continuous version of the previous model, by replacing the network with a domain $\Omega$ in $\mathbb{R}^2$ and the edges of the network with the space $\mathcal{L}$ of Lipschitz curves. The ways in which traffic is distributed over $\Omega$ are modelled through *traffic plans*, i.e. probability measures $Q$ over $\mathcal{L}$, which connect $\mu$ to $\nu$. Each traffic plan generates a *traffic intensity* $i_Q$, whose action over a set $A \subseteq \Omega$ describes how much traffic there is over the curves corresponding to the traffic plan $Q$ and passing through the subregion $A$:

$$\int_\Omega \varphi(x) di_Q(x) := \int_{\mathcal{L}} \int_0^1 \varphi(\sigma(t))|\dot{\sigma}(t)|dtdQ(\sigma), \quad \forall \varphi \in C(\Omega). \tag{2}$$

If at every point $x$ there is an amount of traffic $i_Q(x)$, and the congestion effects are described by a function $g \geq 0$, then the total cost of passing through $x$ will be $G(i_Q(x)) = i_Q(x)g(i_Q(x))$ and a *Wardrop equilibrium* is a minimum of the overall transport cost, i.e. a solution to

$$\inf_Q \int_\Omega G(i_Q(x))dx, \tag{$\mathcal{W}$}$$

whenever the composition $G \circ i_Q$ exists. In this configuration nobody is interested to change its path, since everyone is paying the least.

In [30] Brasco et al. proved that the problem $(\mathcal{W})$ is equivalent to a minimization problem over vector fields with prescribed divergence

$$\inf_{\mathbf{w} \in L^p(\Omega,\mathbb{R}^2)} \left\{ \int_\Omega \mathcal{G}(\mathbf{w}(x))dx : \mathrm{div}\mathbf{w} = \mu - \nu \right\}, \tag{$\mathcal{B}$}$$

where $\mathcal{G}(\mathbf{w}) := G(|\mathbf{w}|)$. The previous problem admits the following dual formulation, as a minimization of a classical functional in Calculus of Variations,

$$\sup \left\{ \int_\Omega \varphi d(\mu - \nu) - \int_\Omega \mathcal{G}^*(\nabla\varphi) : \varphi \in W^{1,q}(\Omega) \right\}, \tag{$\mathcal{D}$}$$

where $\mathcal{G}^*$ is the *Legendre transform* of $\mathcal{G}$ and $q = \frac{p}{p-1}$; the corresponding Euler-Lagrange equation is

$$\operatorname{div}(\nabla \mathcal{G}^*(\nabla \varphi)) = \mu - \nu, \quad \text{in } \Omega.$$

It turns out to be the Laplace or $q$-Laplace equation, in the simplest cases, but it may be much more degenerate in a more realistic traffic model, see [30], [89], [49], [23], [28] and [88]. See also [45], [32] and [44] for more details about the equivalence between the problems $(\mathcal{W})$,$(\mathcal{B})$ and $(\mathcal{D})$.

This thesis deals with the problem of continuous congested optimal transport in the Heisenberg Group $\mathbb{H}^n$. This is the simplest non commutative Lie group: it coincides with $\mathbb{R}^{2n+1}$ with a totally degenerate metric defined on a sub-bundle of the tangent bundle. This subbundle is called *horizontal bundle* and it can be assigned through a family $X_1, \ldots, X_{2n}$ of *horizontal vector fields*: all the differential calculus in this setting is expressed in terms of these vector fields, for which the following non-trivial bracket relation holds

$$X_{2n+1} := [X_i, X_{n+i}] \neq 0, \quad \forall i \in \{1, \ldots, n\}.$$

In particular, displacement is allowed only along *horizontal curves*, i.e. integral curves of the vector fields $(X_i)_{i=1}^{2n}$: the sub-Riemannian distance between two points $x, y \in \mathbb{H}^n$ is given by

$$d_{SR}(x,y) := \inf \left\{ \int_0^1 |\dot{\sigma}(t)|_H dt : \sigma \text{ horizontal curve}, \sigma(0) = x, \sigma(1) = y \right\},$$

where $|\cdot|_H$ is the norm associated with the metric on the horizontal bundle. See for instance [1], [19], [35], [36] and [69] for a general overview on this type of spaces.

The Monge-Kantorovich problem in the Heisenberg group

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y)^\alpha d\gamma(x,y). \tag{3}$$

was first studied in [5], where L. Ambrosio and S. Rigot proved the existence and the uniqueness of the solution to (3) for $\alpha = 2$; this result was extended to more general non-homogeneous settings in [58]. The first work on the problem (3) for $\alpha = 1$ is [52], where the authors proved the existence of solutions induced by maps. Afterwards the analogous result was proven in more general measure metric spaces, see for instance [16] and [46].

In this direction, we introduced the *horizontal transport density* $a_\gamma$ defined as

$$\int_{\mathbb{H}^n} \varphi(x) da_\gamma := \int_{\mathbb{H}^n \times \mathbb{H}^n} \left( \int_0^1 \varphi(\sigma_{x,y}(t)) |\dot{\sigma}_{x,y}(t)|_H dt \right) d\gamma(x,y), \tag{4}$$

for any $\varphi$ continuous, where $\gamma$ is a solution to (3) with $\alpha = 1$, and $\sigma_{x,y}$ is a minimizing geodesic between the points $x$ and $y$. In (4) the exterior integral is computed over the product space $\mathbb{H}^n \times \mathbb{H}^n$, while in the definition of traffic intensity it runs over curves: in this sense the traffic intensity can be interpreted as a dynamic version of the transport density. We adapted some summability results contained in [44] (which are specific to the Euclidean setting) to $\mathbb{H}^n$, paying attention to the fact that in $\mathbb{H}^n$ the geodesic dimension does not coincide neither with the topological dimension, nor with the homogeneous one (see [66]). Passing through a vector version of the transport density, we showed that the Monge-Kantorovich problem for $\alpha = 1$ can also be formulated as a minimization problem with horizontal divergence-type constraint

$$\inf \left\{ \int |\mathbf{w}| dx : \mathrm{div}_H \mathbf{w} = \mu - \nu \right\}, \tag{5}$$

where $\mathrm{div}_H$ is defined in duality with the *horizontal gradient* $\nabla_H \varphi = \sum_{i=1}^{2n} X_i \varphi X_i$, with $\varphi \in C^\infty$. Its dual formulation is the well-known *Kantorovich duality formula*

$$\sup \left\{ \int u d(\mu - \nu) : \|\nabla_H u\|_\infty \leq 1 \right\}. \tag{6}$$

The congested optimal transport problem in this setting was still an open problem. In order to to take into account the geometry of the Heisenberg group we have significantly reduced the set of admissible paths, allowing the agents to move only along horizontal curves. If $\Omega \subset \mathbb{H}^n$, a *horizontal traffic plan* is a probability measure over the space of horizontal curves with value in $\Omega$, which connect $\mu$ to $\nu$. Any horizontal traffic plan $Q$ induces a *horizontal traffic intensity* $i_Q$, defined as (2), where the integral w.r.t. $Q$ is computed over the horizontal curves, and the Euclidean norm is replaced by the horizontal one $|\cdot|_H$.

Given any two points $x, y \in \Omega$ and a horizontal traffic plan $Q$, we will denote by $d_Q$ the horizontal length weighted with $g \circ i_Q$

$$d_Q(x,y) := \inf \left\{ \int_0^1 g(i_Q(\sigma(t)))|\dot{\sigma}(t)|_H dt : \right. \\ \left. \sigma \text{ horizontal}, \sigma(0) = x, \sigma(1) = y \right\}; \quad (7)$$

then, a *horizontal Wardrop equilibrium*, analogous to the one introduced in [94], is a horizontal traffic plan $Q$ such that

1. $Q$ is concentrated on the geodesics w.r.t. the metric $d_Q$, i.e.

$$\int_0^1 g(i_Q(\sigma(t)))|\dot{\sigma}(t)|_H dt = d_Q(\sigma(0), \sigma(1))$$

for $Q$-a.e. horizontal curve $\sigma$;

2. $Q$ induces a measure $\gamma_Q := (e_0, e_1)_\# Q \in \Pi(\mu, \nu)$ which solves the Monge-Kantorovich problem

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\Omega \times \Omega} d_Q(x,y) d\gamma(x,y), \quad (8)$$

associated with the cost $d_Q$, depending on $Q$ itself.

If the congestion function $g(i) \sim i^{p-1}$, a horizontal traffic plan $Q$ turns out to be a Wardrop equilibrium if and only if it solves

$$\inf_{Q \in \mathcal{Q}_H^p(\mu,\nu)} \int_\Omega G(i_Q(x))dx, \quad (\mathcal{W}_H)$$

where $\mathcal{Q}_H^p(\mu,\nu)$ is the set of horizontal traffic plans $Q$ such that $i_Q \in L^p(\Omega)$. In the last chapter of this thesis, we get a necessary and sufficient condition for the non emptiness of this set; in particular, if one assume that $\mu$ and $\nu$ are $p$-summable, then the set $\mathcal{Q}_H^p(\mu,\nu)$ is not empty.

In the limit case $p = 1$, the problem $(\mathcal{W}_H)$ reads as

$$\inf_{Q \in \mathcal{Q}_H(\mu,\nu)} \int l_{SR}(\sigma) dQ(\sigma), \tag{9}$$

which turns out to be equivalent to the Monge-Kantorovich problem for $\alpha = 1$, and hence to (5) and (6).

In the spirit of [30], under the $p$-summability assumption for both $\mu$ and $\nu$, one can also prove that the minimization problem over $p$-summable horizontal vector fields, with prescribed horizontal divergence,

$$\inf_{\mathbf{w} \in L^p(\Omega, H\Omega)} \left\{ \int_\Omega \mathcal{G}(\mathbf{w}(x)) dx : \operatorname{div}_H \mathbf{w} = \mu - \nu \right\} \tag{$\mathcal{B}_H$}$$

admits a solution. Here $\mathcal{G}(\mathbf{w}) := G(|\mathbf{w}|_H)$, $H\Omega$ is the restriction of $H\mathbb{H}^n$ to the set $\Omega$. Moreover, it holds that $(\mathcal{W}_H) = (\mathcal{B}_H)$ and there is a way to find a solution to one problem, starting from a solution to the other one, and vice versa: in order to find a solution to $(\mathcal{W}_H)$, starting from a solution to $(\mathcal{B}_H)$, we passed through a sequence of approximating Riemannian manifolds, where the non-horizontal direction is increasingly penalized. In the spirit [88], one actually can prove that to get the existence of solutions to $(\mathcal{B}_H)$ it is enough that $\mu$ and $\nu$ belong to the more general dual space $(HW^{1,q}(\Omega))'$. Here $HW^{1,q}(\Omega)$ is the space of function $\varphi \in L^q(\Omega)$, whose horizontal gradient (in the weak sense) $\nabla_H \varphi \in L^q(\Omega, H\Omega)$. In addition, under this hypothesis on $\mu$ and $\nu$, the problem

$$\sup \left\{ -\int_\Omega \varphi d(\mu - \nu) - \int_\Omega \mathcal{G}^*(\nabla_H \varphi) : \varphi \in HW^{1,q}(\Omega) \right\} \tag{$\mathcal{D}_H$}$$

admits solutions and its value equals $(\mathcal{B}_H)$. Moreover, if $\mathbf{w}$ is the solution to $(\mathcal{B}_H)$ and $u \in HW^{1,p}(\Omega)$ is a solution to $(\mathcal{D}_H)$, then the relation between them is

$$\mathbf{w} = \nabla \mathcal{G}^*(\nabla_H u).$$

In particular, the function $u$ solves in the weak sense the the Euler-Lagrange equation

$$\operatorname{div}_H(\nabla \mathcal{G}^*(\nabla_H \varphi)) = \mu - \nu \quad \text{in } \Omega. \tag{10}$$

The main difficult in studying regularity for PDEs in this setting is due to the fact that derivatives do not commute: $[X_i, X_{i+n}] = X_{2n+1}$; hence, when we derive the equation and we switch derivatives (in order to get higher regularity estimates for solutions), we get a derivative of solutions in the direction $X_{2n+1}$, which has to be managed.

If the congestion function $g$ has the standard form $g(i) = i^{p-1}$, then $\mathcal{G}^*(z) = \frac{1}{q}|z|^q$ and (10) becomes the degenerate $q$-Laplace equation in $\mathbb{H}^n$. This equation has been widely studied in literature, both in homogeneous and non-homogeneous case. In both case, the optimal regularity for solutions is the Hölder continuity of the horizontal derivatives: it has been established in [95], for $2 \le q < \infty$, and in [75] for $1 < q < 2$, in homogeneous case; see also [79] for an alternative proof in $\mathbb{H}^1$, for $q > 4$. For the analogous result in non-homogeneous case see [74]. See [80] and references therein for a general overview on this topic.

In the spirit of [9] and [29], we study the equivalence of the three problems $(\mathcal{W}_H)$, $(\mathcal{B}_H)$ and $(\mathcal{D}_H)$ in the orthotropic case: more precisely, if $\mathbf{w} = \sum_{i=1}^{2n} \mathbf{w}_i X_i$ is a horizontal vector field, we consider a function

$$\mathcal{G}(\mathbf{w}) = \sum_{i=1}^{2n} G(|\mathbf{w}_i|)$$

having $p$-growth in each horizontal direction. This gives rise to a function $\mathcal{G}^*$ that has $q$-growth in each horizontal direction. See [7], [22], [24], [25], [26], [27], [29], [31], [67] and [81] for regularity results of this type of equations in the Euclidean setting. Here we prove the local Lipschitz regularity for solutions to (10) for such a function $\mathcal{G}^*$, assuming that $i$-th eigenvalue of the Hessian of $\mathcal{G}^*$ is comparable with the $n+i$-th one. The model function is

$$\mathcal{G}^*(z) = \sum_{i=1}^{n} \frac{1}{q} \left( z_i^2 + z_{n+i}^2 \right)^{\frac{q}{2}} :$$

it has $q$-growth in the horizontal directions of the complexified tangent bundle, according to [90, Chapters XII and XIII], and it leads to the equation

$$\sum_{i=1}^{n} \Big[ X_i \left( \left(|X_i u|^2 + |X_{n+i} u|^2\right)^{\frac{q-2}{2}} X_i u \right) \\ + X_{n+i} \left( \left(|X_i u|^2 + |X_{n+i} u|^2\right)^{\frac{q-2}{2}} X_{n+i} u \right) \Big] = 0.$$

As is common in the study of regularity theory for PDEs, we first approximated the equation in order to make it non-degenerate; then, in the spirit of [39], [38], [40] and [41], we introduced a Riemannian approximation of the non-degenerate equation, in order to have the smoothness of solutions. Adapting the technique introduced by X. Zhong in [95], we obtained a Caccioppoli-type inequality similar to that for solutions to Riemannian elliptic equations, where the derivative of the solution in the direction $X_{2n+1}$ has disappeared. Using the well-known Moser iteration scheme, a uniform (with respect to the approximating parameters) bound for the $L^\infty$ norm of the gradient of solutions to the approximating equation follows: passing to the limit, we get a bound on the $L^\infty$ norm of the horizontal gradient of solutions to the starting equation and hence the local Lipschitz continuity follows.

The thesis is organized as follows.

Chapter 1 contains an overview of existing results: first, we recall the definition and the main properties of the Heisenberg group. Then, we recall some known results about the optimal transport theory in this setting.

The aim of Chapter 2 is twofold: on the one hand, we prove the existence of horizontal transport densities that are $L^p$ functions, for some particular $p$'s. On the other, we prove that even the Monge-Kantorovich problem, associated with the sub-Riemannian distance, admits a divergence-type formulation. In order to do this, we introduce a vector version of the horizontal transport density and we prove the differentiability of the Kantorovich potential. In the end we show that the same problem can be also formulated as a minimization problem over measures on curves.

In Chapter 3 we define the problem of continuous congested optimal transport in the Heisenberg group. First, we give the definition of traffic plans and associated traffic intensities, and we collect some of their properties. Second, we rigorously define the congested metric. In the end, we introduce the congested optimal transport problem as a minimization problem over a suitable set of horizontal traffic plans. We show that it admits a solution, which turns out to be an equilibrium configuration.

Chapter 4 deals with the equivalent formulations of the continuous congested optimal transport problem in the Heisenberg group. We introduce the divergence-type problem and we show that it admits a solution. Moreover we introduce its dual formulation and we show that two problems are actually equivalent. In the end, under some additional hypothesis, we show that the divergence-type problem is equivalent to the minimization problem introduced in Chapter 3.

Chapter 5 is devoted to the three problems in the orthotropic case. We investigate the equivalence of these problems, by using techniques that are different from those of the previous two chapters. In the end, we study the local Lipschitz regularity for solutions to a PDE arising from the dual formulation of the problem. This result holds true in the homogeneous case.

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# Chapter 1

# Preliminaries

The first chapter of this thesis deals with some preliminary results that will be useful later on. In the first section we introduce the setting of the Heisenberg group, while in the second one we recall some well-known results about the optimal transport theory in this setting.

Given a locally compact and separable metric space $(M, d)$, we denote by $\mathcal{B}(M)$ its Borel $\sigma$-algebra. A (Borel) *measure* $\lambda$ on $M$ is a $\sigma$-additive function $\lambda : \mathcal{B}(M) \to \mathbb{R}$. We denote by

- $\mathcal{M}(M)$ the set of finite Radon measures on $M$;
- $\mathcal{M}_c(M)$ the subset of $\mathcal{M}(M)$ consisting of finite Radon measures on $M$ with compact support;
- $\mathcal{M}_+(M)$ the subset of $\mathcal{M}(M)$ consisting of positive and finite Radon measures on $M$;
- $\mathcal{P}(M)$ the subset of $\mathcal{M}_+(M)$ consisting of probability measures on $M$;
- $\mathcal{P}_c(M)$ the subset of $\mathcal{P}(M)$ consisting of probability measures on $M$ with compact support.

Given a sequence of measures $(\lambda_n)_{n\in\mathbb{N}}$ on $M$, we say that $\lambda_n$ weakly converges to $\lambda$, we write $\lambda_n \rightharpoonup \lambda$, if

$$\int_M f d\lambda_n \underset{n\to\infty}{\to} \int_M f d\mu,$$

for any bounded continuous function $f \in C_b(M)$. If $(M, d)$ is a compact metric space, then the Riesz Theorem implies that the weak convergence of measure is nothing but the weak* convergence.

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two locally compact and separable metric spaces, $f : M_1 \to M_2$ be a Borel map and $\lambda$ be a measure on $M_1$. Then, the *push-forward of* $\lambda$ is the measure on $M_2$ defined as

$$f_{\#}\lambda(A) := \lambda(f^{-1}(A)),$$

for any $A \in \mathcal{B}(M_2)$. If $\varphi : M_2 \to [0, +\infty]$ is any Borel function, then

$$\int_{M_2} \varphi d\,(f_{\#}\lambda) = \int_{M_1} (\varphi \circ f)\, d\mu.$$

## 1.1 The Heisenberg group $\mathbb{H}^n$

In this section we introduce the Heisenberg Group and we equip it with a natural intrinsic sub-Riemannian distance, which makes it a polish, geodesic and non-branching metric space and a doubling metric measure space, when equipped with its Haar measure.

Given $n \geq 1$, the *n-th Heisenberg group* $\mathbb{H}^n$ is the simplest non-commutative Carnot group[1]. Its Lie algebra $\mathfrak{h}^n$ is stratified of step 2

$$\mathfrak{h}^n = \mathfrak{h}^n_1 \oplus \mathfrak{h}^n_2,$$

[1]Given a Lie group $(\mathbb{G}, \star)$, we denote by $\mathcal{X}(\mathbb{G})$ the vector space of smooth vector fields and by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$, that is the subset of $\mathcal{X}(\mathbb{G})$ of smooth left-invariant vector fields. We say that $(\mathbb{G}, \star)$ is a *Carnot group* if it is connected, simply connected and its Lie algebra $\mathfrak{g}$ admits a *stratification*, that is

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r,$$

with

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \forall 1 \leq i \leq r, \quad \mathfrak{g}_r \neq 0, \quad \mathfrak{g}_{r+1} = 0,$$

where $[\mathfrak{g}_1, \mathfrak{g}_i] = \operatorname{span}\{[X, Y] \mid X \in \mathfrak{g}_1,\ Y \in \mathfrak{g}_i\}$, where $\mathfrak{h}_1^n = \text{span}\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}\}$, $\mathfrak{h}_2^n = \text{span}\{X_{2n+1}\}$ and the only non-trivial bracket-relation is

$$[X_j, X_{n+j}] = X_{2n+1}, \quad \forall j = 1, \ldots n. \tag{1.1}$$

The subspace $\mathfrak{h}_1$ is called *horizontal layer*. We recall that the Lie algebra $\mathfrak{h}^n$ is isomorphic to the tangent space at the origin $T_e\mathbb{H}^n$ through the vector space isomorphism

$$X \mapsto X(e) \in T_e\mathbb{H}^n.$$

See for instance [92, Chapter 4]. Hence, the horizontal layer $\mathfrak{h}_1^n$ of the algebra is isomorphic to a linear subspace $V$ of $T_e\mathbb{H}^n$. The disjoint union of

$$H\mathbb{H}^n := \bigsqcup_{q\in\mathbb{H}^n} (d\ell_q)_e(V) = \bigcup_{q\in\mathbb{H}^n} (q, (d\ell_q)_e(V)), \tag{1.2}$$

defines a sub-bundle of the tangent bundle that we will call *horizontal bundle*, where $d\ell_q$ is the differential of the left translation

$$\ell_q : \mathbb{G} \to \mathbb{G}, \quad \ell_q(g) := q \star g, \ q \in \mathbb{G}.$$

The vector fields $X_1, \ldots, X_{2n}$ defines a global frame for $H\mathbb{H}^n$, and at each $q \in \mathbb{H}^n$ the fiber is generated by these vector fields evaluated at $q$

$$H_q\mathbb{H}^n = \text{span}\{X_1(q), \ldots, X_{2n}(q)\}.$$

A *horizontal vector field* is a section of the horizontal bundle $\phi : \mathbb{H}^n \to H\mathbb{H}^n$,

$$\phi = \sum_{j=1}^{2n} \phi_j X_j.$$

Since the horizontal bundle $H\mathbb{H}^n$ admits a global trivialization, we will sometimes identify a section of $H\mathbb{H}^n$ with its canonical coordinate w.r.t. the moving frame $\{X_1, \ldots, X_{2n}\}$. In this way a section will be identified with a function

$$\phi = (\phi_1, \ldots, \phi_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}.$$

We fix an inner product $\beta$ on the horizontal layer $\mathfrak{h}_1^n \cong H_e\mathbb{H}^n$ such that $\{X_1, \dots, X_{2n}\}$ is an orthonormal basis: this induces a left-invariant Riemannian metric $b$ on $H\mathbb{H}^n$

$$\mathbb{H}^n \ni q \mapsto b_q := \left(\ell^*_{q^{-1}}\beta\right)_q : H_q\mathbb{H}^n \times H_q\mathbb{H}^n \to \mathbb{R}, \quad q \in \mathbb{H}^n,$$

where $\ell^*_{q^{-1}}\beta$ is the pullback of $\beta$ by the left translation $\ell_{q^{-1}}$,

$$\left(\ell^*_{q^{-1}}\beta\right)_q (v, w) = \beta\left(\left(dl_{q^{-1}}\right)_q (v), \left(dl_{q^{-1}}\right)_q (w)\right), \quad \forall v, w \in H_q\mathbb{H}^n.$$

From now on, we will drop the index $q$ for the metric and we will use the notation $\langle \cdot, \cdot \rangle_H$ to denote both $\beta$ and $b_q$. We adopt the same notational convention for the norm

$$| \cdot |_H := \sqrt{\langle \cdot, \cdot \rangle_H}. \tag{1.3}$$

associated to the scalar product $\beta$, which makes $\{X_1, \dots, X_{2n}\}$ an orthonormal basis for $\mathfrak{h}_1^n$ and hence a global orthonormal frame for $H\mathbb{H}^n$.

The triple $(\mathbb{H}^n, H\mathbb{H}^n, \langle \cdot, \cdot \rangle_H)$ is the simplest example of sub-Riemannian manifold: it plays the role of $\mathbb{R}^n$ in Riemannian geometry. See [35] and [58].

Given a smooth vector field $X \in \mathcal{X}(\mathbb{H}^n)$ and $q_0 \in \mathbb{H}^n$, we denote by $\exp(tX)(q_0)$ the integral curve of the vector field $X$, passing trough $q_0$ at time $t = 0$, that is the map $\sigma : I \subseteq \mathbb{R} \to \mathbb{H}^n$, such that

$$\begin{cases} \dot{\sigma}(t) = X(\sigma(t)), \\ \sigma(0) = q_0. \end{cases}$$

This map is well-defined for all whole $\mathbb{R}$, see for instance [19, Proposition 2.1.53] or [76, Appendix]. Moreover, for any $q_0 \in \mathbb{H}^n$ and any $X \in \mathfrak{h}^n$ it holds that

$$\exp(tX)(q_0) = q_0 \star \exp(tX)(e).$$

The *exponential map of the Heisenberg group* $\mathbb{H}^n$ is the map

$$\mathrm{Exp} : \mathfrak{h}^n \to \mathbb{H}^n$$

defined as

$$\mathrm{Exp}(X) := \exp(X)(e). \tag{1.4}$$

Since $\mathbb{H}^n$ is a Carnot group, in particular it is connected and simply connected, hence the exponential map is a global diffeomorphism between $\mathfrak{h}^n$ (equipped with its trivial manifold structure of finite dimensional vector space) and $\mathbb{H}^n$: hence every $q \in \mathbb{H}^n$ can be written in an unique way as

$$q = \mathrm{Exp}(x_1X_1 + \cdots + x_nX_n + x_{n+1}X_{n+1} + \ldots x_{2n}X_{2n} + x_{2n+1}X_{2n+1}), \quad (1.5)$$

with $(x_1, \ldots, x_{2n+1}) \in \mathbb{R}^{2n+1}$. Through the Campbell-Hausdorff formula (see [19, Theorem 2.2.18]) in this system of coordinates the group law reads as

$$x \cdot y := \left(x_1 + y_1, \ldots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \frac{1}{2}\sum_{j=1}^{2n}(x_j y_{n+j} - x_{n+j} y_j)\right), \quad (1.6)$$

the unit element is $0_{\mathbb{R}^{2n+1}}$ and the center of the group is

$$L := \{(0, x_{2n+1}) \in \mathbb{R}^{2n+1};\ x_{2n+1} \in \mathbb{R}\}.$$

The stratification of the algebra $\mathfrak{h}^n$ induces a family of dilations $\delta_s$ on $\mathbb{H} \equiv \mathbb{R}^{2n+1}$

$$\delta_s((x_1, \ldots, x_{2n+1})) := (sx_1, \ldots, sx_{2n}, s^2 x_{2n+1}).$$

The vector fields $X_1, \ldots, X_{2n+1} \in \mathfrak{h}^n$ in coordinates read as

$$X_j = \partial_{x_j} - \frac{x_{n+j}}{2}\partial_{x_{2n+1}}, j = 1, \ldots, n,$$

$$X_{n+j} = \partial_{x_{n+j}} + \frac{x_j}{2}\partial_{x_{2n+1}}, \quad j = 1, \ldots, n,$$

$$X_{2n+1} = \partial_{x_{2n+1}}.$$

Moreover, the $(2n+1)$-dimensional Lebesgue measure $\mathcal{L}^{2n+1}$ is the left-Haar measure (up to multiplicative constants) of the Carnot group $(\mathbb{R}^{2n+1}, \cdot)$, and its push-forward through the inverse of the map (1.4) is the left-Haar measure (up to multiplicative constants) of the Carnot group $(\mathbb{H}^n, \star)$. The homogeneous dimension of the group is

$$N := \sum_{i=1}^{2} i \dim \mathfrak{h}_i^n = 2n + 2 \quad (1.7)$$

and it holds that

$$\mathcal{L}^{2n+1}(\delta_s(B)) = s^N \mathcal{L}^{2n+1}(B)$$

for all Borel set $B$ and all $s > 0$. From now on we will always work in coordinates: with an abuse of notation, we denote by $\mathbb{H}^n$ the group $(\mathbb{R}^{2n+1}, \cdot)$ and we will write $x = (x_1, \dots, x_{2n+1}) \in \mathbb{H}^n$ to indicate a generic point $\mathbb{H}^n$.

### 1.1.1 Horizontal curves and Carnot-Caratheodory distance

We denote by $C([a,b], \mathbb{R}^{2n+1})$ the set of continuous curves on $\mathbb{R}^{2n+1}$ and by $\mathrm{Lip}([a,b], \mathbb{R}^{2n+1})$ its subset of Lipschitz curves.

**Definition 1.1** (Horizontal curves). A *horizontal curve* is a curve $\sigma \in \mathrm{Lip}([a,b], \mathbb{R}^{2n+1})$ such that

$$\dot{\sigma}(t) \in H_{\sigma(t)}\mathbb{H}^n, \quad \text{for a.e. } t \in [a,b].$$

We denote by

$$H([a,b], \mathbb{R}^{2n+1}) := \left\{ \sigma \in C([a,b], \mathbb{R}^{2n+1}) : \sigma \text{ is horizontal} \right\},$$

and by

$$H^{x,y}([a,b], \mathbb{R}^{2n+1}) := \left\{ \sigma \in H([a,b], \mathbb{R}^{2n+1}) : \sigma(a) = x, \sigma(b) = y \right\}$$

for any two points $x, y \in \mathbb{H}^n$.

We recall the following important result that enable us to define a natural intrinsic sub-Riemannian distance on $\mathbb{H}^n$.

**Theorem 1.1.1** (Chow-Rashevsky's Theorem). *For any two points in $\mathbb{H}^n$ there exists a horizontal curve between them.*

See [35, Chapter 3] for the proof of the previous result.

Given a horizontal curve $\sigma \in H([a,b], \mathbb{R}^{2n+1})$, one can define its *sub-Riemannian length*

$$l_{SR}(\sigma) := \int_a^b |\dot{\sigma}(t)|_H dt, \tag{1.8}$$

where $|\cdot|_H$ is defined in (1.3). Let us remark that any horizontal curve has finite sub-Riemannian length.

Given two points $x, y \in \mathbb{H}^n$, one can define the *sub-Riemannian distance* or *Carnot-Caratheodory distance* $d_{SR}$ between them as

$$d_{SR}(x,y) := \inf \left\{ l_{SR}(\sigma) : \sigma \in H^{x,y}([a,b], \mathbb{R}^{2n+1}) \right\}. \tag{1.9}$$

Theorem 1.1.1 implies that the distance

$$d_{SR} : \mathbb{H}^n \times \mathbb{H}^n \to [0, +\infty)$$

is well-defined.

Given $x \in \mathbb{H}^n$ and $r > 0$ the sub-Riemannian ball $B(x,r)$ centred at $x$ with radius $r$ is the set

$$B(x,r) := \{ y \in \mathbb{H}^n : d_{SR}(x,y) < r \}.$$

One can prove that the topology induced by $d_{SR}$ coincides with the original topology of $\mathbb{H}^n$, i.e. the Euclidean one. In particular, $d_{SR} : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ is continuous, see [1, Theorem 3.31] and [58, Proposition 1.16], and this in turn implies that the metric space $(\mathbb{H}^n, d_{SR})$ is locally compact, see [70, Proposition 1.4.3]. More precisely, $(\mathbb{H}^n, d_{SR}, \mathcal{L}^{2n+1})$ is a locally compact *Polish*, i.e. a complete and separable, metric measure space. Let us just remark that completeness and separability can be lifted from the metric $(\mathbb{H}^n, d_{SR})$ to the metric space $(C([a,b], \mathbb{R}^{2n+1}), d_\infty)$, where

$$d_\infty(\sigma_1, \sigma_2) = \sup_{t \in [a,b]} \|\sigma_1(t) - \sigma_2(t)\|_{\mathbb{R}^{2n+1}}, \quad \forall \sigma_1, \sigma_2 \in C([a,b], \mathbb{R}^{2n+1}).$$

The topology induced by this distance on $C([a,b], \mathbb{R}^{2n+1})$ is called *topology of uniform convergence.*

On the metric space $(\mathbb{H}^n, d_{SR})$ one can consider the length structure induced by the distance $d_{SR}$, see [34, Chapter 2]: one can consider the functional

$$l : C\left([a,b], \mathbb{R}^{2n+1}\right) \to [0, +\infty],$$

where

$$l(\sigma) := \sup \left\{ \sum_{i=1}^{k} d_{SR}(\sigma(t_{i-1}), \sigma(t_i)) : \right.$$
$$\left. k \in \mathbb{N}, a = t_0 < t_1 < \ldots < t_{k-1} < t_k = b \right\} \in [0, \infty], \quad (1.10)$$

for any continuous curve $\sigma \in C([a,b], \mathbb{R}^{2n+1})$. We denote by

$$R\left([a,b], \mathbb{R}^{2n+1}\right) := \left\{\sigma \in C([a,b], \mathbb{R}^{2n+1}) : l(\sigma) < +\infty\right\}$$

the subspace of the *rectifiable curves*. Let us just recall that any $\sigma \in R\left([a,b], \mathbb{R}^{2n+1}\right)$ can be parametrized proportionally to the arc-length.

In particular, the previous functional is lower semicontinuous on the space $R\left([a,b], \mathbb{R}^{2n+1}\right)$, with respect to the topology of the uniform convergence, since it is the supremum of lower semicontinuous functions. The definition of length in (1.10) extends the definition (1.8) in the following sense.

**Lemma 1.1.2.** *If* $\sigma \in H\left([a,b], \mathbb{R}^{2n+1}\right)$*, then*

$$l(\sigma) = l_{SR}(\sigma).$$

See [70, Theorem 1.3.5] for the proof.

### Geodesics in $\mathbb{H}^n$

A *minimizing horizontal curve* between two points $x, y \in \mathbb{H}^n$ is a horizontal curve $\sigma \in H^{x,y}([a,b], \mathbb{R}^{2n+1})$ such that

$$d_{SR}(x,y) = l_{SR}(\sigma).$$

As for Riemannian manifolds, a minimizing horizontal path $\sigma$ with constant speed, i.e. such that $|\dot{\sigma}(t)|_H$ is constant, minimizes the *sub-Riemannian energy functional*

$$\mathcal{E} : H([a,b], \mathbb{R}^{2n+1}) \to \mathbb{R},$$

defined as

$$\mathcal{E}(\sigma) = \int_a^b |\dot{\sigma}(t)|_H^2 dt.$$

Let us denote by

$$e_{SR}(x,y) := \inf \left\{ \mathcal{E}(\sigma) : \sigma \in H^{x,y}([a,b], \mathbb{R}^{2n+1}) \right\}$$

the *sub-Riemannian energy* between two points $x, y \in \mathbb{H}^n$.

We call *geodesic* any horizontal curve $\sigma \in H([a,b], \mathbb{R}^{2n+1})$ such that

$$e_{SR}(\sigma(a), \sigma(b)) = \mathcal{E}(\sigma).$$

One can prove that $\sigma$ is a geodesic if and only if it is a minimizing horizontal curve with constant speed, see [58, Proposition 2.2].

**Theorem 1.1.3** (Hopf-Rinow's theorem)**.** *[58, Theorem 2.4] Let us consider the complete metric space $(\mathbb{H}^n, d_{SR})$, then*

1. *any bounded and closed subset is compact;*
2. *for any $x, y \in \mathbb{H}^n$ there exists a geodesic between them. In this case we say that $(\mathbb{H}^n, d_{SR})$ is a geodesic space.*

We denote by

$$\mathrm{Geo}(\mathbb{H}^n) := \left\{ \sigma \in H([0,1], \mathbb{R}^{2n+1}) : \sigma \text{ is a geodesic} \right\}$$

the set of geodesics parametrized on $[0,1]$. Notice that

$$\sigma \in \mathrm{Geo}(\mathbb{H}^n) \iff d_{SR}(\sigma(t), \sigma(t')) = |t - t'| d_{SR}(\sigma(0), \sigma(1)), \quad \forall t, t' \in [0,1]$$

and this implies that $\mathrm{Geo}(\mathbb{H}^n)$ is a closed subset of $C([0,1], \mathbb{R}^{2n+1})$, equipped with the topology of uniform convergence.

Minimizing horizontal curves in $\mathbb{H}^n$, and more generally in any sub-Riemannian manifold, can be founded solving the following system of Hamiltonian equations

$$\begin{cases} \dot{x} = \frac{\partial \mathcal{H}(x,p)}{\partial p} \\ \dot{p} = -\frac{\partial \mathcal{H}(x,p)}{\partial x}, \end{cases} \tag{1.11}$$

with Hamiltonian

$$\mathcal{H}(x,p) = \frac{1}{2} \sum_{j=1}^{n} \left( \left( p_{x_j} - \frac{x_{n+j}}{2} p_{x_{2n+1}} \right)^2 + \left( p_{x_{n+j}} + \frac{x_j}{2} p_{x_{2n+1}} \right)^2 \right),$$

where $(p, x)$ are the canonical coordinates on $T^*\mathbb{H}^n$. See for instance [1] for more details about this topic.

In the case of the Heisenberg group, one can compute explicitly the equations of geodesics. We set

$$E := \{(x, y) \in \mathbb{H}^n \times \mathbb{H}^n;\ x^{-1} \cdot y \notin L\}, \tag{1.12}$$

then it holds the following characterization for minimizing geodesics parametrized on $[0, 1]$.

**Theorem 1.1.4.** (Geodesics parametrized on $[0, 1]$) *A non trivial geodesic starting from* $0$ *is the restriction to* $[0, 1]$ *of the curve*

$$\sigma_{\chi,\theta}(t) = (x_1(t), \dots, x_{2n+1}(t))$$

*either of the form*

$$x_j(t) = \frac{\chi_j \sin(\theta t) - \chi_{n+j}\,(1 - \cos(\theta t))}{\theta}, \quad j = 1, \dots, n \tag{1.13}$$

$$x_{n+j}(t) = \frac{\chi_{n+j} \sin(\theta t) + \chi_j\,(1 - \cos(\theta t))}{\theta}, \quad j = 1, \dots, n \tag{1.14}$$

$$x_{2n+1}(t) = \frac{|\chi|^2}{2\theta^2}\,(\theta t - \sin(\theta t))\,, \tag{1.15}$$

*for some* $\chi \in \mathbb{R}^{2n} \setminus \{0\}$ *and* $\theta \in [-2\pi, 2\pi] \setminus \{0\}$*, or of the form*

$$(x_1(t), \dots, x_{2n+1}(t)) = (\chi_1 t, \dots, \chi_{2n} t, 0)\,,$$

*for some* $\chi \in \mathbb{R}^{2n} \setminus \{0\}$ *and* $\theta = 0$*. In particular, it holds*

$$|\chi|_{\mathbb{R}^{2n}} = |\dot\sigma|_H = d_{SR}(0, \sigma(1)).$$

*Moreover it holds:*

1. *For all* $(x, y) \in E$*, there is a unique geodesic* $x \cdot \sigma_{\chi,\theta}$ *parametrized on* $[0, 1]$ *between* $x$ *and* $y$*, for some* $\chi \in \mathbb{R}^{2n} \setminus \{0\}$ *and some* $\varphi \in (-2\pi, 2\pi)$*.*
2. *If* $(x, y) \notin E$*, then* $x^{-1} \cdot y = (0, \dots, 0, z_{2n+1})$ *for some* $z_{2n+1} \in \mathbb{R} \setminus \{0\}$*. Hence, there are infinitely many geodesics parametrized on* $[0, 1]$ *between* $x$ *and* $y$*: they are all the curves of the form* $x \cdot \sigma_{\chi,2\pi}$*, if* $z_{2n+1} > 0$*,* $x \cdot \sigma_{\chi,-2\pi}$*, if* $z_{2n+1} < 0$*, for any* $\chi \in \mathbb{R}^{2n}$ *such that* $|\chi|_{\mathbb{R}^{2n}} = \sqrt{4\pi|z_{2n+1}|}$*.*

See for instance [5] and [71].

*Remark* 1. By definition of geodesics, the graph of the multi-valued map

$$\mathcal{S} : \mathbb{H}^n \times \mathbb{H}^n \to C([0,1], \mathbb{R}^{2n+1}),$$

that associates with any pair of points $(x, y) \in \mathbb{H}^n \times \mathbb{H}^n$ the set of geodesics $\mathcal{S}(x, y) \subset \mathrm{Geo}(\mathbb{H}^n)$ between $x$ and $y$, is closed in $\mathbb{H}^n \times \mathbb{H}^n \times C([0,1], \mathbb{R}^{2n+1})$. Hence, from the theory of Souslin sets and general theorems about measurable selections, see for instance [18, Theorem 6.9.2 and Theorem 7.4.1], it follows that there exists a *selection of geodesics*, i.e. a map

$$S : \mathbb{H}^n \times \mathbb{H}^n \to \mathrm{Geo}(\mathbb{H}^n)$$

such that $S(x, y) = \sigma_{x,y} \in \mathcal{S}(x, y)$, that is $\gamma$- measurable for any Borel measure $\gamma$ on $\mathbb{H}^n \times \mathbb{H}^n$. Moreover $S$ is continuous, hence Borel, on the set $E$, defined in (1.12).

If $e_t$ is the evaluation map at time $t \in [0,1]$, that is the map $e_t(\sigma) := \sigma(t)$ for all $\sigma \in C\left([0,1], \mathbb{R}^{2n+1}\right)$, then one can consider the map

$$S_t := e_t \circ S : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n,$$

that associates with any two points $x, y \in \mathbb{H}^n$ the point $S_t(x, y) := \sigma_{x,y}(t)$ of $\mathbb{H}^n$ at distance $t\, d_{SR}(x, y)$ from $x$, on the selected geodesic $\sigma_{x,y}$ between $x$ and $y$.

From [93, Theorem 7.29] it follows that $(\mathbb{H}^n, d_{SR})$ is a non-branching metric space: any two minimizing geodesics which coincide on a non trivial interval coincide on the whole intersection of their intervals of definition.

### 1.1.2 Measure contraction property $MCP(0, 2n+3)$ in $\mathbb{H}^n$

Let $S : \mathbb{H}^n \times \mathbb{H}^n \to \mathrm{Geo}(\mathbb{H}^n)$ be a selection of geodesics, let $e_t$ be the evaluation map, for $t \in [0,1]$, and let us consider the map

$$S_t := e_t \circ S : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n.$$

If we fix $\overline{y} \in \mathbb{H}^n$ and $t \in (0,1)$, then the function $S_t(\cdot, \overline{y})$ is $C^\infty$ on $\mathbb{H}^n \setminus (\overline{y} \cdot L)$, see [5] or [71]. Moreover it holds that

$$\det D_x(S_t(x, \overline{y})) \geq (1-t)^{2n+3} \tag{1.16}$$

for all $x \in \mathbb{H}^n \setminus (\overline{y} \cdot L)$. Hence, for any $y \in \mathbb{H}^n$ and any Borel set $A \subset \mathbb{H}^n \setminus (y \cdot L)$ it holds that

$$\mathcal{L}^{2n+1}(A) \leq \frac{1}{(1-t)^{2n+3}} \mathcal{L}^{2n+1}(S_t(A, y)). \tag{1.17}$$

For the proof of the previous inequalities see [66, Section 2]. Instead of the exponent $2n+3$ one would expect the topological dimension $2n+1$, as in the Euclidean setting, or at most the homogeneous dimension $N = 2n+2$. However the author in [66, Remark 2.3] shows that this exponent is sharp.

Roughly speaking, inequality (1.17) means that the metric measure space $(\mathbb{H}^n, d_{SR}, \mathcal{L}^{2n+1})$ satisfies the Measure Contraction Property $MCP(0, 2n+3)$: this is a generalization to metric measure spaces of the concept of Ricci curvature bounded by below. This notion was introduced by Otha in [77]: it controls the distortion of measures along geodesics. Recently in [8] the authors proved that every two-step compact sub-Riemannian manifold and every Lipschitz Carnot group satisfy $MCP(0, R)$, for some $R > 0$. See also [12], [82], [84] and references therein for further results in this direction.

### 1.1.3 Intrinsic differentiability

We start by saying that a group homomorphism $F : \mathbb{H}^n \to \mathbb{R}$ is homogeneous if $F(\delta_s(x)) = s\,F(x)$ for all $x \in \mathbb{H}^n$ and all $s > 0$. From now on, in this section, $\Omega$ will be an open subset of $\mathbb{H}^n$. Now we give the definition of Lie derivative in the direction of left-invariant vector fields.

**Definition 1.2** (Derivative along vector fields)**.** Let $\varphi : \Omega \to \mathbb{R}$, we call Lie Derivative of $\varphi$ at $x \in \Omega$ in the direction $X_j$, $j = 1, \ldots, 2n+1$

$$X_j \varphi(x) := \frac{d}{dt}\Big|_{t=0} \varphi(\exp(tX_j)(x)),$$

if it exists. Moreover we denote by $\nabla_H \varphi(x) := \sum_{j=1}^{2n} X_j\varphi(x)X_j(x) \in H_x\Omega$. We say that $\varphi \in C_H^k(\Omega)$, for $k \geq 1$, if $X_j^i\varphi \in C(\Omega)$, for any $j = 1, \dots, 2n$, for any $i = 1, \dots, k$.

In the previous definition $H\Omega$ denotes the restriction of $H\Omega$ to the open set $\Omega$, whose horizontal fibers $H_x\Omega$ are restricted to all points $x \in \Omega$.

A simple computation based on the classical chain rule shows that

$$\frac{d}{dt}(\varphi \circ \sigma)\Big|_{t=s} = \sum_{j=1}^{2n} X_j\varphi(\sigma(s))\dot{\sigma}_j(s), \tag{1.18}$$

provided $\sigma \in H([a,b], \mathbb{R}^{2n+1})$ is differentiable at $s \in [a,b]$, $\varphi \in C^\infty(\Omega)$.

**Definition 1.3.** A map $\varphi : \Omega \to \mathbb{R}$ is Pansu-differentiable at $x \in \mathbb{H}^n$ if there exists an homogeneous group homomorphism $F : \mathbb{H}^n \to \mathbb{R}$ such that

$$\lim_{y \to x} \frac{\varphi(y) - \varphi(x) - L(x^{-1} \cdot y)}{d_{SR}(x,y)} = 0.$$

If the map $L$ exists, it is unique and will be denoted by $D_H\varphi(x)$.

If $\varphi : \Omega \to \mathbb{R}$ is Pansu-differentiable at $x \in \Omega$ then $\varphi$ is differentiable at $x$ in the directions $X_j, \forall j = 1, \dots, 2n$ and

$$D_H\varphi(x)(y) = \sum_{j=1}^{2n} y_j X_j\varphi(x),$$

see [60, Proposition 5.6].

**Theorem 1.1.5** (Pansu-differentiability theorem [78])**.** *Let $\varphi \in \mathrm{Lip}(\Omega, d_{SR})$ be a $C$-Lipschitz function. Then, for $\mathcal{L}^{2n+1}$-a.e. $x \in \Omega$, the function $\varphi$ is Pansu-differentiable at $x$ and $|\nabla_H\varphi(x)|_H \leq C$.*

For every $1 \leq q \leq \infty$, the space

$$HW^{1,q}(\Omega) := \left\{\varphi : \Omega \to \mathbb{R} : \varphi \in L^q(\Omega), \nabla_H\varphi \in L^q(\Omega, H\Omega)\right\}, \tag{1.19}$$

where the derivatives $X_j\varphi$ has to be understood in the sense of distributions, is a Banach space equipped with the norm

$$\|\varphi\|_{HW^{1,q}(\Omega)} := \|\varphi\|_{L^q(\Omega)} + \|\nabla_H\varphi\|_{L^q(\Omega, H\Omega)}.$$

If $1 \le q \le \infty$, we denote by

$$HW_0^{1,q}(\Omega) := \overline{C_c^\infty(\Omega)}^{HW^{1,q}(\Omega)},$$

while we denote by

$$HW^{-1,p}(\Omega) := (HW_0^{1,q}(\Omega))',$$

where $p = \frac{q}{q-1}$ and $1 \le q < \infty$.

### Poincaré-Sobolev inequalities

Let us suppose that $\Omega$ is a Poincaré-Sobolev domain in $\mathbb{H}^n$, i.e. an open bounded subset of $\mathbb{H}^n$ such that there exists a covering of sub-Riemannian balls $(B)_{B\in\mathcal{F}}$ and numbers $L > 0$ $\alpha, \beta \ge 1$:

1. for every $x \in \mathbb{H}^n$
$$\sum_{B\in\mathcal{F}} \mathbb{1}_{(\alpha+1)B}(x) \le L\mathbb{1}_\Omega(x);$$
2. there exists a ball $B_0 \in \mathcal{F}$ such that for any $B \in \mathcal{F}$ one may find a chain $B_0, B_1, \dots, B_{s(B)} = B$ such that $B_i \cap B_{i+1} \ne \emptyset$ and $\mathcal{L}^{2n+1}(B_i \cap B_{i+1}) \ge L^{-1} \max\left(\mathcal{L}^{2n+1}(B_i), \mathcal{L}^{2n+1}(B_{i+1})\right);$
3. for any $i = 0, \dots, s(B)$ it holds
$$B \subseteq \beta B_i.$$

Let us just remark that the class of PS-domains in $\mathbb{H}^n$ is very large: in particular sub-Riemannian balls and any bounded domain with $C^{1,1}$ boundary (in Euclidean sense) are PS-domains. See [61] and references therein for a detailed overview on this topic. These sets support a $q$-Poincaré inequality, i.e. there exists $c = c(n, q, \Omega) > 0$ such that

$$\int_\Omega |\varphi(x) - \varphi_\Omega|^q dx \le c \int_\Omega |\nabla_H \varphi(x)|^q dx, \quad \forall \varphi \in HW^{1,q}(\Omega) \tag{1.20}$$

for any $1 \le q < +\infty$. See [64] and [61].

Moreover, the following Sobolev inequality holds true.

**Theorem 1.1.6.** *Let* $1 \leq q < N$. *For any* $\varphi \in HW_0^{1,q}(B(x,r))$, $B(x,r) \subset \mathbb{H}^n$, *it holds that*

$$\left( \fint_{B(x,r)} |\varphi|^{\frac{Nq}{N-q}} dx \right)^{\frac{N-q}{Nq}} \leq cr \left( \fint_{B(x,r)} |\nabla_H \varphi|_H^q dx \right)^{\frac{1}{q}},$$

*where* $c = c(n,q)$.

### 1.1.4 Horizontal vector measures in $H\mathbb{H}^n$

Here we recall the notion of horizontal vector measure in $H\mathbb{H}^n$, see [10] and [50].

Let $\Omega \subseteq \mathbb{H}^n$. We denote by $C_c(\Omega, H\Omega)$ the class of continuous horizontal vector fields with compact support in $\Omega$, and by $C_0(\Omega, H\Omega)$ its completion with respect to the uniform norm

$$\|\phi\|_\infty = \sup_{x \in \Omega} |\phi(x)|_H,$$

where $\phi : \Omega \to H\Omega$ is a horizontal vector field. The space $C_0(\Omega, H\Omega)$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space.

Let $\lambda \in \mathcal{M}(\Omega)$ and $\alpha : \Omega \to H\Omega$ be a locally bounded $\lambda$-measurable horizontal vector field. Hence, one can define the following bounded linear functional $T_{\alpha\lambda}$ on $C_c(\Omega, H\Omega)$, w.r.t. $\|\cdot\|_\infty$-norm,

$$C_c(\Omega, H\Omega) \ni \phi \mapsto T_{\alpha\lambda}(\phi) := \int_\Omega \langle \phi, \alpha \rangle_H d\lambda.$$

As a consequence, one can define a notion of *vector measure* $\alpha\lambda$ in $H\Omega$ by setting

$$C_c(\Omega, H\Omega) \ni \phi \mapsto \int_\Omega \phi \cdot d(\alpha\lambda) := T_{\alpha\lambda}(\phi).$$

By density, this functional can be extended in a unique way to a continuous linear functional on $C_0(\Omega, H\Omega)$.

We denote by $\mathcal{M}(\Omega, H\Omega)$ the space of all vector measures in $H\Omega$ in the previous sense.

Since the vector field $\alpha$ can be written as $\alpha = \sum_{i=1}^{2n} \alpha_i X_i$, where the components $\alpha_i : \Omega \to \mathbb{R}$ w.r.t. the horizontal frame are locally bounded

$\lambda$-measurable functions, then the vector measure $\mathbf{w} = \alpha\lambda$ can be also written in components as $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_{2n}) = (\alpha_1\lambda, \dots, \alpha_{2n}\lambda)$, where this notation means that $\alpha_i$ is the density of the measure $\mathbf{w}_i$ w.r.t. $\lambda$. That is

$$T_{\mathbf{w}}(\phi) = \int_\Omega \phi \cdot d\mathbf{w} = \sum_{i=1}^{2n} \int_\Omega \phi_i(x) d\mathbf{w}_i(x) = \sum_{i=1}^{2n} \int_\Omega \phi_i(x)\alpha_i(x) d\lambda.$$

One may endow the space $\mathcal{M}(\Omega, H\Omega)$ with the norm

$$\|\mathbf{w}\|_{\mathcal{M}(\Omega,H\Omega)} := |\mathbf{w}|(\Omega) < +\infty,$$

where the *variation measure* $|\mathbf{w}| \in \mathcal{M}_+(\Omega)$ is defined as

$$|\mathbf{w}|(A) := \sup\left\{ \int_\Omega \phi \cdot d\mathbf{w} : \phi \in C_c(\Omega, H\Omega), \text{supp } \phi \subseteq A, \|\phi\|_\infty \leq 1 \right\}, \tag{1.21}$$

for any Borel set $A \subseteq \Omega$.

Since the horizontal bundle $H\Omega$ has a global trivialization, one can always argue component-wise. Hence from classical results, see for instance [4], any $T \in C_0(\Omega, H\Omega)'$ can be represented by a vector measure $\mathbf{w}$ in $H\Omega$ as

$$T(\phi) = \int_\Omega \phi \cdot d\mathbf{w}, \quad \forall \phi \in C_0(\Omega, H\Omega),$$

and

$$\|\mathbf{w}\|_{\mathcal{M}(\Omega,H\Omega)} = \|T\|_{(C_0(\Omega,H\Omega))'}.$$

The identification between the space $\mathcal{M}(\Omega, H\Omega)$ of finite Radom vector measures and $(C_0(\Omega, H\Omega))'$ can be proved using the map $\Theta : \mathcal{M}(\Omega, H\Omega) \longrightarrow (C_0(\Omega, H\Omega))'$ defined by

$$\Theta(\mathbf{w})(\phi) := \int_\Omega \phi \cdot d\mathbf{w} = T_{\mathbf{w}}(\phi) \quad \forall\, \phi \in C_0(\Omega, H\Omega).$$

Let us just remark that, if $\Omega$ is compact, then $C_c(\Omega, H\Omega) = C_0(\Omega, H\Omega) = C(\Omega, H\Omega)$ and

$$\mathcal{M}(\Omega, H\Omega) = (C(\Omega, H\Omega))'.$$

### 1.1.5 Mollification in $\mathbb{H}^n$

Let us consider a mollifier for the group structure, i.e. a function $\rho \in C^\infty(\mathbb{H}^n)$, such that $\rho \geq 0$, supp $\rho \subset B(0,1)$ and $\int_{\mathbb{H}^n} \rho(x)dx = 1$; for any $\epsilon > 0$ we denote by

$$\rho_\varepsilon(x) := \varepsilon^{-N}\rho\left(\delta_{1/\varepsilon}(x)\right).$$

If $\varphi \in L^1_{loc}$, then

$$\rho_\varepsilon * \varphi(x) := \int_{\mathbb{H}^n} \rho(xy^{-1})\varphi(y)dy,$$

is smooth. Due to the non-commutativity nature one may also define a different convolution

$$\varphi * \rho_\varepsilon(x) := \int_{\mathbb{H}^n} \varphi(y)\rho(y^{-1}x)dy.$$

The mollified functions $\rho_\varepsilon * \varphi$ and $\varphi * \rho_\varepsilon$ enjoy many standard properties, see for instance [59], [50, Subsection 2.3] and references therein.

More generally, one may also mollify Radon measures: given a finite Radon measure $\lambda \in \mathcal{M}(\mathbb{H}^n)$, resp. a finite vector Radon measure $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_{2n}) \in \mathcal{M}(\mathbb{H}^n, H\mathbb{H}^n)$, then the mollified functions $\lambda^\varepsilon \in C^\infty(\mathbb{H}^n)$, resp. $\mathbf{w}^\varepsilon \in C^\infty(\mathbb{H}^n, H\mathbb{H}^n)$, are defined as

$$\lambda^\varepsilon(x) := \rho_\varepsilon * \lambda(x) = \int_{\mathbb{H}^n} \rho_\varepsilon(x \cdot y^{-1})d\lambda(y),$$

and

$$\mathbf{w}^\varepsilon := \sum_{j=1}^{2n} \mathbf{w}_j^\varepsilon X_j,$$

where $\mathbf{w}_j^\varepsilon(x) := \rho_\varepsilon * \mathbf{w}_j(x)$, for any $j = 1, \ldots, 2n$.

### 1.1.6 Riemannian approximation of the sub-Riemannian Heisenberg group

The Heisenberg group $(\mathbb{H}^n, d_{SR})$ arises as the pointed Hausdorff-Gromov limit of a sequence of Riemannian manifolds, in which the non-horizontal direction is increasingly penalized.

One can equip $\mathbb{H}^n$ with a left-invariant Riemannian metric: for any $\epsilon > 0$, one can consider the metric tensor $b_\epsilon$ that has at any point $x \in \mathbb{H}^n$

$$X_1(x), \ldots, X_{2n}(x), \epsilon X_{2n+1}(x)$$

as orthonormal basis for $T_x\mathbb{H}^n$.

We relabel

$$X_i^\epsilon := X_i, \forall i = 1, \ldots, 2n \text{ and } X_{2n+1}^\epsilon := \epsilon X_{2n+1},$$

and we denote by $|\cdot|_\epsilon := \sqrt{b_\epsilon(\cdot,\cdot)}$ the norm associated to the metric $b_\epsilon$ and by $d_\epsilon$ the associated control distance: this distance turns out to be left-invariant with respect to (1.6), since $X_1^\epsilon, \ldots, X_{2n}^\epsilon, X_{2n+1}^\epsilon$ are themselves left invariant. For any $x, y \in \mathbb{H}^n$

$$d_\epsilon(x,y) \le d_{\epsilon'}(x,y) \le d_{SR}(x,y),$$

where $0 < \epsilon' \le \epsilon$, hence $\forall x, y \in \mathbb{H}^n$

$$d_{SR}(x,y) = \lim_{\epsilon \to 0} d_\epsilon(x,y),$$

and the convergence is uniform on compact subsets of $\mathbb{H}^n \times \mathbb{H}^n$, see [53, Lemma 2.7]. In particular the balls $B_\epsilon \to B$, in the Hausdorff-Gromov sense.

Moreover, one can consider a regularized gauge function

$$G_\epsilon^2(x) := \sum_{i=1}^{2n} |x_i|^2 + \min\left\{ \frac{|x_{2n+1}|^2}{\epsilon^2}, |x_{2n+1}| \right\}, \quad x \in \mathbb{H}^n,$$

and [37, Lemma 2.13] implies that the distance function $d_{G,\epsilon}(x,y) := G_\epsilon(y^{-1} \cdot x)$ is equivalent to $d_\epsilon$, i.e. there exists a constant $A > 0$ such that $\forall \epsilon > 0$ and $\forall x, y \in M_\epsilon$

$$A^{-1} d_{G,\epsilon}(x,y) \le d_\epsilon(x,y) \le A d_{G,\epsilon}(x,y).$$

This implies that the doubling property

$$\mathcal{L}^{2n+1}(B_\epsilon(x, 2r)) \le C \mathcal{L}^{2n+1}(B_\epsilon(x,r)),$$

holds uniformly in $\epsilon$, with a constant $C > 1$ independent of $\epsilon$.

If $\Omega \subseteq \mathbb{H}^n$ is an open set and $\varphi : \Omega \to \mathbb{R}$ is a measurable function, the gradient associated with the Riemannian metric $b_\epsilon$ is

$$\nabla_\epsilon \varphi := \sum_{i=1}^{2n+1} X_i^\epsilon \varphi X_i^\epsilon = \sum_{i=1}^{2n} X_i \varphi X_i + \epsilon^2 X_{2n+1} u X_{2n+1},$$

which has to be understood in the sense of distributions. It is obvious that

$$\nabla_\epsilon \varphi \longrightarrow \nabla_H \varphi, \text{ as } \epsilon \to 0$$

and, if we denote by $|\cdot|_\epsilon$ the norm associated to the scalar product $b_\epsilon$, then

$$|\nabla_\epsilon \varphi|_\epsilon^2 = \sum_{i=1}^{2n} (X_i \varphi)^2 + \epsilon^2 (X_{2n+1} \varphi)^2 \to |\nabla_H \varphi|_H^2, \text{ as } \epsilon \to 0.$$

For every $1 \le q \le \infty$, one can define the Banach space

$$W_\epsilon^{1,q}(\Omega) := \{\varphi : \Omega \to \mathbb{R} : \varphi \in L^q(\Omega), \nabla_\epsilon \varphi \in L^q(\Omega, T\Omega)\}, \tag{1.22}$$

equipped with the norm

$$\|\varphi\|_{W_\epsilon^{1,q}(\Omega)} := \|\varphi\|_{L^q(\Omega)} + \|\nabla_\epsilon \varphi\|_{L^q(\Omega, T\Omega)}.$$

In the end let us recall that Sobolev's inequalities hold with constants independent of $\epsilon$, see for instance [37] and [53].

**Theorem 1.1.7.** *Let $1 \le q < N$. For any $\varphi \in W_\epsilon^{1,q}(B_\epsilon(x,r))$, $B_\epsilon(x,r) \subset \mathbb{H}^n$, it holds that*

$$\left( \fint_{B_\epsilon(x,r)} |\varphi|^{\frac{Nq}{N-q}} dx \right)^{\frac{N-q}{Nq}} \le \bar{c} r \left( \fint_{B_\epsilon(x,r)} |\nabla_\epsilon \varphi|_\epsilon^q dx \right)^{\frac{1}{q}},$$

*where $\bar{c} = \bar{c}(n,q)$, independent of $\epsilon$.*

## 1.2 Optimal transport theory in $\mathbb{H}^n$

In this section we collect some known results about optimal transport theory in the Heisenberg Group. We are interested in particular in the Monge problem associated with the sub-Riemannian distance: hence, most of the following results are taken from [52]. See also [16] and [46].

### 1.2.1 The Monge-Kantorovich problem with generic cost

In this subsection we write down in $\mathbb{H}^n$ some classical results about optimal transport theory, which hold in the more general structure of Polish metric spaces. See for instance [3] and [87].

Let us denote by $\mathcal{P}(\mathbb{H}^n)$, resp. $\mathcal{P}(\mathbb{H}^n \times \mathbb{H}^n)$, the set of Borel probability measures on $\mathbb{H}^n$, resp. on $\mathbb{H}^n \times \mathbb{H}^n$.

Let $\mu, \nu \in \mathcal{P}(\mathbb{H}^n)$, we denote by

$$\Pi(\mu,\nu) = \left\{\gamma \in \mathcal{P}(\mathbb{H}^n \times \mathbb{H}^n) : (\pi_1)_\# \gamma = \mu, (\pi_2)_\# \gamma = \nu\right\}$$

the set of *transport plans between* $\mu$ *and* $\nu$, where $\pi_1$ and $\pi_2$ are the projection on the first and second factor, respectively. Let us just remark that $\Pi(\mu,\nu)$ is compact w.r.t. the weak convergence of measures.

Given a lower semicontinuous cost function $k : \mathbb{H}^n \times \mathbb{H}^n \to [0,+\infty]$, the *Monge-Kantorovich transport problem* between $\mu$ and $\nu$ associated with the cost $k$

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{H}^n \times \mathbb{H}^n} k(x,y)\, d\gamma(x,y), \tag{1.23}$$

admits solutions, see [87, Theorem 1.7].

We denote by

$$\Pi_k(\mu,\nu) := \{\gamma \in \Pi(\mu,\nu) : \gamma \text{ solves } (1.23)\}$$

the set of *optimal transport plan* for the generic cost function $k$: it is a closed subset of the compact set $\Pi(\mu,\nu)$, w.r.t. the weak convergence of measures. Moreover, if $\gamma \in \Pi_k(\mu,\nu)$ and

$$\int_{\mathbb{H}^n \times \mathbb{H}^n} k d\gamma < +\infty,$$

then $\gamma$ is concentrated on a $k$-cyclically monotone $\sigma$-compact set $\Gamma \subseteq \mathbb{H}^n \times \mathbb{H}^n$, i.e.

$$\sum_{i=1}^N k(x_i, y_i) \le \sum_{i=1}^N k(x_{i+1}, y_i)$$

whenever $N \ge 2$ and $(x_1, y_1), \ldots, (x_N, y_N) \in \Gamma$, see [3, Theorem 4.1].

**Transport Plans and Transport Maps**

We say that a transport plan $\gamma \in \Pi(\mu, \nu)$ is *induced by a transport map* if there exists a Borel map

$$T : \mathbb{H}^n \to \mathbb{H}^n \text{ such that } (I \otimes T)_{\#}\mu = \gamma,$$

where $(I \otimes T)(x) := (x, T(x))$. We will refer to such a map $T : \mathbb{H}^n \to \mathbb{H}^n$ solving the following *Monge problem*

$$\inf_{T_{\#}\mu=\nu} \int_{\mathbb{H}^n} k(x, T(x)) d\mu(x), \tag{1.24}$$

as an *optimal transport map* for the cost function $k$.

In the end let us just remark that, if an optimal transport plan $\gamma \in \Pi_k(\mu, \nu)$ is induced by a transport map $T$, then $T$ is an optimal transport map for the cost $k$. Moreover, if any optimal transport plan $\gamma \in \Pi_k(\mu, \nu)$ is induced by a transport map, then there exists a unique optimal transport map. Hence, $\gamma \in \Pi_k(\mu, \nu)$ is unique.

## 1.2.2 The Monge-Kantorovich problem with cost depending on the sub-Riemannian distance

From now on we consider two compactly supported Borel probability measures $\mu, \nu \in \mathcal{P}_c(\mathbb{H}^n)$ and a cost functions $k$ of the type

$$k = h \circ d_{SR},$$

where $h : [0, \infty] \to [0, \infty]$ is a strictly convex function.

The following uniqueness result holds. See [47, Theorem 5.3 and Corollary 5.4] for the proof.

**Theorem 1.2.1.** *Let us suppose that $\mu \ll \mathcal{L}^{2n+1}$. Then, for any optimal transport plan $\gamma \in \Pi_k(\mu, \nu)$ there exists a Borel map $T : \mathbb{H}^n \to \mathbb{H}^n$ such that $T_{\#}\mu = \nu$. Hence, both the optimal transport map and the optimal transport plan are unique.*

*Remark* 2. If $h(s) = s^2$, the optimal transport problem reads as

$$\inf_{\gamma\in\Pi(\mu,\nu)} \int_{\mathbb{H}^n\times\mathbb{H}^n} d_{SR}(x,y)^2 d\gamma(x,y).$$

This problem was first investigated in [5], where the authors prove the Heisenberg version of the well known *Brenier Theorem*, see [87, Theorem 1.22]. This result was first extended to the $H$-type Carnot groups in [83], and then to a sub-Riemannian manifold for which the distance is locally Lipschitz (or locally semi-concave) outside of the diagonal, in [58].

Theorem 5.3 and Corollary 5.4 in [47] extend the previous result to non-branching metric measure spaces satisfying the Measure Contraction Property, without an explicit form of the optimal map. Recently, in [48] the authors replaced the non-branching hypothesis with the weaker *essentially non-branching* hypothesis, see [48, Definition 2.2].

**The Monge problem with $k(x,y) = d_{SR}(x,y)$**

Let us suppose that

$$k(x,y) = d_{SR}(x,y), \quad \forall (x,y) \in \mathbb{H}^n \times \mathbb{H}^n.$$

From the arguments in Subsection 1.2.1 it follows that the Monge-Kantorovich transport problem

$$\inf_{\gamma\in\Pi(\mu,\nu)} \int_{\mathbb{H}^n\times\mathbb{H}^n} d_{SR}(x,y)\, d\gamma(x,y), \tag{1.25}$$

admits solutions: any solution to (1.25) is concentrated on a $d_{SR}$-cyclically monotone set.

From now on we denote by

$$\Pi_1(\mu,\nu) := \{\gamma \in \Pi(\mu,\nu) : \gamma \text{ solves } (1.25)\}$$

and by

$$\mathrm{Lip}_1(\mathbb{H}^n, d_{SR}) := \{u : \mathbb{H}^n \to \mathbb{R} : |u(x) - u(y)| \le d_{SR}(x,y), \forall x,y \in \mathbb{H}^n\}\,. \tag{1.26}$$

The problem (1.25) is equivalent to the following *Kantorovich dual problem*:

$$\sup\left\{\int_{\mathbb{H}^n} ud(\mu-\nu) : u \in \mathrm{Lip}_1(\mathbb{H}^n, d_{SR})\right\}. \tag{1.27}$$

Moreover, (1.27) admits a solution and the following theorem holds.

**Theorem 1.2.2.** *There exists a function $u \in \mathrm{Lip}_1(\mathbb{H}^n, d_{SR})$ such that*

$$\min_{\gamma\in\Pi(\mu,\nu)} \int_{\mathbb{H}^n\times\mathbb{H}^n} d_{SR}(x,y)\, d\gamma(x,y) = \int_{\mathbb{H}^n} u(x)\, d\mu(x) - \int_{\mathbb{H}^n} u(y)\, d\nu(y),$$

*and $\gamma \in \Pi(\mu,\nu)$ is optimal if and only if*

$$u(x) - u(y) = d_{SR}(x,y) \qquad \gamma - a.e. \text{ in } \mathbb{H}^n \times \mathbb{H}^n.$$

We call such a $u \in \mathrm{Lip}_1(\mathbb{H}^n, d_{SR})$ a *Kantorovich potential.*

If the measure $\mu$ is absolutely continuous with respect to the Haar measure of the group, from the explicit representation of minimizing geodesics (1.1.4) it follows that any optimal transport plan $\gamma \in \Pi_1(\mu,\nu)$ is concentrated on the set

$$E := \{(x,y) \in \mathbb{H}^n \times \mathbb{H}^n;\ x^{-1}\cdot y \notin L\}, \tag{1.28}$$

of pairs of points connected by a unique minimizing geodesic, see [52, Lemma 4.1].

**Proposition 1.2.3.** *Let $\mu \ll \mathcal{L}^{2n+1}$. Given $\gamma \in \Pi_1(\mu,\nu)$, then for $\gamma$-a.e. $(x,y) \in \mathbb{H}^n \times \mathbb{H}^n$, there exists a unique minimizing geodesic between $x$ and $y$, i.e.*

$$\gamma(\mathbb{H}^n \times \mathbb{H}^n \setminus E) = 0.$$

If $u \in \mathrm{Lip}_1(\mathbb{H}^n, d_{SR})$ is a Kantorovich potential and $\gamma \in \Pi_1(\mu,\nu)$ is an optimal transport plan concentrated on some set $\Gamma \subset \mathbb{H}^n \times \mathbb{H}^n$, then for any $(x,y) \in \Gamma$ it holds that

$$u(x) - u(y) = d_{SR}(x,y),$$

see Theorem 1.2.2.

Let $(x, y) \in \mathbb{H}^n \times \mathbb{H}^n$ and $\sigma_{x,y}$ be a geodesic between $x$ and $y$, the Lipschitzianity of $u$ implies

$$u(\sigma_{x,y}(t)) = u(x) - d_{SR}(x, \sigma_{x,y}(t)), \quad \forall t[0,1].$$

In this way one can define an order relation on the minimizing geodesic $\sigma_{x,y}$ in the following way: let $t_1, t_2 \in [0,1]$, $x' = \sigma_{x,y}(t_1)$ and $x'' := \sigma_{x,y}(t_2)$, then

$$x' \leq x'' \Leftrightarrow u(x') \geq u(x''). \tag{1.29}$$

Following the existence literature for the Riemannian setting, see [87, Chapter 3] and [56, Section 2], we fix a Kantorovic potential $u \in \mathrm{Lip}_1(\mathbb{H}^n, d_{SR})$ and we use it to check the optimality of transport plans. In this way one can select some optimal transport plans that satisfy a monotonicity condition, according to (1.29).

More precisely, we denote by $\Pi_2(\mu, \nu)$ the set of transport plans solving the secondary variational problem

$$\inf_{\gamma \in \Pi_1(\mu,\nu)} \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y)^2 \, d\gamma(x,y). \tag{1.30}$$

This problem admits solutions since the functional

$$\Pi(\mu,\nu) \ni \gamma \mapsto \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y)^2 \, d\gamma(x,y)$$

is continuous w.r.t. the weak convergence of measures and $\Pi_1(\mu, \nu)$ is compact w.r.t. the same convergence. Again from Theorem 1.2.2, we can rephrase problem (1.30) as a classical Kantorovich transport problem

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{H}^n \times \mathbb{H}^n} \beta(x,y) d\gamma(x,y)$$

where the cost $\beta(x, y)$ is defined in the following way

$$\beta(x,y) = \begin{cases} d_{SR}(x,y)^2 & \text{if } u(x) - u(y) = d_{SR}(x,y), \\ +\infty & \text{otherwise.} \end{cases}$$

Since $\beta$ is lower semicontinuous and $\int_{\mathbb{H}^n\times\mathbb{H}^n}\beta(x,y)\,d\gamma(x,y) < +\infty$ for all $\gamma\in\Pi_2(\mu,\nu)$, it follows that any $\gamma\in\Pi_2(\mu,\nu)\subset\Pi_1(\mu,\nu)$ is concentrated on a $\beta$-cyclically monotone set $\Gamma$, i.e.

$$u(x)-u(y)=d_{SR}(x,y),\quad \forall (x,y)\in\Gamma, \tag{1.31}$$

and

$$\beta(x,y)+\beta(x',y')\le\beta(x,y')+\beta(x',y),\quad \forall (x,y),(x',y')\in\Gamma. \tag{1.32}$$

Using the non-branching property of $(\mathbb{H}^n, d_{SR})$ one can prove that geodesics used by optimal transport plans cannot bifurcate. Moreover, if an optimal plan solves also (1.30) then, using (1.31) and (1.32), one can get a one-dimensional monotonicity condition along minimizing geodesics. More precisely, the following result holds [52, Lemma 4.2 and Lemma 4.3]:

**Proposition 1.2.4.** *Let $\gamma\in\Pi_1(\mu,\nu)$. Then $\gamma$ is concentrated on a set $\Gamma$ such that for all $(x,y)$, $(x',y')\in\Gamma$ such that $x\neq y$ and $x\neq x'$, if $x'$ lies on a geodesic between $x$ and $y$ then all points $x$, $x'$, $y$ and $y'$ lie on the same geodesic. Moreover if $\gamma\in\Pi_2(\mu,\nu)$, then*

$$x<x'\Rightarrow y\le y'. \tag{1.33}$$

As far as we know, in the Heisenberg Group has not been proven that any $\gamma\in\Pi_2(\mu,\nu)$ is induced by a transport map, and hence $\gamma\in\Pi_2(\mu,\nu)$ is unique. See [87, Theorem 3.18] and [56, Theorem 28] for the analogous result in the Euclidean and Riemannian setting, respectively. Anyway in [52] the authors proved that some particular transport plans in $\Pi_2(\mu,\nu)$, more precisely the ones that can be selected through the variational approximation below, are induced by transport maps.

### Variational approximation and existence of optimal maps

Let $K$ be a compact subset of $\mathbb{H}^n$ such that

$$\mathrm{supp}(\mu)\cup\mathrm{supp}(\nu)\subset K,$$

and let us denote by

$$\Pi := \{\gamma \in \mathcal{P}(\mathbb{H}^n \times \mathbb{H}^n) : (\pi_1)_\# \gamma = \mu,\ \operatorname{supp}((\pi_2)_\# \gamma) \subset K\}.$$

For any $\varepsilon \in \mathbb{R}^+$, we can consider the family of minimization problems

$$\min\{C_\varepsilon(\gamma) : \gamma \in \Pi\}, \tag{$P_\varepsilon$}$$

where

$$C_\varepsilon(\gamma) := \frac{1}{\varepsilon} W_1((\pi_2)_\#\gamma, \nu) + \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y)\, d\gamma(x,y)$$
$$+ \varepsilon \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y)^2\, d\gamma(x,y) + \varepsilon^{6n+8} \operatorname{card}(\operatorname{supp}((\pi_2)_\#\gamma)),$$

where $W_1((\pi_2)_\#\gamma, \nu)$ denotes the *1-Wasserstein distance* between the two measures $(\pi_2)_\#\gamma$ and $\nu$,

$$W_1((\pi_2)_\#\gamma, \nu) := \min\left\{\int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y) : \gamma \in \Pi((\pi_2)_\#\gamma, \nu)\right\}.$$

One can prove that, for any $\varepsilon > 0$ the minimization problem $P_\varepsilon$ admits at least one finite solution. Moreover it holds the following result:

**Lemma 1.2.5.** *Let $(\varepsilon_k)_{k\in\mathbb{N}} \subset \mathbb{R}^+$ be a sequence such that $\varepsilon_k \underset{k\to\infty}{\longrightarrow} 0$ and $\gamma_{\varepsilon_k}$ be a solution to $(P_{\varepsilon_k})$ $\forall k$, such that $\gamma_{\varepsilon_k} \rightharpoonup \gamma \in \mathcal{P}(\mathbb{H}^n \times \mathbb{H}^n)$. Then $\nu_{\varepsilon_k} := (\pi_2)_\#\gamma_{\varepsilon_k} \rightharpoonup \nu$ and $\gamma \in \Pi_2(\mu, \nu)$.*

In particular, if the measure $\mu$ is absolutely continuous w.r.t. the Haar measure of the group, the optimal transport plans that are weak limits of solutions $(\gamma_{\varepsilon_k})_{k\in\mathbb{N}}$ to $(P_{\varepsilon_k})_{k\in\mathbb{N}}$, for some $\varepsilon_k \to 0$ as $k \to \infty$, turn out to be induced by maps. Hence, the problem

$$\inf_{T_\#\mu=\nu} \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y)$$

admits a solution. Moreover, these optimal plans are monotone in the sense of (1.33).

**Theorem 1.2.6** (Theorem 8.1 in [52])**.** *Assume that $\mu \ll \mathcal{L}^{2n+1}$. Then there exists an optimal transport map $T : \mathbb{H}^n \to \mathbb{H}^n$ such that $\gamma = (\mathrm{Id} \otimes T)_\# \mu \in \Pi_2(\mu, \nu)$.*

We preferred to treat the existence of solutions to (1.24) following [52] because in the next chapter we will need the monotonicity property (1.33). Anyhow more recent existence results are also available in [16] and [46] in the more general setting of non -branching geodesic metric measure spaces, satisfying the Measure Contraction Property.

# Chapter 2

# Equivalent formulations of optimal transport problem in $\mathbb{H}^n$

The aim of this chapter is twofold: first we introduce the concept of *horizontal transport density* in $\mathbb{H}^n$; second, we introduce some equivalent formulations of the Monge-Kantorovich problem

$$\min_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y),$$

where $\mu, \nu \in \mathcal{P}_c(\mathbb{H}^n)$. These reformulations will turn out to be the limit cases of two other problems we will introduce in the next two chapters.

We first prove some geometric properties of the *transport rays*: we will use them both to introduce in Section 2.3 the *Beckmann formulation* of the aforementioned Monge-Kantorovich problem and to prove some summability results about horizontal transport densities. In the end, in Section 2.4 we show that this problem also admits a *Lagrangian reformulation*.

## 2.1 Geometric properties of transport rays

In the spirit of [87, Chapter 3] and [56, Section 2], we collect some properties of the transport rays and the transport set, see Definition 2.1 and (2.9) for rigorous definitions. We also show a differentiability result for Kantorovich potentials.

Let us consider $\mu, \nu \in \mathcal{P}_c(\mathbb{H}^n)$ and a Kantorovich potential $u \in \mathrm{Lip}_1(\mathbb{H}^n, d_{SR})$.

Let us just remark that if $x \in$ supp $(\mu)$ and $y \in \mathrm{supp}(\nu)$ are such that

$$u(x) - u(y) = d_{SR}(x, y), \tag{2.1}$$

then

$$u(\sigma_{x,y}(t)) = u(x) - d_{SR}\left(x, \sigma_{x,y}(t)\right), \quad \forall t[0,1],$$

where $\sigma_{x,y}$ is a geodesic between $x$ and $y$. This means that $\sigma_{x,y}$ is a curve along which an optimal transport may occur, see Theorem 1.2.2.

We will call *transport ray* any non-trivial geodesic along which an optimal transport may occur.

**Definition 2.1.** A transport ray is a non-trivial geodesic $\sigma : [0,1] \to \mathbb{H}^n$ such that

1. $\sigma(0) \in \mathrm{supp}(\mu)$ and $\sigma(1) \in \mathrm{supp}(\nu)$;
2. $u(\sigma(0)) - u(\sigma(1)) = d_{SR}(\sigma(0), \sigma(1))$;

### 2.1.1 Pansu differentiability of the Kantorovich potential

Following [56, Lemma 10], one can prove that the Kantorovich potential $u$ is Pansu differentiable in the interior of transport rays.

**Proposition 2.1.1.** *Let $u \in \mathrm{Lip}_1(\mathbb{H}^n, d_{SR})$ and $x, y \in \mathbb{H}^n, x \neq y$ such that $u(x) - u(y) = d_{SR}(x, y)$. Let $\sigma : [0,1] \to \mathbb{H}^n$ be a geodesic between $x$ and $y$, starting from $x$, then $u$ is Pansu differentiable in $\sigma(t)$, for all $t \in ]0,1[$, and*

$$\nabla_H u(\sigma(t)) = -\frac{\dot{\sigma}(t)}{|\dot{\sigma}(t)|_H}. \tag{2.2}$$

For the proof of this result we need the following two lemmas. The next one collects a differentiability property of the Carnot-
Charatheodory distance function from a fixed point $y \in \mathbb{H}^n$. Let $L_y := y \cdot L$.

**Lemma 2.1.2.** *The function $d_y(\cdot) := d_{SR}(\cdot, y)$ is of class $C^\infty$ in the euclidean sense on $\mathbb{H}^n \setminus L_y$. In particular $d_y$ is Pansu differentiable on $\mathbb{H}^n \setminus L_y$. Moreover, if $x \in \mathbb{H}^n \setminus L_y$ and $\sigma : [0,1] \to \mathbb{H}^n$ is the geodesic between $x$ and $y$, starting from $y$, then*

$$\nabla_H d_y(x) = \frac{\dot\sigma(1)}{|\dot\sigma(1)|_H}.$$

*Proof.* Let us set $\Phi(\chi, \varphi) := y \cdot \sigma(1)$, where $\sigma : [0,1] \to \mathbb{H}^n$ is a geodesic starting from 0, as in Theorem 1.1.4. This map is a $C^\infty$-diffeomorphism from $\mathbb{R}^{2n} \setminus \{0\} \times (-2\pi, 2\pi)$ onto $\mathbb{H}^n \setminus L_y$ in the usual euclidean sense, see e.g. [71], [6], [66]. In particular it is Pansu differentiable and, from the fact that $d_y \in \mathrm{Lip}_1(\Omega, d_{SR})$, it holds $|\nabla_H d_y(x)| \le 1, \forall x \in \mathbb{H}^n \setminus L_y$. Let us denote by $x := \sigma(1) \in \mathbb{H}^n \setminus L_y$: since $d_{SR}(\sigma(t)) = t d_{SR}(x,y)$ for all $t \in [0,1]$, and $\sigma(t) \in \mathbb{H}^n \setminus L_y$ for all $t \in ]0,1]$, we can differentiate w.r.t. $t$:

$$\begin{aligned} d_{SR}(x,y) = \frac{d}{dt} d_y(\sigma(t)) = \sum_{j=1}^{2n} X_j(d_y(\sigma(t)))\dot\sigma_j(t) \\ \le |\nabla_H d_y(\sigma(t))|_H |\dot\sigma(t)|_H \le d_{SR}(x,y), \end{aligned} \tag{2.3}$$

where we used the fact that $\sigma$ is a geodesic, then $|\dot\sigma(t)|_H = d_{SR}(x,y)$. Hence, all the inequalities in (2.3) are equalities: in particular we get that

$$\nabla_H d_y(\sigma(t)) = \frac{\dot\sigma(t)}{|\dot\sigma(t)|_H}, \quad \forall t \in ]0,1].$$

□

**Lemma 2.1.3.** *Lef $f, g, h : \mathbb{H}^n \to \mathbb{R}$ three functions such that $f(x) \le g(x) \le h(x)$, for all $x \in \mathbb{H}^n$. Let $y \in \mathbb{H}^n$ such that $f(y) = g(y) = h(y)$ and $f, h$ are Pansu differentiable at $y$, with $\nabla_H f(y) = \nabla_H h(y)$. Then, $g$ is Pansu differentiable in $y$ and $\nabla_H g(y) = \nabla_H f(y) = \nabla_H h(y)$.*

*Proof.* From [60, Proposition 5.6] it follows that,

$$D_H f(y)(x) = \langle \nabla_H f(y), \pi_y(x) \rangle_H = \langle \nabla_H h(y), \pi_y(x) \rangle_H = D_H h(y)(x),$$

where the map $y \to \pi_y(x)$ is the smooth section of $H\mathbb{H}^n$ defined as

$$\pi_y(x) := \sum_{i=1}^{2n} x_i X_i(y). \tag{2.4}$$

Then

$$f(x) - f(y) - D_H f(y)(y^{-1} \cdot x) \leq g(x) - g(y) - \underbrace{D_H f(y)(y^{-1} \cdot x)}_{=D_H h(y)(y^{-1}\cdot x)} \leq$$
$$\leq h(x) - h(y) - D_H h(y)(y^{-1} \cdot x).$$

If we divide the previous inequalities by $d_{SR}(x, y)$ and we let $x$ tend to $y$, by using Pansu differentiability of $f$ and $g$ and [60, Proposition 5.6] again, we get the thesis. □

*Proof of Proposition 2.1.1.* Let $t_0 \in ]0, 1[$ and $a, b \in \sigma([0, 1])$ such that $u(a) > u(\sigma(t_0)) > u(b)$. From Lemma 2.1.2 it follows that the functions $d_a(\cdot)$ and $d_b(\cdot)$ are smooth in a neighborhood of $\sigma(t_0)$. Since $u \in \mathrm{Lip}_1(\mathbb{H}^n, d_{SR})$, it holds that

$$u(a) - u(z) \leq d_a(z), \quad \forall z \in \mathbb{H}^n.$$

Moreover $a$ and $b$ lie on the geodesic between $x$ and $y$, then $u(a) = u(b) + d_{SR}(a, b)$, and hence

$$d_b(z) \geq u(z) - u(b) \geq d_{SR}(a, b) - d_a(z), \quad \forall z \in \mathbb{H}^n,$$

where equalities hold if $z = \sigma(t_0)$, since $\sigma$ is a geodesic. Moreover, from Theorem 2.1.2 again, it follows that $\nabla_H d_b(\sigma(t_0)) = -\nabla_H d_a(\sigma(t_0)) = -\frac{\dot\sigma(t_0)}{|\dot\sigma(t_0)|_H}$. Hence, from Lemma 2.1.3, it follows that $u$ is Pansu differentiable in $\sigma(t_0)$ and

$$\nabla_H u(\sigma(t_0)) = -\frac{\dot\sigma(t_0)}{|\dot\sigma(t_0)|_H}.$$

□

### 2.1.2 Disjointness of transport rays

We now introduce the following lemma, which guarantees that transport rays cannot intersect at points that are in the interior of both of them.

**Lemma 2.1.4.** *Let $\gamma \in \Pi_1(\mu, \nu)$. Then $\gamma$ is concentrated on a set $\Gamma$ such that $\forall (x,y),(x',y') \in \Gamma$ with $(x,y) \neq (x',y')$, if two transport rays between these two pairs of points intersect at an interior point $z \in \mathbb{H}^n$, then all points $x$, $x'$, $y$, $y'$ and $z$ lie on the same transport ray. Moreover if $\gamma \in \Pi_2(\mu,\nu)$, then either $x \leq x' \leq z \leq y \leq y'$ or $x' \leq x \leq z \leq y' \leq y$.*

*Proof.* We first recall that (1.2.1) reads as

$$d_{SR}(x,y) + d_{SR}(x',y') \leq d_{SR}(x,y') + d_{SR}(x',y), \tag{2.5}$$

$\forall (x,y),(x',y') \in \Gamma$. Let $\sigma : [0, d_{SR}(x,y)] \to \mathbb{H}^n$ be a geodesic between $x$ and $y$, $\tilde{\sigma} : [0, d_{SR}(x',y')] \to \mathbb{H}^n$ a geodesic between $x'$ and $y'$, $z \in \sigma(0, d_{SR}(x,y)) \cap \tilde{\sigma}(0, d_{SR}(x',y'))$, so $z = \sigma(d_{SR}(x,z)) = \tilde{\sigma}(d_{SR}(x',z))$. We denote by $\alpha$ the curve between $x$ and $y'$ defined in the following way:

$$\alpha(t) := \begin{cases} \sigma\left(\frac{d_{SR}(x,z)}{d_{SR}(x',z)}t\right), & \text{if } t \in [0, d_{SR}(x',z)], \\ \tilde{\sigma}(t), & \text{if } t \in (d_{SR}(x',z), d_{SR}(x',y')]. \end{cases}$$

$$\alpha(t) := \begin{cases} \sigma(t), & \text{if } t \in [0, d_{SR}(x,z)], \\ \tilde{\sigma}(t), & \text{if } t \in [d_{SR}(x,z), d_{SR}(x',y')]. \end{cases}$$

We will prove that $\alpha$ is geodesic between $x$ and $y'$. Indeed, otherwise we would have

$$\begin{aligned} d_{SR}(x,y') < l(\alpha) &= l(\alpha_{|[0,d_{SR}(x',z)]}) + l(\alpha_{|[d_{SR}(x',z),d_{SR}(x',y')]}) \\ &= d_{SR}(x,z) + d_{SR}(z,y'). \end{aligned} \tag{2.6}$$

Since $z$ lies on both the geodesic between $x$ and $y$ and the geodesic between $x'$ and $y'$, it follows that

$$\begin{cases} d_{SR}(x,y) = d_{SR}(x,z) + d_{SR}(z,y); \\ d_{SR}(x',y') = d_{SR}(x',z) + d_{SR}(z,y'). \end{cases} \tag{2.7}$$

By replacing (2.7) in (2.6), we obtain:

$$d_{SR}(x,y') + d_{SR}(z,y) + d_{SR}(x',z) < d_{SR}(x,y) + d_{SR}(x',y'). \tag{2.8}$$

By the triangle inequality follows that:

$$d_{SR}(x',y) \leq d_{SR}(x',z) + d_{SR}(z,y),$$

and then, by replacing this last inequality in (2.8), we obtain

$$d_{SR}(x,y') + d_{SR}(x',y) < d_{SR}(x,y) + d_{SR}(x',y'),$$

and this contraddicts (2.5). It follows that $\tilde{\sigma}$ and $\alpha$ are geodesics that coincide on the non-trivial interval $[d_{SR}(x',z), d_{SR}(x',y')]$. Since $\mathbb{H}^n$ is non-branching, this implies that $\tilde{\sigma}$ and $\alpha$ are sub-arcs of the same geodesic, namely $\alpha$ if $d_{SR}(x',z) \leq d_{SR}(x,z)$ and $\tilde{\sigma}$ otherwise, on which all points $x, x', z, y'$ lie.

The thesis follows from Proposition 1.2.4. □

### 2.1.3 Compactness of the transport set

We denote by $\mathcal{T}_1$ the set of all points which lie on transport rays

$$\mathcal{T}_1 := \bigcup \{x \in \sigma([0,1]) : \sigma \text{ is transport ray}\},$$

and by $\mathcal{T}_0$ the complementary set of *rays of length zero*

$$\mathcal{T}_0 := \Big\{ z \in \operatorname{supp}(\mu) \cap \operatorname{supp}(\nu) : |u(z) - u(z')| < d_{SR}(z,z'), \\ \forall z' \in \operatorname{supp}(\mu) \cup \operatorname{supp}(\nu), z \neq z' \Big\}.$$

We will call *transport set* the set

$$\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_0. \tag{2.9}$$

We observe that

$$\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu) \subseteq \mathcal{T}. \tag{2.10}$$

Moreover, as in [56, Lemma 8] the following result holds.

**Theorem 2.1.5.** *The transport set $\mathcal{T}$ is compact.*

*Proof.* Thanks to Hopf-Rinow theorem it is enough to prove that $\mathcal{T}$ is a closed and bounded set. Let us consider the function $v : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$, $v(x, y) = u(x) - u(y)$. Since $v$ is continuous, it attains a maximum $L < \infty$ on $\text{supp}(\mu) \times \text{supp}(\nu)$, which is a compact set. Let us prove that $L \geq 0$. If $\text{supp}(\mu) \cap \text{supp}(\nu) \neq \emptyset$, then $\forall x \in \text{supp}(\mu) \cap \text{supp}(\nu)$ we have that $(x, x) \in \text{supp}(\mu) \times \text{supp}(\nu)$ and $v(x, x) = 0$. Otherwise, from (2.10) it follows that $\mathcal{T}_1 \neq \emptyset$, hence there exists at least one transport ray $\sigma$. If $x = \sigma(0)$ and $y = \sigma(1)$, then $v(x, y) = d_{SR}(x, y) > 0$. Hence $L \geq 0$.

We can suppose that $A := \mathcal{T} \setminus \text{supp}(\mu) \cup \text{supp}(\nu) \neq \emptyset$; otherwise, from the previous theorem it follows that $\mathcal{T} = \text{supp}(\mu) \cup \text{supp}(\nu)$, which is compact. Hence, any $z \in A$ lies on a transport ray $\sigma_z$. Let us denote by $a = \sigma_z(0)$ and $b = \sigma_z(1)$, then

$$d_{SR}(a, z) + d_{SR}(b, z) = d_{SR}(a, b) = v(a, b) \leq L.$$

Hence, $A$ lies in the union of the $L$-neighborhoods of the compact sets $\text{supp}(\mu)$ and $\text{supp}(\nu)$, thus $\mathcal{T}$ is bounded.

Let us prove that $\mathcal{T}$ is closed. Let us consider $(z_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}$, converging to some $z$, we prove that $z \in \mathcal{T}$. If there exists a subsequence $(z_{n_k})_{k \in \mathbb{N}} \subseteq \text{supp}(\mu) \cup \text{supp}(\nu)$, then $z \in \text{supp}(\mu) \cup \text{supp}(\nu)$ by compactness of $\text{supp}(\mu)$ and $\text{supp}(\nu)$. Let us suppose that $z_n \in A, \forall n \in \mathbb{N}$: there exists a transport ray $\sigma_n$, whose endpoints we denote by $a_n = \sigma_n(0)$ and $b_n := \sigma_n(1)$. There exist two subsequences

$$a_{n_j} \to a \in \text{supp}(\mu) \text{ and } b_{n_j} \to b \in \text{supp}(\nu), \text{ when } j \to \infty. \tag{2.11}$$

Since

$$d_{SR}(z_{n_j}, a_{n_j}) + d_{SR}(z_{n_j}, b_{n_j}) = d_{SR}(a_{n_j}, b_{n_j}) = u(a_{n_j}) - u(b_{n_j}), \quad \forall n_j,$$

we have that

$$d_{SR}(z, a) + d_{SR}(z, b) = d_{SR}(a, b) = u(a) - u(b). \tag{2.12}$$

Hence there are two possibilities

1. $a = b \underset{(2.12)}{\Longrightarrow} z = a = b \in \mathcal{T}$;
2. $a \neq b \underset{(2.11)+(2.12)}{\Longrightarrow} z \in \mathcal{T}_1$.

□

## 2.2 Horizontal transport densities

The notion of transport density has been introduced in the Euclidean setting by Bouchitté and Buttazzo in [20], [21] and by Evans and Gangbo in [55]. In [20], [21] it was connected to some shape optimization-problems. In [55] it was used to get the existence of optimal maps for the Monge-Kantorovich probem associated with the Euclidean distance (see for instance [87, Section 3.1] for more details about this topic). In [57] and [86] for instance, the authors gave some sufficient conditions to get, respectively, results of uniqueness and summability for the transport density.

In this section we introduce the notion of *horizontal transport density*, adapting to the Heisenberg group setting the presentation provided in [44]. A horizontal transport density is a measure representing the density of transport along horizontal curves and it is computed using geodesics of the space. Then, we will investigate conditions under which the transport density is Lebesgue absolutely continuous, with $L^p$ density for some particular $p$'s.

Let $\mu, \nu \in \mathcal{P}_c(\mathbb{H}^n)$ be two compactly supported Borel probability measures over $\mathbb{H}^n$. From now on, we fix a selection of geodesics

$$S : \mathbb{H}^n \times \mathbb{H}^n \to \mathrm{Geo}(\mathbb{H}^n),$$

$S(x, y) = \sigma_{x,y} \in \mathrm{Geo}(\mathbb{H}^n)$, that is $\gamma$-measurable for any $\gamma \in \Pi(\mu, \nu)$. See Remark 1.

According to the terminology used in literature, we can give the following definition.

**Definition 2.2** (Horizontal transport density)**.** Given an optimal transport plan $\gamma \in \Pi_1(\mu, \nu)$, one can define a positive and finite Radon measure $a_\gamma \in$

$\mathcal{M}_+(\mathbb{H}^n)$,

$$\int_{\mathbb{H}^n} \varphi(x) da_\gamma(x) := \int_{\mathbb{H}^n \times \mathbb{H}^n} \left( \int_0^1 \varphi(\sigma_{x,y}(t)) |\dot{\sigma}_{x,y}(t)|_H dt \right) d\gamma(x,y), \qquad (2.13)$$

for any function $\varphi \in C_0(\mathbb{H}^n)$.

Here $C_0(\mathbb{H}^n)$ denotes the completion of the space of continuous function with compact support $C_c(\mathbb{H}^n)$, with respect to the norm

$$\|\varphi\|_\infty := \sup_{x \in \mathbb{H}^n} |\varphi(x)|.$$

This measure represents the amount of transport taking place in each region of $\mathbb{H}^n$. If we look at the action of $a_\gamma$ on sets, we have that for every Borel set $A$,

$$a_\gamma(A) = \int_{\mathbb{H}^n \times \mathbb{H}^n} \mathcal{H}^1(A \cap \sigma_{x,y}([0,1])) d\gamma(x,y).$$

One can define $a_\gamma \in \mathcal{M}_+(\mathbb{H}^n)$ as in (2.13), for any transport plan $\gamma \in \Pi(\mu, \nu)$, not necessarily optimal: in particular, if $\varphi \in C_0(\mathbb{H}^n)$ then

$$\left| \int_{\mathbb{H}^n} \varphi(x) da_\gamma(x) \right| \leq \|\varphi\|_\infty \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y) = C \|\varphi\|_\infty,$$

where $C > 0$ is a finite constant because $\mu$ and $\nu$ have compact support. Moreover, if $\gamma \in \Pi_1(\mu, \nu)$, then $a_\gamma$ is supported on the compact set $\mathcal{T}$, see (2.9) and Theorem 2.1.5, and its total mass satisfies

$$a_\gamma(\mathbb{H}^n) \leq \min_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y).$$

This in turn implies that one can define the transport densities $a_\gamma \in \mathcal{M}_+(\mathbb{H}^n)$ in duality with the continuous functions $C(\mathbb{H}^n)$.

In the end, let us remark that if $\mu \ll \mathcal{L}^{2n+1}$ and $\gamma \in \Pi_1(\mu, \nu)$, then $\gamma(\mathbb{H}^n \times \mathbb{H}^n \setminus E) = 0$, see Proposition 1.2.3, and therefore the definition of $a_\gamma$ is independent of the choice of the selection $S$.

### 2.2.1 Absolute continuity of horizontal transport densities

The first goal is to prove the existence of at least one horizontal transport density, absolutely continuous w.r.t. the Haar measure of the group.

Given an optimal transport plan $\gamma \in \Pi_1(\mu, \nu)$, we consider the interpolation measures between $\mu$ and $\nu$

$$(\mu_t)_{t\in[0,1]} := ((S_t)_\# \gamma)_{t\in[0,1]};$$

hence, the horizontal transport density $a_\gamma$ may be written as

$$a_\gamma = \int_0^1 (S_t)_\# (d_{SR}\gamma) dt$$

where $d_{SR}\gamma$ is a positive measure on $\mathbb{H}^n \times \mathbb{H}^n$. Since $\gamma$ has compact support, then there exists $C > 0$ such that $d_{SR}(x, y) \leq C$ for any $(x, y) \in \operatorname{supp}(\gamma)$ and hence

$$a_\gamma \leq C \int_0^1 \mu_t dt. \tag{2.14}$$

In order to prove that $a_\gamma$ is absolutely continuous w.r.t. $\mathcal{L}^{2n+1}$, it is sufficient to prove that $\mu_t$ is absolutely continuous w.r.t. $\mathcal{L}^{2n+1}$ for almost every $t \in [0, 1]$. In this way we get that, whenever $\mathcal{L}^{2n+1}(A) = 0$, then

$$a_\gamma(A) \leq C \int_0^1 \mu_t(A) dt = 0. \tag{2.15}$$

In particular if $\gamma \in \Pi_1(\mu, \nu)$ is induced by a transport map, i.e. is of the form $\gamma := (\mathrm{Id} \otimes T)_\# \mu \in \Pi(\mu, \nu)$, where $T : \mathbb{H}^n \to \mathbb{H}^n$ is a measurable map, we denote by

$$T_t := S_t \circ (\mathrm{Id} \otimes T) : \mathbb{H}^n \to \mathbb{H}^n, \quad \forall t \in [0, 1].$$

Here $T_t(x)$ is the point at distance $t d_{SR}(x, T(x))$ from $x$, on the selected geodesic $S(x, T(x))$ between $x$ and $T(x)$. Then

$$\mu_t = (T_t)_\# \mu.$$

Also in this case, if $\mu \ll \mathcal{L}^{2n+1}$, then all the above quantities are independent of the choice of the selection $S$.

If $\mu \ll \mathcal{L}^{2n+1}$, we are able to find at least an optimal transport plan $\gamma \in \Pi_1(\mu,\nu)$, such that the interpolation measures $\mu_t$ constructed from $\gamma$ are absolutely continuous for $t<1$.

**Proposition 2.2.1.** *Suppose that $\mu \ll \mathcal{L}^{2n+1}$ then, there exists $\gamma \in \Pi_1(\mu,\nu)$ such that*

$$\mu_t := (S_t)_\# \gamma \ll \mathcal{L}^{2n+1}, \quad \forall t \in [0,1). \tag{2.16}$$

*Proof.* First we suppose that $\nu$ is finitely atomic, with atoms $(y^i)_{i=1}^M$. Let $\gamma \in \Pi_2(\mu,\nu) \subset \Pi_1(\mu,\nu)$, as in Theorem 1.2.6, which is monotone in the sense of (1.29) and induced by a transport map $T$. Let us denote by $\Gamma \subseteq \mathbb{H}^n \times \mathbb{H}^n$ the set $\gamma$ is concentrated on and (1.31) and (1.32) hold.

We denote by $\Omega_i := T^{-1}(\{y^i\}) \cap \pi_1(\Gamma)$: obviously these sets are mutually disjoint and $\mu(\Omega)=1$, where $\Omega := \bigcup_{i=1}^M \Omega_i$.

Now we denote by $\Omega_i(t) := T_t(\Omega_i)$: if we fix $t \in [0,1[$, then $\Omega_i(t) \cap \Omega_j(t) = \emptyset$ for every $i,j=1,\dots,M$. Indeed, if $\exists\, z \in \Omega_i(t) \cap \Omega_j(t)$ then $\exists\, x^i \in \Omega_i$ and $x^j \in \Omega_j$ such that $(x^i,y^i),(x^j,y^j) \in \Gamma$, $(x^i,y^i) \neq (x^j,y^j)$ and the geodesics between these two pairs of points intersect at $z$. Since $\gamma \in \Pi_2(\mu,\nu)$, by Theorem 2.1.4 we can suppose that $x^i, y^i, x^j, y^j, z$ belong to the same unit-speed geodesic and $x^i \le x^j \le z \le y^i \le y^j$. In particular this means, on the one hand, that $t d_{SR}(x^i,y^i) = d_{SR}(x_i,z) \ge d_{SR}(x^j,z) = t d_{SR}(x^j,y^j)$, hence $d_{SR}(x^i,y^i) \ge d_{SR}(x^j,y^j)$. On the other hand $(1-t)d_{SR}(x^i,y^i) = d_{SR}(z,y_i) \le d_{SR}(z,y^j) = (1-t)d_{SR}(x^j,y^j)$, hence $d_{SR}(x^i,y^i) \le d_{SR}(x^j,y^j)$. It follows that $d_{SR}(x^i,y^i) = d_{SR}(x^j,y^j)$ and hence $d_{SR}(x^i,z) = d_{SR}(x^j,z)$ and $d_{SR}(z,y^i) = d_{SR}(z,y^j)$, which in turn implies that $x^j = x^i$ and $y^i = y^j$ and gives a contradiction. However it may happen that $x^i = y^i$ or $x^j = y^j$. Let us suppose that $x^i = y^i = z$: the same computation above implies that $d_{SR}(x^j,y^j) = 0$, which in turns implies that $y^i = y^j$ and gives a contradiction.

Remember also that $\mu$ is absolutely continuous and hence there exists a correspondence $\varepsilon \mapsto \delta = \delta(\varepsilon)$ such that

$$\mathcal{L}^{2n+1}(A) < \delta(\varepsilon) \Rightarrow \mu(A) < \varepsilon.$$

Let $A \subset \mathbb{H}^n$ be a Borel set, $t \in [0,1)$, then $\mu_t := (T_t)_\# \mu$ is concentrated on $T_t(\text{supp}(\mu))$ and

$$\mu_t(A) = \sum_{i=1}^{M} \mu_t(A \cap \Omega_i(t)) = \sum_{i=1}^{M} \mu(T_t^{-1}(A \cap \Omega_i(t))) = \mu\left(\bigcup_{i=1}^{M} (T_t^{-1}(A \cap \Omega_i(t)))\right),$$

since the sets $T_t^{-1}(A \cap \Omega_i(t)) \subseteq \Omega_i$ are disjoint. We observe that for any $x \in \Omega_i$, $T_t(x) = S_t(x, y^i)$, hence by (1.17) follows that

$$\mathcal{L}^{2n+1}(U) \leq \frac{1}{(1-t)^{2n+3}} \mathcal{L}^{2n+1}(T_t(U)),$$

for any $U \subset \Omega_i$. This in turn implies that

$$\mathcal{L}^{2n+1}(T_t^{-1}(A \cap \Omega_i(t))) \leq \frac{1}{(1-t)^{2n+3}} \mathcal{L}^{2n+1}(A \cap \Omega_i(t)),$$

and so

$$\mathcal{L}^{2n+1}\left(\bigcup_{i=1}^{M} (T_t^{-1}(A \cap \Omega_i(t)))\right) \leq \frac{1}{(1-t)^{2n+3}} \mathcal{L}^{2n+1}(A).$$

Hence, it is sufficient to suppose that $\mathcal{L}^{2n+1}(A) < (1-t)^{2n+3}\delta(\varepsilon)$ to get $\mu_t(A) < \varepsilon$. This proves that $\mu_t \ll \mathcal{L}^{2n+1}$.

Now, if $\nu$ is not finitely atomic, we can take a sequence $(\nu_k)_{k\in\mathbb{N}}$ of atomic measures weakly converging to $\nu$. For any $k \in \mathbb{N}$ we consider optimal plan $\gamma_k$, as in the first part of the proof. In particular $(\gamma_k)_{k\in\mathbb{N}}$ is a sequence of optimal transport plans weakly converging to an optimal transport plan $\gamma$; moreover the sequence $(\mu_t^k)_{k\in\mathbb{N}}$ weakly converges to the corresponding $\mu_t := (S_t)_\# \gamma$, thanks to Proposition 1.2.3 and [52, Lemma 7.3]. Take a set $A$ such that $\mathcal{L}^{2n+1}(A) < (1-t)^{2n+3}\delta(\varepsilon)$. Since the Lebesgue measure is regular,$A$ is included in an open set $B$ such that $\mathcal{L}^{2n+1}(B) < (1-t)^{2n+3}\delta(\varepsilon)$. Hence $\mu_t^k(B) < \varepsilon, \forall k \in \mathbb{N}$. Passing to the limit and using Portmanteau's Theorem, see [17, Theorem 2.1], we get

$$\mu_t(A) \leq \mu_t(B) \leq \liminf_k \mu_t^k(B) \leq \varepsilon.$$

This proves that $\mu_t \ll \mathcal{L}^{2n+1}$. □

**Theorem 2.2.2.** *Suppose that $\mu \ll \mathcal{L}^{2n+1}$, then there exists $\gamma \in \Pi_1(\mu,\nu)$ such that $a_\gamma \ll \mathcal{L}^{2n+1}$.*

*Proof.* Let $\gamma \in \Pi_1(\mu,\nu)$ satisfying (2.16). Then, the thesis follows immediately from (2.15) applied to $a_\gamma$. □

Obviously the previous argument depends only on one of the two marginals and it is completely symmetric: if $\nu \ll \mathcal{L}^{2n+1}$, again one can get the existence of an optimal transport plan $\gamma \in \Pi_1(\mu,\nu)$ such that the associated horizontal transport density $a_\gamma$ is absolute continuous w.r.t. the $(2n+1)$-dimensional Lebesgue measure.

Let us just remark that, in the Riemannian setting, if one between $\mu$ and $\nu$ is absolutely continuous w.r.t. the volume measure, then the trasport density does not depend on the choice of the transport plan, see [56]. See also [57] and [2] for the analogous result in $\mathbb{R}^n$.

### 2.2.2 Summability of horizontal transport densities

The next step is to prove, under some suitable assumptions, the existence of at least one optimal plan $\gamma \in \Pi_1(\mu,\nu)$, whose associated horizontal transport density belongs to $L^p$, for some $p \in [1,\infty]$. From now on, given $\lambda \in \mathcal{M}(\mathbb{H}^n)$ we will write that $\lambda \in L^p$ if $\lambda \ll \mathcal{L}^{2n+1}$, with density $\rho \in L^p$. We will denote by $\|\lambda\|_p := \|\rho\|_{L^p}$.

Let $\gamma \in \Pi_1(\mu,\nu)$ be an optimal transport plan as in Theorem 2.2.2, using Minkowski inequality in (2.14) it holds

$$\|a_\gamma\|_p \le C \int_0^1 \|\mu_t\|_p dt. \tag{2.17}$$

In order to prove $p$-summability of $a_\gamma$, for some $\gamma \in \Pi_1(\mu,\nu)$, it is sufficient to prove that almost every measure $\mu_t$ is in $L^p$ and to estimate their $L^p$ norms, choosing a posteriori $p$ such that the integral above converges.

In the following theorem we will estimate for all $t \in (0,1)$ the $L^p$ norm of the interpolation measures $\mu_t$, associated with transport plans $\gamma \in \Pi_1(\mu,\nu)$ that satisfy the thesis of Theorem 2.2.2.

**Proposition 2.2.3.** *If $\mu \in L^p$, then there exists $\gamma \in \Pi_1(\mu,\nu)$ such that $\mu_t := (S_t)_\#\gamma$ satisfies*

$$\|\mu_t\|_p \le (1-t)^{-(2n+3)/q}\|\mu\|_p, \quad \forall t \in (0,1), \tag{2.18}$$

*where $q := \frac{p}{p-1}$ is the conjugate exponent of $p$.*

*Proof.* Let us denote by $\rho$ the density of $\mu$ w.r.t. $\mathcal{L}^{2n+1}$. Consider first the discrete case: let us assume that the target measure $\nu$ is finitely atomic and let us denote by $(y^i)_{i=1,\dots,M}$ its atoms, as in the previous proof. Let us consider an optimal transport plan $\gamma \in \Pi_2(\mu,\nu)$, as in Theorem 1.2.6, concentrated on a set $\Gamma$. As before, since $\gamma$ is induced by a map $T$, we denote by $\Omega_i := T^{-1}(\{y^i\}) \cap \pi_1(\Gamma)$, for $i \in \{1,\dots,M\}$, so that for $\gamma$-a.e. $(x,y) \in \Omega_i \times \mathbb{H}^n$, we have $y = y_i$. Let us consider the corresponding interpolation measures $\mu_t$. As in the proof of Theorem 2.2.1 we get that $\mu_t \ll \mathcal{L}^{2n+1}$ for every $t \in [0,1)$; moreover, for all $\varphi \in C(\mathbb{H}^n)$, by the definition of push-forward we get that

$$\begin{aligned}\int \varphi(x)d\mu_t(x) &= \sum_{i=1}^{M}\int_{\Omega_i}\varphi(S_t(x,y_i))d\gamma(x,y_i) = \\ &= \sum_{i=1}^{M}\int_{\Omega_i}\varphi(T_t(x))d\mu(x).\end{aligned}$$

Let us fix $i \in \{1,\dots,M\}$ and let us denote by $\rho_t$ the density of $\mu_t$ w.r.t. $\mathcal{L}^{2n+1}$ and by $\rho_t^i := \rho_{t\lfloor\Omega_i}$. Let us take the change of variable $z = S_t(x,y_i) = T_{t\lfloor\Omega_i}(x)$. We know, from Lemma 2.1.4 and disjointness of the sets $\Omega_i(t)$, that this map is injective. Then, for all $\varphi \in C(\mathbb{H}^n)$, we get

$$\begin{aligned}\int_{\Omega_i}\varphi(x)d\mu_t^i(x) &= \int_{\Omega_i}\varphi(T_t(x))\rho(x)dx = \\ &= \int_{\Omega_i(t)}\varphi(z)\rho(T_t^{-1}(z))|\det D_x(S_t(x,y^i))|^{-1}dz.\end{aligned}$$

Hence, we have that

$$\rho_t^i(z) = \rho(T_t^{-1}(z))|\det D_x(S_t(x,y^i))|^{-1}, \quad \text{for a.e. } z \in \Omega_i(t).$$

Consequently, we get

$$\|\rho_t^i\|_{L^p(\Omega_i(t))}^p = \int_{\Omega_i(t)} \rho(T_t^{-1}(z))^p |\det D_x(S_t(x,y^i))|^{-p} dz =$$
$$= \int_{\Omega_i} \rho(x)^p |\det D_x(S_t(x,y^i))|^{1-p} dx.$$

Hence from (1.16) it follows that

$$\|\rho_t^i\|_{L^p(\Omega_i(t))}^p \le (1-t)^{(1-p)(2n+3)} \|\rho\|_{L^p(\Omega_i)}^p, \quad \forall i \in \{1,\dots,M\}.$$

Then, we have

$$\|\mu_t\|_p \le (1-t)^{-(2n+3)/q} \|\mu\|_p, \quad \forall t \in (0,1), \tag{2.19}$$

where $q = \frac{p}{p-1}$. As in the proof of Proposition 2.2.1, if $\nu$ is not finitely atomic, we can take a sequence $(\nu_k)_{k\in\mathbb{N}}$ of atomic measures weakly converging to $\nu$. As in Theorem 2.2.2, we consider the sequence $(\gamma_k)_{k\in\mathbb{N}}$ of optimal plans satisfying (2.19): this sequence weakly converges to an optimal plan $\gamma$ and $\mu_t^k$ weakly converge to the corresponding $\mu_t := (S_t)_\#\gamma$. Hence, we get that

$$\|\mu_t\|_p \le \liminf_{k\to 0} \|\mu_t^k\|_p \le (1-t)^{-(2n+3)/q} \|\mu\|_p.$$

□

Now we are able to prove the following theorem:

**Theorem 2.2.4.** *If $\mu \in L^p$ for some $p \in [1,\infty]$, it holds: if $p < \frac{2n+3}{2n+2}$ then there exists $\gamma \in \Pi_1(\mu,\nu)$ such that $a_\gamma \in L^p$; otherwise there exists $\gamma \in \Pi_1(\mu,\nu)$ such that $a_\gamma \in L^s$ for $s < \frac{2n+3}{2n+2}$.*

*Proof.* Let $\gamma \in \Pi_1(\mu,\nu)$ satisfying (2.18). Then, it follows from (2.17) applied to $a_\gamma$ that

$$\|a_\gamma\|_p \le C \int_0^1 \|\mu_t\|_p dt \le C\|\mu\|_p \int_0^1 (1-t)^{-(2n+3)/q} dt.$$

The last integral is finite whenever $q > 2n+3$, i.e. $p < \frac{2n+3}{2n+2}$.

If $p \ge \frac{2n+3}{2n+2}$ the thesis follows from the fact that any density in $L^p$ also belongs to any $L^s$ space for $s < p$. □

By symmetry, if also $\nu \in L^p$ then one can find $\tilde{\gamma} \in \Pi_1(\mu,\nu)$, possibly different from the optimal transport plan $\gamma \in \Pi_1(\mu,\nu)$ obtained in Proposition 2.2.3, such that $\tilde{\mu}_t := (S_t)_{\#}\tilde{\gamma}$ satisfies

$$\|\tilde{\mu}_t\|_p \leq t^{-(2n+3)/q}\|\nu\|_p, \quad \forall t \in (0,1). \tag{2.20}$$

Hence, one cannot glue together (2.18) and (2.20) and deduce anything about the $p$-summability of transport densities.

## 2.3 Beckmann's formulation of the optimal transport problem in $\mathbb{H}^n$

In this section we introduce another formulation of the Monge-Kantorovich problem

$$\min_{\gamma\in\Pi(\mu,\nu)} \left\{ \int_{\mathbb{H}^n\times\mathbb{H}^n} d_{SR}(x,y)d\gamma(x,y) \right\}, \tag{MKP}$$

known in literature as *Beckmann problem*, see [13] and [87, Section 4.2]. This new formulation is set on the space of compactly supported horizontal vector measures in $H\mathbb{H}^n$, subjected to a horizontal divergence-type constraint

$$\min \Bigg\{ \|\mathbf{w}\|_{\mathcal{M}(\mathbb{H}^n,H\mathbb{H}^n)} : \mathbf{w} \in \mathcal{M}_c(\mathbb{H}^n, H\mathbb{H}^n), \\ \int_{\mathbb{H}^n} \nabla_H\varphi \cdot d\mathbf{w} = -\int_{\mathbb{H}^n} \varphi d(\mu-\nu), \forall\varphi \in C^\infty(\mathbb{H}^n) \Bigg\}. \tag{2.21}$$

We will show that the values of (MKP) and (2.21) are the same and how to construct a solution to (2.21) starting from a solution to (MKP).

### 2.3.1 Vector horizontal transport densities

Given an optimal transport plan $\gamma$, one can also define a vector version of the horizontal transport density $a_\gamma$, i.e. a vector measure in $H\mathbb{H}^n$ that we will call *vector horizontal transport density*.

**Definition 2.3** (Vector horizontal transport density)**.** Given an optimal transport plan $\gamma \in \Pi_1(\mu, \nu)$, one can define the measure $\mathbf{w}_\gamma \in \mathcal{M}(\mathbb{H}^n, H\mathbb{H}^n)$ defined as

$$\int_{\mathbb{H}^n} \phi \cdot d\mathbf{w}_\gamma := \int_{\mathbb{H}^n \times \mathbb{H}^n} \left( \int_0^1 \langle \phi\left(\sigma_{x,y}(t)\right), \dot{\sigma}_{x,y}(t) \rangle_H dt \right) d\gamma(x,y), \tag{2.22}$$

for any $\phi \in C_0(\mathbb{H}^n, H\mathbb{H}^n)$.

Even in this case, the measure $\mathbf{w}_\gamma$ is well-defined and compactly supported for any $\gamma \in \Pi(\mu, \nu)$. If $\gamma \in \Pi_1(\mu, \nu)$, then $\mathbf{w}_\gamma$ is compactly supported on the transport set $\mathcal{T}$ and it holds that

$$\|\mathbf{w}_\gamma\|_{\mathcal{M}(\mathbb{H}^n, H\mathbb{H}^n)} \leq (\text{MKP}). \tag{2.23}$$

Hence, as for the scalar horizontal transport density, we can define the vector measure $\mathbf{w}_\gamma$ in duality with the continuous horizontal vector fields $C(\mathbb{H}^n, H\mathbb{H}^n)$.

### 2.3.2 Absolute continuity and summability of vector horizontal transport densities

There is a deep relation between the vector horizontal transport density and the scalar one, introduced in Section 2.2. First, we observe that given $\gamma \in \Pi(\mu, \nu)$, by definition it follows that

$$|\mathbf{w}_\gamma| \leq a_\gamma$$

as measures. Indeed, let $A \subseteq \mathbb{H}^n$ be a Borel set and let $\phi \in C_c(\mathbb{H}^n, H\mathbb{H}^n)$ such that supp $\phi \subseteq A$ and $\|\phi\|_\infty \leq 1$, then

$$\int_{\mathbb{H}^n} \phi \cdot d\mathbf{w}_\gamma \leq \left| \int_{\mathbb{H}^n} \phi \cdot d\mathbf{w}_\gamma \right| \leq \int_{\mathbb{H}^n} |\phi|_H da_\gamma \leq a_\gamma(A).$$

Taking the sup among all the admissible $\phi$ on the left hand side and having in mind (1.21), one get

$$|\mathbf{w}_\gamma|(A) \leq a_\gamma(A), \quad \forall A \subseteq \mathbb{H}^n \text{ Borel set}.$$

If in addition $\gamma \in \Pi_1(\mu,\nu)$, using Proposition 2.1.1 we may write

$$\dot{\sigma}_{x,y}(t) = |\dot{\sigma}_{x,y}(t)|_H \frac{\dot{\sigma}_{x,y}(t)}{|\dot{\sigma}_{x,y}(t)|_H} = -d_{SR}(x,y)\nabla_H u(\sigma_{x,y}(t)),$$

where $u \in \mathrm{Lip}(\mathbb{H}^n, d_{SR})$ is a Kantorovich potential. The previous equality holds for every $t \in ]0,1[$ and for $\gamma$-almost every $(x,y)$, with $x \neq y$ (otherwise both expressions vanish). This allows us to write

$$\begin{aligned}
&\int_{\mathbb{H}^n} \phi \cdot d\mathbf{w}_\gamma \\
&\quad = \int_{\mathbb{H}^n\times\mathbb{H}^n} \left( \int_0^1 -d_{SR}(x,y) \left\langle \nabla_H u(\sigma_{x,y}(t)), \phi(\sigma_{x,y}(t)) \right\rangle_H dt \right) d\gamma(x,y) \\
&\quad = -\int_0^1 \left( \int_{\mathbb{H}^n\times\mathbb{H}^n} \left\langle \nabla_H u(\sigma_{x,y}(t)), \phi(\sigma_{x,y}(t)) \right\rangle_H d_{SR}(x,y) d\gamma(x,y) \right) dt \\
&\qquad\qquad = -\int_0^1 \left( \int_{\mathbb{H}^n} \left\langle \nabla_H u(z), \phi(z) \right\rangle_H d(S_t)_\#(d_{SR}\gamma)(z) \right) dt,
\end{aligned}$$

for every $\phi \in C(\mathbb{H}^n, H\mathbb{H}^n)$. With the same kind of computation one gets that

$$\int_{\mathbb{H}^n} \varphi da_\gamma = \int_0^1 \left( \int_{\mathbb{H}^n} \varphi(z) d(S_t)_\#(d_{SR}\gamma)(z) \right) dt, \forall \varphi \in C(\mathbb{H}^n).$$

Then,

$$\int_{\mathbb{H}^n} \phi \cdot d\mathbf{w}_\gamma = -\int_{\mathbb{H}^n} \langle \phi, \nabla_H u \rangle_H da_\gamma = -\int_{\mathbb{H}^n} \phi \cdot d(\nabla_H u) a_\gamma,$$

for every $\phi \in C(\mathbb{H}^n, H\mathbb{H}^n)$. Since by definition $\mathbf{w}_\gamma$ and $a_\gamma$ are concentrated on the set of differentiability points of the Kantorovich potential $u$, it follows that $-\nabla_H u$ is the density of the measure $|\mathbf{w}_\gamma|$ w.r.t. $a_\gamma$,

$$\mathbf{w}_\gamma = -(\nabla_H u) a_\gamma. \tag{2.24}$$

Since $|\nabla_H u|_H \leq 1$ in the points where it exists, this confirms that $|\mathbf{w}_\gamma| \leq a_\gamma$.

**Theorem 2.3.1.** *If $\mu \ll \mathcal{L}^{2n+1}$, then there exists $\gamma \in \Pi_1(\mu,\nu)$ such that $\mathbf{w}_\gamma \in L^1(\mathbb{H}^n, H\mathbb{H}^n)$.*

*If $\mu \in L^p$ for some $p \in [1,\infty]$, it holds: if $p < \frac{2n+3}{2n+2}$ then there exists $\gamma \in \Pi_1(\mu,\nu)$ such that $\mathbf{w}_\gamma \in L^p(\mathbb{H}^n, H\mathbb{H}^n)$; otherwise there exists $\gamma \in \Pi_1(\mu,\nu)$ such that $\mathbf{w}_\gamma \in L^s(\mathbb{H}^n, H\mathbb{H}^n)$ for $s < \frac{2n+3}{2n+2}$.*

*Proof.* The thesis follows from the argument above, Theorem 2.2.2 and Theorem 2.2.4. □

### 2.3.3 The Beckmann problem in $\mathbb{H}^n$

Let $\gamma \in \Pi_1(\mu,\nu)$ be an optimal transport plan: we can test the vector horizontal transport density $\mathbf{w}_\gamma$ against $\phi = \nabla_H\varphi$, for any $\varphi \in C^\infty$

$$\begin{aligned}\int_{\mathbb{H}^n} \nabla_H\varphi \cdot d\mathbf{w}_\gamma &= \int_{\mathbb{H}^n\times\mathbb{H}^n} \left(\int_0^1 \langle \nabla_H\varphi\left(\sigma_{x,y}(t)\right), \dot\sigma_{x,y}(t)\rangle_H dt\right) d\gamma(x,y) \\ &= \int_{\mathbb{H}^n\times\mathbb{H}^n} \left(\int_0^1 \frac{d}{dt}\left[\varphi(\sigma_{x,y}(t))\right] dt\right) d\gamma(x,y) \\ &= \int_{\mathbb{H}^n\times\mathbb{H}^n} (\varphi(y)-\varphi(x))d\gamma(x,y) = -\int_{\mathbb{H}^n} \varphi d(\mu-\nu), \quad (2.25)\end{aligned}$$

see (1.18).

Now, given a compactly supported vector measure $\mathbf{w} \in \mathcal{M}_c(\mathbb{H}^n, H\mathbb{H}^n)$ we can define its *distributional horizontal divergence* $\mathrm{div}_H\mathbf{w}$ by the rule

$$\langle \mathrm{div}_H\mathbf{w}, \varphi\rangle := -\int_{\mathbb{H}^n} \nabla_H\varphi(x)\cdot d\mathbf{w}, \quad \forall\varphi \in C^\infty.$$

With this definition in mind we can rewrite (2.25) in the following way:

$$\mathrm{div}_H\mathbf{w}_\gamma = \mu - \nu,$$

for any $\gamma \in \Pi_1(\mu,\nu)$, i.e. the horizontal divergence of the measure $\mathbf{w}_\gamma$ is the signed Radon measure $\mu-\nu$.

*Remark* 3. Obviously, (2.25) is independent of the optimality of the transport plan. It holds for any transport plan $\gamma \in \Pi(\mu,\nu)$.

In [13] the author introduced a wide class of problems in the euclidean setting, called *continuous models of transportation.* The Heisenberg version of the simplest case of these problems reads as

$$\min\left\{\|\mathbf{w}\|_{\mathcal{M}(\mathbb{H}^n,H\mathbb{H}^n)} : \mathbf{w}\in\mathcal{M}_c(\mathbb{H}^n,H\mathbb{H}^n), \mathrm{div}_H\mathbf{w} = \mu-\nu\right\}, \quad \text{(BP)}$$

while we will give a more general formulation in Section 4.2. The aim of the next theorem is to prove that (BP) admits the following dual reformulation

$$\sup_{u \in HW^{1,\infty}} \left\{ \int_{\mathbb{H}^n} u d(\mu - \nu) : \|\nabla_H u\|_\infty \leq 1 \right\}. \tag{DP}$$

In order to do this, we will prove that (BP) is equivalent to the Monge-Kantorovic problem (MKP), whose dual formulation is precisely the well-known *Kantorovich duality formula* (DP). The equivalence between (BP) and (DP) was first investigated in the euclidean setting in [91].

**Theorem 2.3.2.** *The problem* (BP) *admits a solution. Moreover,*

$$(\text{BP}) = (\text{MKP}) \tag{2.26}$$

*where*

$$(\text{MKP}) = \min_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y),$$

*and a solution to* (BP) *can be built from a solution to* (MKP).

*Proof.* First we prove the equality between (MKP) and (BP). We start by proving that (BP)$\geq$(MKP). Take an arbitrary function $\varphi \in C^\infty \cap \text{Lip}_1(\mathbb{H}^n, d_{SR})$. Theorem 1.1.5 implies that $\|\nabla_H \varphi\|_\infty \leq 1$; hence, for any $\mathbf{w}$ admissible we get

$$\|\mathbf{w}\|_{\mathcal{M}(\mathbb{H}^n, H\mathbb{H}^n)} = |\mathbf{w}|(\mathbb{H}^n) \geq \int_{\mathbb{H}^n} (-\nabla_H \varphi) \cdot d\mathbf{w} = \int_{\mathbb{H}^n} \varphi d(\mu - \nu).$$

Now we take $\varphi^\varepsilon = \rho_\varepsilon * u$, where $u$ is a Kantorovich potential and $\rho_\varepsilon$ is a mollifier for the group structure. It follows that $\varphi^\varepsilon \in C^\infty \cap \text{Lip}_1(\mathbb{H}^n, d_{SR})$ and converges uniformly on compact sets to the Kantorovich potential $u$, see [50, Proposition 2.14]. By letting $\varepsilon$ tend to 0 we get that

$$\|\mathbf{w}\|_{\mathcal{M}(\mathbb{H}^n, H\mathbb{H}^n)} \geq \int_{\mathbb{H}^n} u d(\mu - \nu) = (\text{MKP}), \tag{2.27}$$

where we used Theorem 1.2.2 in the last equality. Since the previous inequality holds for any admissible $\mathbf{w}$, we may take the minimum in the left hand-side and get

$$(\text{BP}) \geq (\text{MKP}).$$

Now we will prove the converse inequality: given $\gamma \in \Pi_1(\mu, \nu)$, we know that the vector horizontal transport density $\mathbf{w}_\gamma$, defined in (2.22), is compactly supported and it satisfies the divergence constraint thanks to (2.25). Moreover from (2.23) we know that

$$\|\mathbf{w}_\gamma\|_{\mathcal{M}(\mathbb{H}^n, H\mathbb{H}^n)} \leq (\text{MKP}).$$

Hence

$$(\text{BP}) \leq \|\mathbf{w}_\gamma\|_{\mathcal{M}(\mathbb{H}^n, H\mathbb{H}^n)} \leq (\text{MKP}).$$

□

**Corollary 2.3.3.** *If* $\mu \ll \mathcal{L}^{2n+1}$*, either* $\nu \ll \mathcal{L}^{2n+1}$*, then there exists* $\mathbf{w} \in L^1(\mathbb{H}^n, H\mathbb{H}^n)$ *that solves* (BP)*.*

*Proof.* The thesis follows from Theorem 2.3.1 and Theorem 2.3.2. □

*Remark* 4. For any $\gamma \in \Pi_1(\mu, \nu)$ and any Kantorovich potential $u$, the pair $(a_\gamma, u)$ solves the Monge-Kantorovich system

$$\begin{cases} \text{div}_H((\nabla_H u) a_\gamma) = \mu - \nu, \\ |\nabla_H u|_H = 1, \quad a_\gamma - a.e., \\ |\nabla_H u|_H \leq 1. \end{cases}$$

The second condition holds because $|\nabla_H u|_H = 1$ on the transport set and $a_\gamma$ is supported on it.

## 2.4 Lagrangian formulation of the optimal transport problem in $\mathbb{H}^n$

In this section we introduce a Lagrangian formulation of the Monge-Kantorovich problem (MKP), following [3, Lecture 9, section 3].

Let $\mu, \nu \in \mathcal{P}_c(\mathbb{H}^n)$, we consider the problem

$$\inf \left\{ \int_{C([0,1],\mathbb{R}^{2n+1})} l(\sigma) dQ(\sigma) : \right.$$
$$\left. Q \in \mathcal{P}(C([0,1],\mathbb{R}^{2n+1})), (e_0)_\# Q = \mu, (e_1)_\# Q = \nu \right\}, \quad (2.28)$$

where the functional $l$ is defined in (1.10).

**Theorem 2.4.1.** *If $\mu, \nu \in \mathcal{P}_c(\mathbb{H}^n)$, then (2.28) admits a solution.*

*Moreover, it holds*

$$(2.28) = \min_{\gamma \in \Pi(\mu,\nu)} \left\{ \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y) \right\}$$

*and $Q \in \mathcal{P}\left(C\left([0,1],\mathbb{R}^{2n+1}\right)\right)$ solves (2.28) if, and only if, $Q$ is concentrated on the set of minimizing horizontal curves and $\gamma_Q := (e_0, e_1)_\# Q \in \Pi_1(\mu,\nu)$.*

*Proof.* Let $Q \in \mathcal{P}(C([0,1],\mathbb{R}^{2n+1}))$ be admissible, then

$$\int_{C([0,1],\mathbb{R}^{2n+1})} l(\sigma) dQ(\sigma) \geq \int_{C([0,1],\mathbb{R}^{2n+1})} d_{SR}(\sigma(0),\sigma(1)) dQ(\sigma) =$$
$$\int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d(e_0,e_1)_\# Q(x,y) \geq \min_{\gamma \in \Pi(\mu,\nu)} \left\{ \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y) \right\}, \quad (2.29)$$

where the first inequality follows from the definition of $l$ and the triangle inequality, the second equality follows from the definition of push-forward and the last inequality from the fact that $(e_0, e_1)_\# Q \in \Pi(\mu,\nu)$ is admissible.

Let us prove the converse inequality. If $\gamma \in \Pi_1(\mu,\nu)$, then the measure

$$Q := S_\# \gamma \in \mathcal{P}(C([0,1],\mathbb{R}^{2n+1}))$$

and it is supported on $\mathrm{Geo}(\mathbb{H}^n)$ by construction. Hence

$$\min_{\gamma \in \Pi(\mu,\nu)} \left\{ \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y) \right\} = \int_{\mathbb{H}^n \times \mathbb{H}^n} d_{SR}(x,y) d\gamma(x,y)$$
$$= \int_{\mathrm{Geo}(\mathbb{H}^n)} d_{SR}(\sigma(0),\sigma(1)) dQ(\sigma) = \int_{\mathrm{Geo}(\mathbb{H}^n)} l(\sigma) dQ(\sigma). \quad (2.30)$$

Moreover $Q \in \mathcal{P}(C([0,1], \mathbb{R}^{2n+1}))$ is optimal for (2.28) if and only if (2.29) holds with equalities, that happens if and only if $Q$ is supported on the set of minimizing horizontal curves and $\gamma_Q := (e_0, e_1)_\# Q \in \Pi_1(\mu, \nu)$. $\square$

*Remark* 5. This theorem is independent of the compactness of supports and on the ambient space: it holds in more general metric space $(M, d)$ with $\mu, \nu \in \mathcal{P}(M)$ such that

$$\int_M d(x,0) d\mu(x) + \int_M d(0,y)(y) d\nu(y) < +\infty,$$

and $l$ is the length induced by the distance $d$ as in (1.10). Pay attention to the fact that in (2.30) the key point is that for horizontal curves $\sigma$ the quantities $l(\sigma)$ and $l_{SR}(\sigma)$ coincide.

*Remark* 6. Let us just remark that, a posteriori, due to the second part of the statement in Theorem 2.4.1 we can restate (2.28) as

$$\inf \left\{ \int_{C([0,1],\mathbb{R}^{2n+1})} l_{SR}(\sigma) dQ(\sigma) : Q \in \mathcal{P}(C([0,1], \mathbb{R}^{2n+1})), (e_0)_\# Q = \mu, \right.$$
$$\left. (e_1)_\# Q = \nu,\ Q \text{ is concentrated on } H([0,1], \mathbb{H}^n) \right\}.$$

In the next chapter we will call such $Q$'s *horizontal traffic plans.*

# Chapter 3

# Congested optimal transport problem in $\mathbb{H}^n$

We consider a bounded domain $\Omega \subset \mathbb{H}^n$, with regular $C^{1,1}$ boundary, which models the geographical area on which the dynamic takes place, and two probability measures $\mu, \nu \in \mathcal{P}(\overline{\Omega})$, which represent, for instance, the distributions of moving agents and destinations, respectively.

From a traffic point of view the classical Monge-Kantorovich problem is not suitable to take into account congestion effects, because it depends only on the initial and final position of the mass, but not on the paths followed by it.

In the spirit of [43], in order to take into account congestion and to describe how agents $\mu$ choose paths to reach destinations $\nu$, we introduce the concepts of *horizontal traffic plans* and *horizontal traffic intensities*: we will prove the existence of equilibrium configurations of *Wardrop-type*, through the minimization of a convex functional.

## 3.1 Weighted length of horizontal curves

Let $\Omega$ be a bounded domain of $\mathbb{H}^n$, with regular $C^{1,1}$ boundary, i.e. it is a $C^{1,1}$ manifold (in the Euclidean sense) and it has a well-defined tangent space at each point. We consider the set of continuous curves $C([0,1],\overline{\Omega})$, equipped with the topology of uniform convergence.

We denote by

$$H := \{\sigma \in AC([0,1],\overline{\Omega}) : \ \sigma \text{ is horizontal}\}$$

the subspace of horizontal curves, where $AC([0,1],\overline{\Omega})$ denotes the subspace of absolutely continuous curves with values in $\overline{\Omega}$. Let $x, y \in \overline{\Omega}$ and

$$H^{x,y} := \{\sigma \in H : \ \sigma(0) = x, \ \sigma(1) = y\}$$

the set of horizontal curves connecting $x$ and $y$.

*Remark* 7. Let us remark that the set

$$H^{x,y} \neq \emptyset,$$

for any $x, y \in \overline{\Omega}$. Indeed, if $x, y \in \Omega$, since $\Omega$ is open and connected, then the statement follows from the Chow-Rashevsky Theorem applied to the manifold $\Omega$, equipped with the sub-Riemannian structure inherited from $\mathbb{H}^n$, see [35, Chapter 3].

Let us denote by $\mathcal{C}(\partial\Omega)$ the set of *characteristic points* of $\partial\Omega$, that is

$$\mathcal{C}(\partial\Omega) := \{x \in \partial\Omega : T_x\partial\Omega = H_x\mathbb{H}^n\},$$

and let us remark that at a non-characteristic point $z$ the fiber $H_z\mathbb{H}^n$ can be represented as

$$H_z\mathbb{H}^n = H_z\partial\Omega \oplus \operatorname{span}\{\mathbf{n}_H(z)\}.$$

Here, $H_z\partial\Omega = T_z\partial\Omega \cap H_z\mathbb{H}^n$ denotes the horizontal tangent space to $\partial\Omega$ at the point $z$, and $\mathbf{n}_H(z)$ is the horizontal normal at the point $z \in \partial\Omega$, that is the orthogonal projection of the Euclidean normal to $\Omega$ at $z$ on $H_z\mathbb{H}^n$. Then, for any horizontal vector field $Z$, the vector $Z(z) \in H_z\mathbb{H}^n$ admits an unique projection $\pi(Z(z))$ on the space $H_z\partial\Omega$. One can consider the horizontal vector field $\pi(Z) : \Omega \to H\Omega$,

$$z \mapsto \pi(Z)(z) := \pi(Z(z)),$$

where $\pi(Z)(z) = Z(z)$ if $z$ is a characteristic point.

Let us consider the case $x \in \partial\Omega$ and $y \in \Omega$. If $x \in \partial\Omega$ is a non-characteristic point, then the horizontal normal $\mathbf{n}_H$ at the point $x$ does not vanish. As a result, if $\mathbf{n}_H(x) := \sum_{i=1}^{2n} \mathbf{n}_i X_i(x)$, one can consider the horizontal vector field $Z := \sum_{i=1}^{2n} \mathbf{n}_i X_i \in \mathfrak{h}_1^n$, $\delta > 0$ and the horizontal curve

$$\sigma(t) := \exp(-tZ)(x), \tag{3.1}$$

such that $\sigma([0,\delta]) \subset \overline{\Omega}$ and $z := \sigma(\delta) \in \Omega$. Now one can consider a horizontal curve between $z$ and $y$, fully contained in $\Omega$.

If $x \in \mathcal{C}(\partial\Omega)$ is a characteristic point, from [11, Theorem 1.2] it follows that the Hausdorff dimension w.r.t. the Euclidean metric is $\dim_E \mathcal{C}(\partial\Omega) < 2n$. Then, there exists $v = \sum_{i=1}^{2n} v_i X_i(x) \in T_x\partial\Omega = H_x\mathbb{H}^n$ and $\delta > 0$ such that the horizontal curve

$$\sigma(t) = \exp\left(t\sum_{i=1}^{2n} v_i \pi(X_i)\right)(x) \tag{3.2}$$

is well-defined, it belongs to $\partial\Omega$ and it is non-characteristic for all $t \in (0,\delta]$. Hence, one can find a horizontal curve between $z = \sigma(\delta)$ and $y \in \Omega$, using the argument above.

If $x, y \in \partial\Omega$, using the previous arguments one can connect them with $x', y' \in \Omega$ and then find a horizontal curve between $x'$ and $y'$, which is contained in $\Omega$.

Given $\sigma \in H$ we will denote by $\tilde{\sigma}$ its constant-speed reparametrization: hence, $|\dot{\tilde{\sigma}}(t)|_H = l_{SR}(\sigma) = l_{SR}(\tilde{\sigma})$ for a.e. $t \in [0,1]$. We denote by

$$\tilde{H} := \{\tilde{\sigma} : \sigma \in H\},$$

and by

$$\tilde{H}^{x,y} := \{\tilde{\sigma} : \sigma \in H^{x,y}\}.$$

Given $\varphi \in C(\overline{\Omega})$ and $\sigma \in H$, we denote by

$$L_\varphi(\sigma) := \int_0^1 \varphi(\sigma(t))|\dot{\sigma}(t)|_H dt = l_{SR}(\sigma)\int_0^1 \varphi(\tilde{\sigma}(t))dt, \tag{3.3}$$

the *weighted horizontal length* of $\sigma$, with weight $\varphi$. This quantity is well define since the curvilinear integral does not depend on the parametrization.

Let us notice that, if $\varphi \equiv 1$ and $\sigma \in H$, then

$$L_1(\sigma) = l_{SR}(\sigma).$$

Following [43, Lemma 2.7], if $\varphi \in C(\overline{\Omega})$, $\varphi \geq 0$, one can define the weighted sub-Riemannian length also for curves $\sigma \in C([0,1], \overline{\Omega}) \setminus H$. We denote by

$$\overline{L}_\varphi : C([0,1], \overline{\Omega}) \to [0, +\infty]$$

the function

$$\overline{L}_\varphi(\sigma) := \sup \Bigg\{ \sum_{i=1}^{k} \left( \inf_{[t_i, t_{i+1}]} (\varphi \circ \sigma) \right) d_{SR}(\sigma(t_i), \sigma(t_{i+1})) : \\ ([t_i, t_{i+1}])_i \text{ is a subdivision of } [0,1] \Bigg\}. \quad (3.4)$$

The space where $\overline{L}_\varphi$ is well-defined, with real values, is the space $R\left([0,1], \overline{\Omega}\right)$ of rectifiable curves. Moreover, $\sigma \mapsto \overline{L}_\varphi(\sigma)$ is lower semi-continuous, hence Borel, on $R([0,1], \overline{\Omega})$ w.r.t. the topology of uniform convergence, since it is the supremum of a family of l.s.c. functions.

Definition (3.4) extends (3.3): indeed, the following result holds.

**Lemma 3.1.1.** *For any $\varphi \in C(\overline{\Omega})$, $\varphi \geq 0$, and any $\sigma \in H$, it holds that*

$$L_\varphi(\sigma) = \overline{L}_\varphi(\sigma).$$

*Proof.* For any subdivision $([t_i, t_{i+1}])_{i=1,\dots k}$, we have

$$L_\varphi(\sigma) = \sum_{i=1}^{k} \int_{t_i}^{t_{i+1}} \varphi(\sigma(t)) |\dot{\sigma}(t)|_H \, dt \geq \sum_{i=1}^{k} \inf_{[t_i, t_{i+1}]} (\varphi \circ \sigma) \int_{t_i}^{t_{i+1}} |\dot{\sigma}(t)|_H \, dt \geq \\ \geq \sum_{i=1}^{k} \inf_{[t_i, t_{i+1}]} (\varphi \circ \sigma) d_{SR}(\sigma(t_i), \sigma(t_{i+1})),$$

where the last inequality follows from the definition of $d_{SR}$ and the fact that $\varphi \geq 0$. Taking the supremum over all such divisions one get

$$L_\varphi(\sigma) \geq \sup\Big\{ \sum_{i=1}^{k} \inf_{[t_i,t_{i+1}]} (\varphi\circ\sigma) d_{SR}(\sigma(t_i),\sigma(t_{i+1})) : \\ ([t_i,t_{i+1}])_i \text{ is a subdivision of } [0,1] \Big\}.$$

Let us prove the converse inequality. Let $\varepsilon > 0$, the Heine-Cantor theorem implies that $\varphi\circ\sigma$ is uniformly continuous, hence there exists $\delta > 0$ such that

$$\forall t,t' \in [0,1], |t-t'| \leq \delta \Rightarrow |\varphi(\sigma(t)) - \varphi(\sigma(t'))| \leq \frac{\varepsilon}{2}.$$

For any subdivision $([t_i,t_{i+1}])_{i=1,\dots k}$, we know that $\exists t_\varepsilon \in [t_i,t_{i+1}]$ such that $\inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma) \geq \varphi(\sigma(t_\varepsilon)) - \frac{\varepsilon}{2}$. If we choose $([t_i,t_{i+1}])_{i=1,\dots k}$ such that $|t_i - t_{i+1}| \leq \delta$ for all $i$, then

$$\inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma) \geq \varphi(\sigma(t_\varepsilon)) - \frac{\varepsilon}{2} = \varphi(\sigma(t_\varepsilon)) - \varphi(\sigma(t)) + \varphi(\sigma(t)) - \frac{\varepsilon}{2} \geq \\ \geq \varphi(\sigma(t)) - \varepsilon, \quad \forall t \in [t_i,t_{i+1}].$$

Thus, using Lemma 1.1.2 we get

$$\begin{aligned} L_\varphi(\sigma) &\leq \sum_{i=1}^{k} (\inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma) + \varepsilon) \int_{t_i}^{t_{i+1}} |\dot\sigma(t)|_H \, dt = \\ &= \sum_{i=1}^{n} (\inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma) + \varepsilon) \sup\Big\{ \sum_j d_{SR}(\sigma(\tau_j),\sigma(\tau_{j+1})) : \\ &\qquad ([\tau_j,\tau_{j+1}])_j \text{ is a subdivision of } [t_i,t_{i+1}] \Big\} \\ &\leq \sup\Big\{ \sum_i \sum_j (\inf_{[\tau_j,\tau_{j+1}]}(\varphi\circ\sigma) + \varepsilon) d_{SR}(\sigma(\tau_j),\sigma(\tau_{j+1})) : \\ &\qquad ([\tau_j,\tau_{j+1}])_j \text{ is a subdivision of } [t_i,t_{i+1}] \Big\} \\ &= \sup\Big\{ \sum_{i=1}^{k} (\inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma) + \varepsilon) d_{SR}(\sigma(t_i),\sigma(t_{i+1})) : \\ &\qquad ([t_i,t_{i+1}])_i \text{ is a subdivision of } [0,1] \Big\}. \end{aligned}$$

As this last inequality is true for any $\varepsilon > 0$ we get (3.4). □

If $\varphi \in C(\overline{\Omega})$, one can define

$$\overline{L}_\varphi : R\left([0,1], \overline{\Omega}\right) \to [0, +\infty]$$

as $\overline{L}_\varphi := \overline{L}_{\varphi_+} - \overline{L}_{\varphi_-}$, where $\varphi_+ := \max\{0, \varphi\}$ and $\varphi_- := \max\{0, -\varphi\}$. With an abuse of notation we will keep the symbol $L_\varphi$ to denote the function defined on the whole space $C\left([0,1], \overline{\Omega}\right)$. Moreoever, the function $\sigma \mapsto L_\varphi(\sigma)$ is Borel.

## 3.2 Horizontal traffic plans and horizontal traffic intensities

Let $\mu, \nu \in \mathcal{P}(\overline{\Omega})$ be two probability measures on $\overline{\Omega}$. We want to recover an optimal mass transport problem,

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\overline{\Omega} \times \overline{\Omega}} c(x,y) d\gamma(x,y),$$

in which the cost function $c : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}_+$ depends on how the agents $\mu$ use paths in $H$ to get destinations $\nu$. In order to do this one can consider Borel probability measures $Q \in \mathcal{P}(C([0,1], \overline{\Omega}))$: roughly speaking, if $A$ a Borel set in $C([0,1], \overline{\Omega})$, $Q(A)$ is the proportion of agents that are using a path $\sigma \in A$. In addition we want to take into account the sub-Riemannian structure of $\mathbb{H}^n$: to this aim we consider measures $Q$ concentrated on the set of horizontal curves, which we will call *horizontal traffic plan*, according to the terminology introduced in [94].

**Definition 3.1** (**Horizontal traffic plan**)**.** A *horizontal traffic plan* is a probability measure $Q \in \mathcal{P}(C([0,1], \overline{\Omega}))$ such that $Q(H) = 1$ and

$$\int_{C([0,1],\overline{\Omega})} l_{SR}(\sigma) dQ(\sigma) < +\infty. \tag{3.5}$$

A horizontal traffic plan $Q \in \mathcal{P}(C([0,1], \overline{\Omega}))$ is *admissible* between $\mu$ and $\nu$ if $(e_0)_\# Q = \mu$ and $(e_1)_\# Q = \nu$.

We denote by

$$\mathcal{Q}_H(\mu,\nu) = \{\text{horizontal traffic plans admissible between } \mu \text{ and } \nu\}\,.$$

In order to take into account congestion effects one can associate to any horizontal traffic plan $Q \in \mathcal{Q}_H(\mu,\nu)$ a positive and finite Radon measure on $\overline{\Omega}$, which we will call *horizontal traffic intensity.*

**Definition 3.2 (Horizontal traffic intensity).** Let $Q \in \mathcal{Q}_H(\mu,\nu)$. One can associate with $Q$ a positive and finite Radon measure $i_Q \in \mathcal{M}_+(\overline{\Omega})$ defined as

$$\int_{\overline{\Omega}} \varphi(x) di_Q(x) := \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma) dQ(\sigma), \quad \forall \varphi \in C(\overline{\Omega}).$$

We will call this measure $i_Q$ *horizontal traffic intensity induced by* $Q$. Moreover, its total mass is

$$i_Q(\overline{\Omega}) = \int_{C([0,1],\overline{\Omega})} l_{SR}(\sigma) dQ(\sigma). \tag{3.6}$$

## 3.2.1 Some properties of traffic intensities

Let us denote by $\mathcal{R} : R([0,1],\overline{\Omega}) \to R([0,1],\overline{\Omega})$ the map $\sigma \mapsto \tilde{\sigma}$, where $\tilde{\sigma}$ is a constant-speed reparametrization of $\sigma$.

If $Q \in \mathcal{Q}_H(\mu,\nu)$ is a horizontal traffic plan, we define the measure $\tilde{Q} \in \mathcal{P}(C([0,1],\overline{\Omega}))$ as the push-forward of $Q$ through the map $\mathcal{R}$,

$$\tilde{Q} := \mathcal{R}_\# Q.$$

By definition $\tilde{Q}(\tilde{H}) = 1$ and $i_{\tilde{Q}} = i_Q$, since $L_\varphi(\tilde{\sigma}) = L_\varphi(\sigma)$ for any $\varphi \in C(\overline{\Omega})$. Hence, the horizontal traffic intensity is invariant under reparametrization.

We introduce an important property of horizontal traffic intensities.

**Lemma 3.2.1.** *Let us consider* $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}(C([0,1],\overline{\Omega})$ *such that* $Q_n(H) = 1$, $\forall n \in \mathbb{N}$. *Let us suppose that there exists* $M > 0$ *such that*

$$\sup_{n\in\mathbb{N}} \int_{C([0,1],\overline{\Omega})} l_{SR}(\sigma)\; dQ_n(\sigma) \leq M.$$

*Then,* $(\tilde{Q}_n)_{n\in\mathbb{N}}$ *admits a subsequence weakly converging to some* $Q \in \mathcal{P}(C([0,1],\overline{\Omega}))$ *such that* $Q(H) = 1$.

*Proof.* We will prove the tightness of $(\tilde{Q}_n)_{n\in\mathbb{N}}$ by using the Prokhorov theorem, see for instance [3, Theorem 2.8]. Let us consider the sets

$$H_K := \{\sigma \in H : |\dot{\sigma}(t)|_H \leq K\} \subset C([0,1],\overline{\Omega}),$$

for any $K > 0$. The Ascoli-Arzela Theorem implies that this sets are compact w.r.t. the uniform convergence. Indeed, the equi-boundedness follows from

$$l_{SR}(\sigma) = \int_0^1 |\dot{\sigma}(t)|_H dt \leq K,$$

and the equi-continuity from

$$d_{SR}(\sigma(t),\sigma(t')) \leq \int_t^{t'} |\dot{\sigma}(\tau)|_H d\tau \leq K|t-t'|, \quad \forall \sigma \in H_K, \forall t,t' \in [0,1].$$

Hence, using the fact that $Q_n(H) = 1$, $\forall n \in \mathbb{N}$, and the Markov inequality, it follows that

$$\begin{aligned}
\tilde{Q}_n\big(C([0,1],\overline{\Omega}) &\setminus \{\sigma \in \tilde{H} : |\dot{\sigma}|_H \leq K\}\big)\\
&= Q_n\big(C([0,1],\overline{\Omega}) \setminus \{\sigma \in H : l_{SR}(\sigma) \leq K\}\big)\\
&= Q_n\big(\{\sigma \in H : l_{SR}(\sigma) > K\}\big) \leq \frac{1}{K}\int_{C([0,1],\overline{\Omega})} l_{SR}(\sigma)\, dQ_n(\sigma). \quad (3.7)
\end{aligned}$$

By Prokhorov's Theorem there exists a subsequence weakly converging to some $Q \in \mathcal{P}(C([0,1],\overline{\Omega})$. It remains to show that $Q(H) = 1$ From (3.7), the fact that the measures $\tilde{Q}_n$ are concentrated on $\tilde{H}$ and the previous argument, it follows that

$$\sup_{n\in\mathbb{N}} \tilde{Q}_n(H \setminus H_K) = \sup_{n\in\mathbb{N}} \tilde{Q}_n(\tilde{H} \setminus H_K) \leq \frac{M}{K},$$

for any $K > 0$, which in turn implies

$$\begin{aligned}
1 = \limsup_{n\to\infty} \tilde{Q}_n(H) &\leq \limsup_{n\to\infty} \tilde{Q}_n(H_K) + \limsup_{n\to\infty} \tilde{Q}_n(H \setminus H_K)\\
&\leq Q(H_K) + \frac{M}{K}.
\end{aligned}$$

If $K \to +\infty$, we get that $Q(H) \geq 1$. Since $Q \in \mathcal{P}(C([0,1],\overline{\Omega}))$, then $Q(H) = 1$. □

**Lemma 3.2.2.** *Let $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{Q}_H(\mu,\nu)$ such that $Q_n \rightharpoonup Q \in \mathcal{Q}_H(\mu,\nu)$. If there exists $i \in \mathcal{M}_+\left(\overline{\Omega}\right)$ such that $i_{Q_n} \rightharpoonup i$, then $i_Q \leq i$ as measures.*

*Proof.* Let us consider $\varphi \in C(\overline{\Omega}, \mathbb{R}_+)$, then by definition of $i_{Q_n}$ and weak convergence it follows that

$$\int_{\overline{\Omega}} \varphi(x)di(x) = \lim_{n\to+\infty} \int_{\overline{\Omega}} \varphi(x)di_{Q_n}(x) = \lim_{n\to+\infty} \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma)dQ_n(\sigma). \quad (3.8)$$

From Lemma 3.1.1, we know that for any $\varphi \in C(\overline{\Omega}), \varphi \geq 0$ the function $L_\varphi$ is l.s.c. for the topology of uniform convergence: hence, from the Portmanteau Theorem, see for instance [17], and (3.8) it follows that

$$\begin{aligned}\int_{\overline{\Omega}} \varphi(x)di_Q(x) = \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma)dQ(\sigma) &\leq \liminf_{n\to+\infty} \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma)dQ_n(\sigma) \\ &= \int_{\overline{\Omega}} \varphi(x)di(x), \quad \forall \varphi \in C(\overline{\Omega}), \varphi \geq 0.\end{aligned}$$

□

## 3.3 Congested optimal transport problem in $\mathbb{H}^n$

The aim of this section is to introduce the congested optimal transport problem in $\mathbb{H}^n$ and to prove the existence of equilibrium configurations.

The first step is to define the weighted sub-Riemannian length for non-negative $q$-summable functions, in order to define the congested metric. The second step is to introduce a convex minimization problem over the set of the horizontal traffic plans. Afterwards we will see why one can refer to this problem as the *congested optimal transport problem*: in particular, following [43, Section 3] we will see that solutions to the aforementioned minimization problem are equilibrium configurations.

Given a horizontal traffic plan $Q \in \mathcal{Q}_H(\mu,\nu)$, the congestion effects should be captured by a metric associated with the traffic plan itself. Hence, we consider a non-decreasing *congestion function*

$$g : \mathbb{R}_+ \to \mathbb{R}_+, \quad (3.9)$$

such that

$$\lim_{i\to+\infty} g(i) = +\infty.$$

We denote by

$$\varphi_Q(x) := \begin{cases} g(i_Q(x)), & \text{if } i_Q \ll \mathcal{L}^{2n+1}, \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.10}$$

where, with abuse of notation, $i_Q(x)$ is the density of the measure $i_Q$ with respect to the Lebesgue measure.

*Remark* 8. The hypotheses on the congestion function $g$ are quite natural from a modeling viewpoint: the quantity $g(i(x))$ is the cost to be paid for passing through $x \in \overline{\Omega}$, where there is an amount of traffic $i(x)$. Hence, we want that the cost increases as traffic increases; moreover, the hypothesis $\lim_{i\to\infty} g(i) = +\infty$ models the fact that, if there is too much traffic, one gets stuck in it and won't pass through the point $x$.

*Remark* 9. We remark that it is not very restrictive to suppose the existence of $Q \in \mathcal{Q}_H(\mu,\nu)$ such that $i_Q \ll \mathcal{L}^{2n+1}$. For instance, if either $\mu \ll \mathcal{L}^{2n+1}$ or $\nu \ll \mathcal{L}^{2n+1}$ and the transport set $\mathcal{T}$ (which is compact) is such that

$$\mathcal{T} \subseteq \overline{\Omega}; \tag{3.11}$$

we know from Theorem 2.2.2 that there exists $\gamma \in \Pi_1(\mu,\nu)$ such that $a_\gamma \ll \mathcal{L}^{2n+1}$. Hence, if we consider the horizontal traffic plan

$$Q := S_\#\gamma \in \mathcal{Q}_H(\mu,\nu),$$

its horizontal traffic intensity $i_Q$ is exactly $a_\gamma$: indeed, if $\varphi \in C(\overline{\Omega})$

$$\begin{aligned} \int_{\overline{\Omega}} \varphi(x)di_Q(x) = \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma)dS_\#\gamma(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} L_\varphi(S(x,y))d\gamma(x,y) \\ = \int_{\overline{\Omega}} \varphi(x)da_\gamma(x). \end{aligned} \tag{3.12}$$

The hypothesis (3.11) replaces the more classical geodesically convexity hypothesis on the set $\Omega$, since non-trivial geodesically convex subsets of $\mathbb{H}^n$ do not exist, see [72].

Let $Q \in \mathcal{Q}_H(\mu, \nu)$ such that $i_Q \ll \mathcal{L}^{2n+1}$, hence the quantity

$$\int_\Omega \varphi_Q(x) i_Q(x) dx$$

represents the total cost paid by the agents $\mu$ to get destinations $\nu$, given the traffic assignment $Q$. In an optimal scenario one is interested in minimizing the total cost

$$\inf_{Q \in \mathcal{Q}_H(\mu,\nu)} \int_\Omega \varphi_Q(x) i_Q(x) dx. \tag{3.13}$$

One can also express (3.13) in terms of transport plans between the two probability measures $\mu$ and $\nu$. Any travel from $x$ to $y$, which is performed along a path $\sigma \in H^{x,y}$, costs

$$L_{\varphi_Q}(\sigma) = \int_0^1 g\big(i_Q(\sigma(t))\big) |\dot\sigma(t)|_H dt. \tag{3.14}$$

Again in an optimal scenario, one wants to minimize the previous cost

$$c_{\varphi_Q}(x, y) = \inf \left\{ L_{\varphi_Q}(\sigma) : \sigma \in H^{x,y} \right\}, \tag{3.15}$$

hence the *congested optimal transport problem in* $\mathbb{H}^n$ is

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\overline{\Omega} \times \overline{\Omega}} c_{\varphi_Q}(x, y) d\gamma(x, y). \tag{3.16}$$

Let us remark that both (3.14) and (3.15) are formal because we defined the weighted sub-Riemannian length only for continuous functions. The aim of the next subsection is to define both these quantities for non-negative $\varphi \in L^q(\Omega)$, where $q = \frac{p}{p-1}$ and $p$ is the exponent that will appear in Subsection 3.3.2.

### 3.3.1 Weighted sub-Riemannian length and transport cost for non-negative $L^q$ functions

Let $\varphi \in C(\overline{\Omega})$, $\varphi \geq 0$ be a non-negative continuous function. Let us define

$$c_\varphi(x, y) := \inf\{L_\varphi(\sigma) \,:\, \sigma \in H^{x,y}\}, \quad \forall (x, y) \in \Omega \times \Omega. \tag{3.17}$$

**Proposition 3.3.1.** *If $q > N$, then there exists $C > 0$ such that for any $\varphi \in C(\overline{\Omega})$, $\varphi \geq 0$ and any $(x, y), (x', y') \in \Omega \times \Omega$, it holds*

$$|c_\varphi(x, y) - c_\varphi(x', y')| \leq C\|\varphi\|_{L^q(\Omega)} \left(d_{SR}(x, x')^\alpha + d_{SR}(y, y')^\alpha\right), \tag{3.18}$$

*where $\alpha := 1 - \frac{N}{q}$.*

*Proof.* Let $\varphi \in C(\overline{\Omega})$, $\varphi \geq 0$ and $x, y \in \Omega$. For $k > 0$ let $\sigma_k \in H^{x,y}$ be such that

$$\int_0^1 \varphi(\sigma_k(t))|\dot{\sigma}_k(t)|_H dt \leq c_\varphi(x, y) + \frac{1}{k}.$$

In order to study the regularity of $c_\varphi$ with respect to the second variable $y$, we choose a point $z_\varepsilon$ that can be connected to $y$ by a horizontal segment: i.e. we fix a horizontal vector field $Z \in \mathfrak{h}_1^n$, such that $|Z|_H = 1$, and we choose for all $\varepsilon > 0$ the points

$$z_\varepsilon := \exp\left(\varepsilon Z\right)(y),$$

such that $z_\varepsilon \in \Omega$. Now we modify the curve $\sigma_k$ into a curve $\sigma_{k,t_0} \in H^{x,z_\varepsilon}$: we choose $t_0 \in (0, 1)$ and define

$$\sigma_{k,t_0}(t) := \begin{cases} \sigma_k\left(\frac{t}{t_0}\right) & \text{if } t \in [0, t_0] \\ \tilde{\sigma}_{\varepsilon,y}\left(\frac{t-t_0}{1-t_0}\right) & \text{if } t \in ]t_0, 1], \end{cases}$$

where

$$\tilde{\sigma}_{\varepsilon,y}(t) = \exp\left(t\left(\varepsilon Z\right)\right)(y), \quad t \in [0, 1].$$

We then have, for all $k > 0$

$$\begin{aligned} c_\varphi(x, z_\varepsilon) &\leq \int_0^1 \varphi(\sigma_{k,t_0}(t))|\dot{\sigma}_{k,t_0}(t)|_H dt = \\ &= \int_0^1 \varphi(\sigma_k(t))|\dot{\sigma}_k(t)|_H dt + \int_0^1 \varphi(\tilde{\sigma}_{\varepsilon,y}(t))|\dot{\tilde{\sigma}}_{\varepsilon,y}(t)|_H dt \leq \\ &\leq c_\varphi(x, y) + \frac{1}{k} + \varepsilon \int_0^1 \varphi(\tilde{\sigma}_{\varepsilon,y}(t)) dt. \end{aligned}$$

Now, if $k \to +\infty$, we get

$$\frac{1}{\varepsilon}\left[c_\varphi\left(x, \exp\left(\varepsilon Z\right)(y)\right) - c_\varphi(x, y)\right] \leq \int_0^1 \varphi(\tilde{\sigma}_{\varepsilon,y}(t)) dt,$$

and, by similar argument:

$$\frac{1}{\varepsilon}\left[c_\varphi(x,y) - c_\varphi\left(x, \exp\left(\varepsilon Z\right)(y)\right)\right] \leq \int_0^1 \varphi(\tilde{\sigma}_{\varepsilon,y}(1-t))dt.$$

Integrating with respect to $y$, raising to the power $q$ and using the fact that the function $y \mapsto \tilde{\sigma}_{\varepsilon,y}(t)$ has Jacobian determinant 1, we get that, for any fixed $x$, $Zc_\varphi(x,\cdot) \in L^q_{loc}(\Omega)$, and $\|Zc_\varphi(x,\cdot)\|_{L^q(\Omega)} \leq C\|\varphi\|_{L^q(\Omega)}$. Since this holds for every $Z \in \mathfrak{h}_1^n$ we have $c_\varphi(x,\cdot) \in HW^{1,q}(\Omega)$ and

$$\|\nabla_H c_\varphi(x,\cdot)\|_{L^q(\Omega)} \leq \|\varphi\|_{L^q(\Omega)}, \quad \forall x \in \Omega. \tag{3.19}$$

By symmetry we also get that

$$\|\nabla_H c_\varphi(\cdot,y)\|_{L^q(\Omega)} \leq \|\varphi\|_{L^q(\Omega)}, \quad \forall y \in \Omega. \tag{3.20}$$

Since $q > N$, then if follows from (3.19), (3.20) and [61, Theorem 1.11], that there exists $C > 0$ such that

$$|c_\varphi(x,y) - c_\varphi(x,y')| \leq C\|\varphi\|_{L^q(\Omega)} d_{SR}(y,y')^\alpha, \quad \forall x,y,y' \in \Omega,$$
$$|c_\varphi(x,y) - c_\varphi(x',y)| \leq C\|\varphi\|_{L^q(\Omega)} d_{SR}(x,x')^\alpha, \quad \forall x,x',y \in \Omega,$$

with $\alpha = 1 - \frac{N}{q}$. This proves (3.18). □

**Proposition 3.3.2.** *If $q > N$, then for any $\varphi \in C(\overline{\Omega})$, $\varphi \geq 0$, the function $c_\varphi$ defined in* (3.17) *admits a unique continuous extension as a function*

$$c_\varphi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}_+,$$

*with the same modulus of continuity. Moreover the definition at* (3.17) *extends to all pairs $(x,y) \in \overline{\Omega} \times \overline{\Omega}$.*

*Proof.* The first part of the proof easily follows from (3.18).

Let now consider $\varphi \in C(\overline{\Omega})$, $\varphi \geq 0$, $\varepsilon_0 > 0$ and the continuous function $\varphi + \varepsilon_0 > 0$. Given $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $(x_n,y_n)_{n\in\mathbb{N}} \subset \Omega \times \Omega$, $(x_n,y_n) \to (x,y)$, we have

$$c_{\varphi+\varepsilon_0}(x,y) := \lim_{n\to+\infty} c_{\varphi+\varepsilon_0}(x_n,y_n).$$

It means that $\forall\varepsilon > 0$, there exists $n = n(\varepsilon)$ such that

$$|c_{\varphi+\varepsilon_0}(x,y) - c_{\varphi+\varepsilon_0}(x_n,y_n)| < \varepsilon, \quad \forall n > n(\varepsilon).$$

By definition of $c_{\varphi+\varepsilon_0}(x_n,y_n)$ and by the invariance of $L_{\varphi+\varepsilon_0}$ under reparametrization, there exists $\sigma_n \in \tilde{H}^{x_n,y_n}$ such that

$$|c_{\varphi+\varepsilon_0}(x_n,y_n) - L_{\varphi+\varepsilon_0}(\sigma_n)| < \varepsilon.$$

Hence, for any $n > n(\varepsilon)$ it holds that

$$\varepsilon_0|\dot{\sigma}_n|_H = \varepsilon_0 l_{SR}(\sigma_n) \le L_{\varphi+\varepsilon_0}(\sigma_n) < c_{\varphi+\varepsilon_0}(x_n,y_n) + \varepsilon \le M + \varepsilon,$$

where $M \ge 0$. Then, the Ascoli-Arzelà Theorem implies that $(\sigma_n)_{n>n(\varepsilon_0)} \subset H$ admits a subsequence $\sigma_{n_k} \to \sigma$ uniformly as $k \to +\infty$, with $\sigma \in H^{x,y}$ and $L_{\varphi+\varepsilon_0}(\sigma) \le \liminf_{k\to+\infty} L_{\varphi+\varepsilon_0}(\sigma_{n_k}) = c_{\varphi+\varepsilon_0}(x,y)$, see [1, Theorem 3.41] and Lemma 3.1.1. Then, we can conclude that $\forall\varepsilon_0 > 0$, $\forall(x,y) \in \overline{\Omega}\times\overline{\Omega}$, there exists $\sigma \in H^{x,y}$, such that

$$L_{\varphi+\varepsilon_0}(\sigma) \le c_{\varphi+\varepsilon_0}(x,y).$$

Moreover,

$$\begin{aligned} L_\varphi(\sigma) + \varepsilon_0 l_{SR}(\sigma) = L_{\varphi+\varepsilon_0}(\sigma) &\le c_{\varphi+\varepsilon_0}(x,y) \\ &= \lim_{n\to+\infty} c_{\varphi+\varepsilon_0}(x_n,y_n) \le \lim_{n\to+\infty} c_\varphi(x_n,y_n) + O(\varepsilon_0), \end{aligned}$$

hence, letting $\varepsilon_0 \to 0$, $L_\varphi(\sigma) \le c_\varphi(x,y)$.

It remains to prove that $c_\varphi(x,y) \le L_\varphi(\sigma)$, for any $\sigma \in H^{x,y}$.

Let us suppose that $x \in \Omega$ and $y \in \partial\Omega$, all the other cases can be deduced from this one. Let us consider an arbitrary horizontal curve $\sigma \in H^{x,y}$ and, for any $n \in \mathbb{N}$, we take $y_n \in B\left(y, \frac{1}{n}\right) \cap \Omega$. From Remark 7 it follows that there exists a horizontal curve $\sigma_n \in H^{y,y_n}$, such that $l_{SR}(\sigma_n) \le 2d_{SR}(y,y_n)$. Hence, $L_\varphi(\sigma_n) \le 2\|\varphi\|_\infty d_{SR}(y,y_n)$. Given $t_0 \in [0,1]$, we denote by $\tilde{\sigma}_{n,t_0} \in H^{x,y_n}$ the horizontal curve defined as

$$\tilde{\sigma}_{n,t_0}(t) := \begin{cases} \sigma\left(\frac{t}{t_0}\right), & \text{if } t \in [0,t_0], \\ \sigma_n\left(\frac{t-t_0}{1-t_0}\right), & \text{if } t \in [t_0,1]. \end{cases}$$

From the invariance of the weighted sub-Riemannian length under reparametrization, it follows that

$$c_\varphi(x,y) := \lim_{n\to\infty} c_\varphi(x,y_n) \le \liminf_{n\to\infty} L_\varphi(\tilde\sigma_{n,t_0}) = \liminf_{n\to\infty} \left(L_\varphi(\sigma) + L_\varphi(\sigma_n)\right) = L_\varphi(\sigma).$$

Since $\sigma \in H^{x,y}$ is arbitrary, the thesis follows. □

**Corollary 3.3.3.** *Let* $(\varphi_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega})$*,* $\varphi_n \ge 0\ \forall n \in \mathbb{N}$*, bounded in* $L^q$*, then* $(c_{\varphi_n})_{n\in\mathbb{N}}$ *admits a subsequence that converges in* $C(\overline{\Omega}\times\overline{\Omega})$*.*

*Proof.* The existence of a subsequence of $(c_{\varphi_n})_{n\in\mathbb{N}}$ that converges in $C(\overline{\Omega}\times\overline{\Omega})$ follows from Ascoli-Arzelà's theorem. Indeed, equicontinuity follows from Proposition 3.3.2, while the pointwise boundness is a consequence of the identity $c_{\varphi_n}(x,x) = 0$ and (3.18). □

Let us suppose that $q > N$. For a non-negative function $\varphi \in L^q(\Omega)$ we then define

$$\overline{c}_\varphi(x,y) := \sup\left\{c(x,y)\ : c \in \mathcal{A}(\varphi)\right\}, \tag{3.21}$$

for any $(x,y) \in \overline{\Omega}\times\overline{\Omega}$, where

$$\mathcal{A}(\varphi) = \left\{\lim_{n\to+\infty} c_{\varphi_n}\ \text{ in } C(\overline{\Omega}\times\overline{\Omega})\ :\ (\varphi_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega},\mathbb{R}_+), \varphi_n \to \varphi\ \text{ in } L^q\right\}.$$

*Remark* 10. Let us consider $(x,y) \in \overline{\Omega}\times\overline{\Omega}$. Then, for any $k > 0$ there exists $c_k \in \mathcal{A}(\varphi)$ such that

$$|\overline{c}_\varphi(x,y) - c_k(x,y)| < \frac{1}{k}.$$

Moreover, by definition there exists $\left(\varphi_n^k\right)_{n\in\mathbb{N}} \subset C(\overline{\Omega})$, $\varphi_n^k \ge 0$ for any $n \in \mathbb{N}$, such that $\varphi_n^k \xrightarrow[n\to+\infty]{L^q} \varphi$ and $\lim_{n\to\infty} c_{\varphi_n^k} = c_k$ in $C\left(\overline{\Omega}\times\overline{\Omega}\right)$. Hence, one can find $n = n(k)$ large enough, such that the sequence $\left(\varphi^k_{n(k)}\right)_{k\in\mathbb{N}}$ is bounded in $L^q$ and

$$\left|\overline{c}_\varphi(x,y) - c_{\varphi^k_{n(k)}}(x,y)\right| \le \frac{2}{k}.$$

Corollary 3.3.3 implies that $\left(c_{\varphi^k_{n(k)}}\right)_{k\in\mathbb{N}}$ (or at least a subsequence) converges in $C(\overline{\Omega}\times\overline{\Omega})$, as $k \to +\infty$. Moreover, $\overline{c}_\varphi(x,y) = \lim_{k\to\infty} c_{\varphi^k_{n(k)}}(x,y)$, and therefore the supremum at (3.21) is a maximum.

Moreover, the following lemma holds.

**Lemma 3.3.4.** *If* $\varphi \in L^q(\Omega)$, $\varphi \geq 0$

$$\mathcal{A}^*(\varphi) := \left\{ \lim_{n\to+\infty} c_{\varphi_n} \text{ in } C(\overline{\Omega}\times\overline{\Omega}) \,:\, (\varphi_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega},\mathbb{R}_+), \varphi_n \rightharpoonup \varphi \ \text{ in } L^q \right\}$$

*and*

$$\overline{c}^*_\varphi(x,y) := \sup\left\{c(x,y) \,:\, c \in \mathcal{A}^*(\varphi)\right\}.$$

*Then,* $\overline{c}^*_\varphi = \overline{c}_\varphi$.

*Proof.* The inequality $\overline{c}_\varphi \leq \overline{c}^*_\varphi$ is obvious by definition of weak convergence.

Let us prove the converse inequality. Suppose that $\varphi_n \rightharpoonup \varphi$ in $L^q$ and also that $c_{\varphi_n}$ converges to $c$ in $C(\overline{\Omega}\times\overline{\Omega})$. Hence, using Mazur's Lemma there exists a function $\mathcal{N} : \mathbb{N} \to \mathbb{N}$ and real numbers $\{\alpha_{k,n}\}_{k=n,\dots,\mathcal{N}(n)}$ with $\alpha_{k,n} \geq 0$ and $\sum_{k=n}^{\mathcal{N}(n)} \alpha_{k,n} = 1$ and such that the sequence

$$\eta_n := \sum_{k=n}^{\mathcal{N}(n)} \alpha_{k,n}\varphi_k, \quad \forall n \in \mathbb{N},$$

converges strongly (i.e. in $L^q$) to $\varphi$. Since, for fixed $(x,y) \in \overline{\Omega}\times\overline{\Omega}$, the function $\varphi \mapsto c_\varphi(x,y)$ is concave, then $\forall n \in \mathbb{N}$ there exists $m_n \in \{n,\dots,\mathcal{N}(n)\}$ such that $c_{\eta_n}(x,y) \geq\ c_{\varphi_{m_n}}(x,y)$ and hence

$$c(x,y) = \lim_{n\to\infty} c_{\varphi_{m_n}}(x,y) \leq \limsup_{n\to\infty} c_{\eta_n}(x,y) \leq \overline{c}_\varphi(x,y). \tag{3.22}$$

Taking the sup of the left-hand side over $\mathcal{A}^*(\varphi)$ we obtain that $\overline{c}^*_\varphi \leq \overline{c}_\varphi$. □

*Remark* 11. Note that, if we have a constant coefficient unitary horizontal vector $W_1 := a_1X_1+\ldots+a_nX_n+a_{n+1}X_{n+1}+\ldots+a_{2n}X_{2n} \in \mathfrak{h}_1^1$, it is possible to perform a change of variables which sends the vector $W_1$ to the first element of the canonical orthonormal basis. Indeed, if we denote by $W_2,\dots,W_{2n}$ a basis of the orthogonal complement $W_1^\perp$ in $\mathfrak{h}_1^1$ with respect to $\langle\cdot,\cdot\rangle_H$, and by $x$ a point, we can consider the change of variables $\Psi : \mathbb{R}^{2n+1} \to \mathbb{H}^n$,

$$\Psi(e_1,\dots,e_{2n+1}) = \exp(e_1W_1)\exp\left(\sum_{i=2}^{2n} e_iW_i + e_{2n+1}X_{2n+1}\right)(x). \tag{3.23}$$

The pullback of the vector field $W_1$ by $\Psi$ is $\Psi_*W_1 = \partial_{e_1}$, and the point $x$ will be the origin in the new coordinate system.

*Remark* 12. Any function $c \in \mathcal{A}(\varphi)$ satisfies the triangular inequality. Indeed, if $\varphi$ is a continuous function, then $c_\varphi$ is a distance, so given three points $x, y, z \in \overline{\Omega}$, it follows that

$$c_\varphi(x, z) \leq c_\varphi(x, y) + c_\varphi(y, z).$$

Hence, passing to the limit in the definition of $c$ we obtain

$$\begin{aligned} c(x, z) = & \lim_{n \to +\infty} c_{\varphi_n}(x, z) \\ & \leq \lim_{n \to +\infty} (c_{\varphi_n}(x, y) + c_{\varphi_n}(y, z)) = c(x, y) + c(y, z). \end{aligned} \quad (3.24)$$

Now we prove that if $\varphi$ is continuous, then the two functions $\overline{c}_\varphi$ and $c_\varphi$ coincide.

**Proposition 3.3.5.** *If $\varphi \in C(\overline{\Omega}), \varphi \geq 0$, then $\overline{c}_\varphi = c_\varphi$.*

*Proof.* First we consider the constant sequence $\varphi_n := \varphi, \quad \forall n \in \mathbb{N}$. Then $c_\varphi \in \mathcal{A}(\varphi)$ and we get that $\overline{c}_\varphi \geq c_\varphi$.

Let us prove the converse inequality. Let $x, y \in \overline{\Omega}$, $k > 0$ and $\sigma \in H^{x,y}$ such that $L_\varphi(\sigma) < c_\varphi(x, y) + 1/k$. Let us fix a sequence $\varphi_n \to \varphi$ in $L^q$ such that $c_{\varphi_n}$ converges uniformly to some $c$, we want to prove that $c \leq c_\varphi$. From density of simple functions and continuity of $\varphi$ we can assume that there exists a finite decomposition $\{t_0, t_1, \cdots t_M\}$ of the interval $[0, 1]$ such that $\dot{\sigma}$ is constant and horizontal on the interval $[t_{i-1}, t_i]$; in particular

$$L_{\varphi_n}(\sigma) = \sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} \varphi_n(\sigma(t))|\dot{\sigma}(t)|_H dt.$$

Let us consider a single interval $[t_{i-1}, t_i]$: up to a change of coordinates, we can also assume that $|\dot{\sigma}|_H = 1$ on this interval. For this reason, in the change of coordinates $\Psi_i : \mathbb{R}^{2n+1} \to \mathbb{H}^n$, introduced in (3.23), we can choose $\Phi_i(\sigma(t_{i-1})) = (t_{i-1}, 0)$ so that $\Phi_i(\sigma(t_i)) = (t_i, 0)$, and

$$\Phi_i \circ \sigma : [t_{i-1}, t_i] \to \mathbb{R}^{2n+1}, \quad (\Phi_i \circ \sigma)(t) = (t, 0),$$

where $\Phi_i := \Psi_i^{-1} : \mathbb{H}^n \to \mathbb{R}^{2n+1}$.

We now consider, for every $\delta > 0$ and for every $i$, cylindrical neighborhoods $C_{i,\delta} = \{(t, \hat{e}) \in \mathbb{R}^{2n+1} : t \in [t_{i-1}, t_i], |\hat{e}|_{\mathbb{R}^{2n}} \leq \delta\}$, of the curve $\Phi_i \circ \sigma$, with basis $S_{i-1} = \{(t_{i-1}, \hat{e}) : |\hat{e}|_{\mathbb{R}^{2n}} \leq \delta\}$. For every $\hat{e} \in \mathbb{R}^{2n}$, with $|\hat{e}|_{\mathbb{R}^{2n}} \leq \delta$, we call $\sigma_e(t) = \Psi_i(t, \hat{e})$. By definition

$$c_{\varphi_n}\Big(\Psi_i(t_{i-1}, \hat{e}), \Psi_i(t_i, \hat{e})\Big) \leq L_{\varphi_n}(\sigma_e \circ \theta_i),$$

where $\theta_i$ is a change of coordinate which sends $[0, 1]$ to $[t_{i-1}, t_i]$. Note that

$$L_{\varphi_n}(\sigma_e \circ \theta_i) = L_{\varphi_n \circ \Psi_i}(\Phi_i \circ \sigma_e \circ \theta_i) = \int_{t_{i-1}}^{t_i} (\varphi_n \circ \Psi_i)(t, \hat{e})dt. \tag{3.25}$$

Hence, integrating on $S_{i-1}$ we get

$$\int_{S_{i-1}} c_{\varphi_n}\Big(\Psi_i(t_{i-1}, \hat{e}), \Psi_i(t_i, \hat{e})\Big)d\mathcal{L}^{2n}(\hat{e}) \leq \int_{S_{i-1}} \int_{t_{i-1}}^{t_i} (\varphi_n \circ \Psi_i)(t, \hat{e})dtd\mathcal{L}^{2n}(\hat{e}). \tag{3.26}$$

For $n \to \infty$ using the uniform convergence of $c_{\varphi_n}$ to $c$ and the $L^q$ convergence of $\varphi_n$ to $\varphi$ we get that

$$\int_{S_{i-1}} c\Big(\Psi_i(t_{i-1}, \hat{e}), \Psi_i(t_i, \hat{e})\Big)d\mathcal{L}^{2n}(\hat{e}) \leq \int_{C_i} (\varphi \circ \Psi_i)(t, \hat{e})d\mathcal{L}^{2n+1}(t, \hat{e}).$$

Now we divide by the measure of $S_{i-1}$ and pass to the limit as $\delta \to 0^+$. Using the fact that $c$ is continuous

$$\lim_{\delta \to 0^+} \frac{1}{d\mathcal{L}^{2n}(S_{i-1})} \int_{S_{i-1}} c\Big(\Psi_i(t_{i-1}, \hat{e}), \Psi_i(t_i, \hat{e})\Big))d\mathcal{L}^{2n}(\hat{e})$$
$$= c\Big(\Psi_i(t_{i-1}, 0), \Psi_i(t_i, 0)\Big) = c(x^{i-1}, x^i),$$

where $x^i = \sigma(t_i)$, Analogously the integral over $C_i = [t_{i-1}, t_i] \times S_{i-1}$ divided by the measure of $S_{i-1}$ converges to the integral on $[t_{i-1}, t_i]$, which is the integral along the curve $\Phi_i \circ \sigma(t)$

$$\lim_{\delta \to 0^+} \frac{1}{d\mathcal{L}^{2n}(S_{i-1})} \int_{C_i} (\varphi \circ \Psi_i)(t, \hat{e})d\mathcal{L}^{2n+1}(t, \hat{e})$$
$$= \int_{t_{i-1}}^{t_i} (\varphi \circ \Psi_i)(t, 0)dt = \int_{t_{i-1}}^{t_i} \varphi(\sigma(t))|\dot{\sigma}(t)|_H dt.$$

Then, using (3.26), we get that

$$c(x^{i-1}, x^i) \leq \int_{t_{i-1}}^{t_i} \varphi(\sigma(t))|\dot{\sigma}(t)|_H dt, \quad \forall i = 1, \ldots, M,$$

and then, from (3.24),

$$c(x, y) \leq \sum_{i=1}^{M} c(x^{i-1}, x^i) \leq \sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} \varphi(\sigma(t))|\dot{\sigma}(t)|_H dt = L_\varphi(\sigma).$$

This gives

$$c(x, y) \leq c_\varphi(x, y) + \frac{1}{k}$$

for the choice of $\sigma$ and, since $k$ is arbitrary, it follows that $c(x, y) \leq c_\varphi(x, y)$. □

**Lemma 3.3.6.** *Let $q > N$, $\varphi \in L^q(\Omega)$, $\varphi \geq 0$, then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega}, \mathbb{R}_+)$, $\varphi_n \to \varphi$ in $L^q(\Omega)$, such that $c_{\varphi_n}$ converges to $\overline{c}_\varphi$ in $C(\overline{\Omega} \times \overline{\Omega})$.*

*Proof.* From the Remark 10 it follows that for every $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega}, \mathbb{R}_+)$ converging to $\varphi$ in $L^q(\Omega)$, such that $c_{\varphi_n}$ converges uniformly in $\overline{\Omega} \times \overline{\Omega}$ and $\overline{c}_\varphi(x, y) = \lim_{n\to\infty} c_{\varphi_n}(x, y)$.

Let $I$ be a finite set, $(x_i, y_i) \in \overline{\Omega} \times \overline{\Omega}$ for all $i \in I$, and for every $i$ let $(\varphi_n^i)$ be a sequence of non-negative continuous functions converging to $\varphi$ in $L^q(\Omega)$ such that $\overline{c}_\varphi(x_i, y_i) = \lim_n c_{\varphi_n^i}(x_i, y_i)$. Let us set $\varphi_n := \max_{i\in I} \varphi_n^i$, then we have a sequence $(\varphi_n)_{n\in\mathbb{N}}$ that converges to $\varphi$ in $L^q(\Omega)$, and

$$\overline{c}_\varphi(x_i, y_i) \leq \liminf_n \; c_{\varphi_n}(x_i, y_i) \leq \limsup_n \; c_{\varphi_n}(x_i, y_i) \leq \overline{c}_\varphi(x_i, y_i).$$

We thus have $\overline{c}_\varphi(x_i, y_i) = \lim_n c_{\varphi_n}(x_i, y_i)$ for every $i \in I$.

Let $(x_i, y_i)_{i\in\mathbb{N}}$ be a dense sequence of points of $\overline{\Omega} \times \overline{\Omega}$. From the previous argument, $\forall n \in \mathbb{N}$ there exists $\varphi_n \in C(\overline{\Omega}, \mathbb{R}_+)$ such that

$$\|\varphi_n - \varphi\|_{L^q(\Omega)} \leq \frac{1}{n} \text{ and } |\overline{c}_\varphi(x_k, y_k) - c_{\varphi_n}(x_k, y_k)| \leq \frac{1}{n}, \; \forall k \leq n.$$

As in Corollary 3.3.3, we may assume that $c_{\varphi_n}$ converges in uniformly in $\overline{\Omega}\times\overline{\Omega}$ to some $c$, up to a subsequence. Since it holds that $c(x_k, y_k) = \overline{c}_\varphi(x_k, y_k)$ for all $k$, and $(x_k, y_k)_{k\in\mathbb{N}}$ is dense in $\overline{\Omega} \times \overline{\Omega}$, then $c = \overline{c}_\varphi$. □

Now we can extend $L_\varphi$ to the functions $\varphi \geq 0$ that are only $q$-summable. Moreover this $L_\varphi$ behaves as if $\varphi$ were a continuous function.

Let $p = \frac{q}{q-1}$ and let us denote by

$$\mathcal{Q}_H^p(\mu,\nu) := \{Q \in \mathcal{Q}_H(\mu,\nu) \ : \ i_Q \in L^p(\Omega)\}\,.$$

**Lemma 3.3.7.** *If $q > N$, $Q \in \mathcal{Q}_H^p(\mu,\nu)$ and $\varphi \in L^q(\Omega)$, $\varphi \geq 0$. Let $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n \geq 0$, $\forall n \in \mathbb{N}$ converging to $\varphi$ in $L^q$, it holds that:*

*(i) $(L_{\varphi_n})_{n\in\mathbb{N}}$ converges strongly in $L^1(C([0,1],\overline{\Omega}),Q)$ to some limit, independent of the approximating sequence $(\varphi_n)_{n\in\mathbb{N}}$, which will be denoted by $L_\varphi$.*

*(ii) The following equality holds:*

$$\int_\Omega \varphi(x) i_Q(x)dx = \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma)dQ(\sigma). \tag{3.27}$$

*(iii) The following inequality holds for $Q$-a.e. $\sigma \in C([0,1],\overline{\Omega})$:*

$$L_\varphi(\sigma) \geq \bar{c}_\varphi(\sigma(0),\sigma(1)). \tag{3.28}$$

*Proof.* For all $n$ and $m$ in $\mathbb{N}$ we have:

$$\begin{aligned}
&\int_{C([0,1],\overline{\Omega})} |L_{\varphi_n}(\sigma) - L_{\varphi_m}(\sigma)| \; dQ(\sigma) = \\
= &\int_{C([0,1],\overline{\Omega})} \left| \int_0^1 (\varphi_n(\sigma(t)) - \varphi_m(\sigma(t))) \, |\dot\sigma(t)|_H \; dt \right| dQ(\sigma) \\
\leq &\int_\Omega |\varphi_n(x) - \varphi_m(x)| i_Q(x) \; dx \\
\leq &\ \|\varphi_n - \varphi_m\|_{L^q(\Omega)} \|i_Q\|_{L^p(\Omega)}.
\end{aligned}$$

Hence $(L_{\varphi_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(C([0,1],\overline{\Omega}),Q)$, then it converges to some limit that we denote by $L_\varphi$.

Let us suppose that $(\varphi'_n)_{n\in\mathbb{N}}$ is another sequence of non negative functions converging to $\varphi$ in $L^q(\Omega)$. Let us denote by $L'_\varphi$ the limit of the sequence

$(L_{\varphi'_n})_{n\in\mathbb{N}}$ in $L^1(C([0,1],\overline{\Omega}),Q)$, then

$$\begin{aligned}\int_{C([0,1],\overline{\Omega})} &|L_\varphi(\sigma) - L'_\varphi(\sigma)|dQ(\sigma)\\ &\leq \int_{C([0,1],\overline{\Omega})} |L_{\varphi'_n}(\sigma) - L'_\varphi(\sigma)|dQ(\sigma) + \int_{C([0,1],\overline{\Omega})} |L_{\varphi_n}(\sigma) - L_\varphi(\sigma)|dQ(\sigma)\\ &\qquad + \left(\|\varphi_n - \varphi\|_{L^q(\Omega)} + \|\varphi'_n - \varphi\|_{L^q(\Omega)}\right)\|i_Q\|_{L^p(\Omega)}\end{aligned}$$

Letting $n$ tend to $+\infty$ in the right hand side, we get that $L_\varphi = L'_\varphi$ $Q$-a.e., hence the $L^1(C([0,1],\overline{\Omega}),Q)$ limit does not depend on the approximating sequence $(\varphi_n)_{n\in\mathbb{N}}$.

The proof of $(ii)$ follows from $(i)$:

$$\begin{aligned}\int_\Omega \varphi(x)i_Q(x)\ dx = \lim_{n\to\infty}\int_\Omega \varphi_n(x)i_Q(x)\ dx &= \lim_{n\to\infty}\int_{C([0,1],\overline{\Omega})} L_{\varphi_n}(\sigma)\ dQ(\sigma)\\ &= \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma)\ dQ(\sigma).\end{aligned}$$

Finally, let $(\varphi_n)_{n\in\mathbb{N}}$ be an approximating sequence as in Lemma 3.3.6; from the definition of $c_{\varphi_n}$ it follows that for any $\sigma \in H$

$$L_{\varphi_n}(\sigma) \geq c_{\varphi_n}(\sigma(0),\sigma(1)).$$

Integrating against $Q$ and letting $n \to \infty$ in the previous inequality, we get $(iii)$. □

### 3.3.2 The optimization problem

We consider the following optimization problem

$$\inf_{Q\in\mathcal{Q}_H(\mu,\nu)} \mathcal{F}(i_Q), \tag{3.29}$$

where

$$\mathcal{F}(i) = \begin{cases}\int_\Omega G(i(x))dx, & \text{if } i \ll \mathcal{L}^{2n+1},\\ +\infty, & \text{otherwise,}\end{cases}$$

and the function $G : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, non-decreasing with $p$-growth

$$ai^p \le G(i) \le bi^p + 1, \quad \forall i \in \mathbb{R}_+, \tag{3.30}$$

for some $p \in (1, \infty)$ and such that

$$G(0) = 0. \tag{3.31}$$

Moreover, if we suppose that

$$\mathcal{Q}_H^p(\mu, \nu) = \{Q \in \mathcal{Q}_H(\mu, \nu) \ : \ i_Q \in L^p(\Omega)\} \neq \emptyset, \tag{3.32}$$

then the problem (3.29) reads as

$$\inf_{Q \in \mathcal{Q}_H^p(\mu,\nu)} \int_\Omega G(i_Q(x))dx. \tag{3.33}$$

*Remark* 13. Let us just remark that, if for instance $\mu, \nu \in L^p(\Omega)$, then $\mathcal{Q}_H^p(\mu, \nu) \neq \emptyset$, see Corollary 5.1.6.

At this point, one can prove that the problem (3.33) admits a solution.

**Theorem 3.3.8.** *Let us suppose that*

$$\mathcal{Q}_H^p(\mu, \nu) \neq \emptyset,$$

*then* (3.33) *admits a solution.*

*Proof.* First of all, from (3.30) and (3.32) it follows that (3.33) is finite.

Let $(Q_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (3.33). Since we are interested in the sequence of the associated horizontal traffic intensities, which are invariant under reparametrization, we may assume that $Q_n = \tilde{Q}_n$ for all $n \in \mathbb{N}$. We may assume that there exists $C > 0$ such that

$$\int_\Omega G(i_{Q_n}(x))dx \le C, \quad \forall n \in \mathbb{N},$$

since $(Q_n)_{n\in\mathbb{N}}$ is a minimizing sequence. Hence, from (3.30) it follows that $(i_{Q_n})_{n\in\mathbb{N}}$ is uniformly bounded in $L^p(\Omega)$, which in turn implies uniform boundedness in $L^1(\Omega)$, i.e. there is a constant $M > 0$ such that

$$\int_{C([0,1],\overline{\Omega})} l_{SR}(\sigma)dQ_n(\sigma) = \int_\Omega i_{Q_n}(x)dx \le M.$$

We are in the hypotheses of Lemma 3.2.1, hence we can assume that the minimizing sequence $(Q_n)_{n\in\mathbb{N}}$ (up to subsequences) weakly converges to some horizontal traffic plan $Q \in \mathcal{Q}_H^p(\mu,\nu)$. Moreover, since the uniform boundedness of $(i_{Q_n})_{n\in\mathbb{N}}$ in $L^p(\Omega)$ holds, we can assume that this sequence (up to subsequences) weakly converges to some $i \in L^p(\Omega)$. By lemma 3.2.2 it follows that $i_Q \leq i$, that in turn implies that $i_Q \in L^p(\Omega)$. Now one can use the monotonicity and convexity properties of $G$ to get

$$\int_\Omega G(i_Q(x))dx \leq \int_\Omega G(i(x))dx \leq \liminf_{n\to+\infty} \int_\Omega G(i_{Q_n}(x))dx,$$

which proves that $Q$ is a solution to (3.33). □

If in addition $G$ is strictly convex, and $Q_1$ and $Q_2$ are solutions to (3.33), then it follows that $i_{Q_1} = i_{Q_2}$. Hence the optimal traffic intensity is unique but this doesn't imply that $Q_1 = Q_2$.

### 3.3.3 Variational inequalities

We suppose in addition that $G$ is differentiable and that there exists a positive constant $c$ such that

$$0 \leq G'(i) \leq ci^{p-1} + 1, \quad p \in (1,\infty).$$

Recall that $q$ is the conjugate exponent of $p$, given by $q = \frac{p}{p-1}$.

The variational inequalities characterizing solutions of the convex problem (3.33) can be expressed as follows.

**Proposition 3.3.9.** *$\overline{Q} \in \mathcal{Q}_H^p(\mu,\nu)$ solves* (3.33) *if and only if*

$$\int_\Omega \overline{\varphi}(x) i_{\overline{Q}}(x)dx = \inf\left\{\int_\Omega \overline{\varphi}(x) i_Q(x)dx \ : \ Q \in \mathcal{Q}_H^p(\mu,\nu)\right\}, \tag{3.34}$$

*with $\overline{\varphi} := G'(i_{\overline{Q}}) \in L^q(\Omega)$.*

*Proof.* Let us prove the first implication. If $\overline{Q} \in \mathcal{Q}_H^p(\mu,\nu)$ solves (3.33), then for any $Q \in \mathcal{Q}_H^p(\mu,\nu)$ and any $\varepsilon > 0$, it holds that $\overline{Q} + \varepsilon(Q - \overline{Q}) \in \mathcal{Q}_H^p(\mu,\nu)$.

In particular $i_{\overline{Q}+\varepsilon(Q-\overline{Q})} = i_{\overline{Q}} + \varepsilon(i_Q - i_{\overline{Q}}) \in L^p(\Omega)$. Hence,

$$\begin{aligned} 0 &\leq \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[\mathcal{F}\left(i_{\overline{Q}+\varepsilon(Q-\overline{Q})}\right) - \mathcal{F}(i_{\overline{Q}})\right] = \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[\mathcal{F}(i_{\overline{Q}} + \varepsilon(i_Q - i_{\overline{Q}})) - \mathcal{F}(i_{\overline{Q}})\right] = \\ &= \int_\Omega G'(i_{\overline{Q}}(x))(i_Q(x) - i_{\overline{Q}}(x))dx = \int_\Omega \overline{\varphi}(x)(i_Q(x) - i_{\overline{Q}}(x))dx. \end{aligned}$$

We have just proven that

$$\int_\Omega \overline{\varphi}(x) i_{\overline{Q}}(x)dx \leq \int_\Omega \overline{\varphi}(x) i_Q(x)dx, \quad \forall Q \in \mathcal{Q}^p_H(\mu,\nu),$$

which proves (3.34).

Conversely, if $\overline{Q} \in \mathcal{Q}^p_H(\mu,\nu)$ satisfies (3.34), then for every $Q \in \mathcal{Q}^p_H(\mu,\nu)$, one has

$$\mathcal{F}(i_Q) - \mathcal{F}(i_{\overline{Q}}) \geq \int_\Omega \overline{\varphi}(x)(i_Q(x) - i_{\overline{Q}}(x))dx \geq 0,$$

where we used the convexity assumption on $G$, in the first inequality, and the fact that $\overline{Q}$ solves (3.34), in the second one. □

### 3.3.4 Characterization of minimizers

The aim of this subsection is to characterize minima of (3.29): the first step consists in relating the optimality condition (3.34) with the Monge-Kantorovich problem with cost $\overline{c}_{\overline{\varphi}}$, where $\overline{\varphi} = G'(i_{\overline{Q}}) \in L^q(\Omega)$ for some optimal $\overline{Q} \in \mathcal{Q}^p_H(\mu,\nu)$ optimal for (3.33). In order to do this, we need the following preliminary lemma.

Every $\lambda \in \mathcal{M}_+(\overline{\Omega})$ we will consider is supposed to be defined on the whole $\mathbb{H}^n$, extended by 0 outside $\overline{\Omega}$.

**Lemma 3.3.10.** *Let us consider $\rho \in C^\infty_c(\mathbb{H}^n)$, $\rho \geq 0$ such that $\int_{\mathbb{H}^n} \rho(x)dx =$ 1 and consider the problem*

$$\inf_{Q\in\mathcal{Q}_H(\mu,\nu)} \int_{\mathbb{H}^n} G(\rho * i_Q(x))dx, \tag{3.35}$$

*where $*$ is the convolution in $\mathbb{H}^n$. Then, this problem admits a solution.*

*Moreover,* $\overline{Q} \in \mathcal{Q}_H(\mu,\nu)$ *solves* (3.35) *if and only if*

$$\int_{\mathbb{H}^n} G'(\rho * i_{\overline{Q}})(\rho * i_{\overline{Q}})dx = \inf_{Q\in\mathcal{Q}_H(\mu,\nu)} \int_{\mathbb{H}^n} G'(\rho * i_{\overline{Q}})(\rho * i_Q)dx. \tag{3.36}$$

*Proof.* First of all we observe that (3.35) is finite: this follows from the fact that $\rho * i_Q \in C_c^\infty(\mathbb{H}^n)$ and from (3.30). Moreover, for every $Q \in \mathcal{Q}_H(\mu,\nu)$ the $L^1$ norm of $\rho * i_Q$ equals the total mass of $i_Q$: indeed

$$\int_{\mathbb{H}^n} \rho * i_Q(x)dx = \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \rho(y^{-1}\cdot x)di_Q(y)dx = \int_{\mathbb{H}^n} di_Q(y) = i_Q\left(\overline{\Omega}\right). \tag{3.37}$$

Let $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{Q}_H(\mu,\nu)$ be a minimizing sequence, i.e.

$$\int_{\mathbb{H}^n} G(\rho * i_{Q_n}(x))dx \underset{n\to\infty}{\longrightarrow} \inf_{Q\in\mathcal{Q}_H(\mu,\nu)} \int_{\mathbb{H}^n} G(\rho * i_Q(x))dx.$$

Since $i_Q = i_{\tilde{Q}}$, for any $Q \in \mathcal{Q}_H(\mu,\nu)$, we can assume that $Q_n = \tilde{Q}_n, \forall n \in \mathbb{N}$. From the fact that $(Q_n)_{n\in\mathbb{N}}$ is a minimizing sequence, the fact that (3.35) is finite and (3.30) it follows that $(\rho * i_{Q_n})_{n\in\mathbb{N}}$ is bounded in $L^p$, hence in $L^1$. Thanks to (3.37) we get a uniform bound on $(i_{Q_n}(\overline{\Omega}))_{n\in\mathbb{N}}$ so, using Banach-Alaoglu's Theorem and the compactness of $\overline{\Omega}$, we can assume that $(i_{Q_n})_{n\in\mathbb{N}}$ weakly converges to some finite measure $i \in \mathcal{M}_+(\overline{\Omega})$. Moreover

$$\sup_{n\in\mathbb{N}} \int_{C([0,1],\overline{\Omega})} l_{SR}(\sigma)dQ_n(\sigma) = \sup_{n\in\mathbb{N}} \int_{\Omega} di_{Q_n}(x) < +\infty, \tag{3.38}$$

hence form Lemma 3.2.1 it follows that $Q_n \rightharpoonup \overline{Q} \in \mathcal{Q}_H(\mu,\nu)$. Lemma 3.2.2 implies that $i_{\overline{Q}} \le i$. From the monotonicity and the convexity of $G$ we have

$$\int_{\mathbb{H}^n} G(\rho * i_{\overline{Q}}(x))dx \le \int_{\mathbb{H}^n} G(\rho * i(x))dx \le \liminf_{n\to\infty} \int_{\mathbb{H}^n} G(\rho * i_{Q_n}(x))dx,$$

which proves that $\overline{Q}$ solves (3.35).

As for the second part of the statement, as in Proposition 3.3.9, suppose that $\overline{Q} \in \mathcal{Q}_H(\mu,\nu)$ solves (3.33), then for any $Q \in \mathcal{Q}_H(\mu,\nu)$

$$0 \le \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{H}^n} \left(G(\rho * i_{\overline{Q}+\varepsilon(Q-\overline{Q})}) - G(\rho * i_{\overline{Q}})\right) dx$$
$$= \int_{\mathbb{H}^n} G'(\rho * i_{\overline{Q}})(\rho * i_Q - \rho * i_{\overline{Q}})dx.$$

Conversely, if $Q \in \mathcal{Q}_H(\mu,\nu)$ satisfies (3.36), then by convexity of $G$

$$\int_{\mathbb{H}^n} \left(G(\rho * i_Q) - G(\rho * i_{\overline{Q}})\right) dx \geq \int_{\mathbb{H}^n} G'(\rho * i_{\overline{Q}}) \left(\rho * i_Q - \rho * i_{\overline{Q}}\right) dx \geq 0,$$

for any $Q \in \mathcal{Q}_H(\mu,\nu)$. □

We can do this under the additional hypothesis of strict convexity for the function

$$G : \mathbb{R}_+ \to \mathbb{R}_+.$$

**Proposition 3.3.11.** *Let $q > N$ and $G$ strictly convex. If $\overline{Q}$ solves (3.33) and $\overline{\varphi} := G'(i_{\overline{Q}}) \in L^q(\Omega)$ then*

$$\int_\Omega \overline{\varphi}(x) i_{\overline{Q}}(x)dx = \inf_{Q\in\mathcal{Q}^p_H(\mu,\nu)} \int_\Omega \overline{\varphi}(x) i_Q(x)dx = \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\varphi}}(x,y)d\gamma(x,y). \tag{3.39}$$

*Proof.* The first equality follows from (3.34).

Let us prove the second one. Let $\varphi \in L^q(\Omega), \varphi \geq 0$ and $Q \in \mathcal{Q}^p_H(\mu,\nu)$. Using Lemma 3.3.7, the definition of push-forward measure and $\gamma_Q := (e_0,e_1)_\# Q \in \Pi(\mu,\nu)$

$$\begin{aligned}\int_\Omega \varphi(x) i_Q(x)dx &= \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma)dQ(\sigma) \geq \int_{C([0,1],\overline{\Omega})} \overline{c}_\varphi(\sigma(0)),\sigma(1))dQ(\sigma)\\ &= \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\varphi(x,y)d\gamma_Q(x,y) \geq \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\varphi(x,y)d\gamma(x,y).\end{aligned}$$

Then we get

$$\inf_{Q\in\mathcal{Q}^p_H(\mu,\nu)} \int_\Omega \varphi(x) i_Q(x)dx \geq \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\varphi(x,y)d\gamma(x,y), \quad \forall\varphi \in L^q(\Omega), \varphi \geq 0. \tag{3.40}$$

By a similar argument, if $\varphi \in C(\overline{\Omega}), \varphi \geq 0$ and $Q \in \mathcal{Q}_H(\mu,\nu)$, using Lemma 3.3.5, the fact that $Q$ is an admissible traffic plan between $\mu,\nu$ again, and (3.17)

$$\begin{aligned}\int_\Omega \varphi(x) di_Q(x) &= \int_{C([0,1],\overline{\Omega})} L_\varphi(\sigma)dQ(\sigma) \geq \int_{C([0,1],\overline{\Omega})} c_\varphi(\sigma(0),\sigma(1))dQ(\sigma) =\\ &= \int_{C([0,1],\overline{\Omega})} \overline{c}_\varphi(\sigma(0),\sigma(1))dQ(\sigma) \geq \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\varphi(x,y)d\gamma(x,y).\end{aligned}$$

Then we get

$$\inf_{Q\in\mathcal{Q}_H(\mu,\nu)}\int_{\overline{\Omega}}\varphi(x)di_Q(x)\geq\inf_{\gamma\in\Pi(\mu,\nu)}\int_{\overline{\Omega}\times\overline{\Omega}}\overline{c}_\varphi(x,y)d\gamma(x,y),\quad\forall\varphi\in C(\overline{\Omega}),\varphi\geq 0. \tag{3.41}$$

Let $\varphi\in C(\overline{\Omega})$, $\varphi\geq\varepsilon_0>0$, and $\varepsilon>0$. For any $x$ and $y$ in $\overline{\Omega}$ there exists $\sigma^\varepsilon_{x,y}\in H^{x,y}$ such that

$$L_\varphi(\sigma^\varepsilon_{x,y})\leq c_\varphi(x,y)+\varepsilon=\overline{c}_\varphi(x,y)+\varepsilon, \tag{3.42}$$

where we used Proposition 3.3.2 and Proposition 3.3.5. The set

$$\mathcal{K}^{x,y}:=\left\{\sigma^\varepsilon_{x,y}\in H^{x,y}:L_\varphi(\sigma^\varepsilon_{x,y})\leq c_\varphi(x,y)+\varepsilon\right\}$$

is a closed subset of $C([0,1],\overline{\Omega})$. Indeed, let $(\sigma_n)_{n\in\mathbb{N}}\subset\mathcal{K}^{x,y}$ that converges uniformly to $\sigma\in C([0,1],\overline{\Omega})$. The bound $\varphi\geq\varepsilon_0$ implies the bound

$$\varepsilon_0 l_{SR}(\sigma_n)\leq\underbrace{c_\varphi(x,y)}_{\leq M<+\infty}+\varepsilon.$$

Hence $\sigma\in H^{x,y}$, see [1, Theorem 3.41], and $L_\varphi(\sigma)\leq c_\varphi(x,y)+\varepsilon$, thanks to Lemma 3.1.1. This implies the closedness of the graph of the multivalued map

$$\mathcal{S}_\varepsilon:\overline{\Omega}\times\overline{\Omega}\to C([0,1],\overline{\Omega}),$$

$(x,y)\mapsto\sigma^\varepsilon_{x,y}\in\mathcal{K}^{x,y}$, in $\overline{\Omega}\times\overline{\Omega}\times C([0,1],\overline{\Omega})$. Then, for any $\gamma\in\Pi(\mu,\nu)$, there exists a map $S_\varepsilon:\overline{\Omega}\times\overline{\Omega}\to H$, $S_\varepsilon(x,y)=\sigma^\varepsilon_{x,y}\in\mathcal{K}^{x,y}$, which is $\gamma$-measurable (see for instance [18, Theorem 6.9.13]). One can consider the measure $Q_\varepsilon:=(S_\varepsilon)_\#\gamma\in\mathcal{P}\left(C([0,1],\overline{\Omega})\right)$, concentrated on the set $H$ by construction. Moreover,

$$\int_{C([0,1],\overline{\Omega})}l_{SR}(\sigma)dQ_\varepsilon(\sigma)=\int_{\overline{\Omega}\times\overline{\Omega}}l_{SR}(\sigma^\varepsilon_{x,y})d\gamma(x,y)<+\infty,$$

thanks to the bound on $\varphi$. Hence $Q_\varepsilon\in\mathcal{Q}_H(\mu,\nu)$ and

$$\int_{\overline{\Omega}}\varphi(x)di_{Q_\varepsilon}(x)=\int_{\overline{\Omega}\times\overline{\Omega}}L_\varphi(S_\varepsilon(x,y))d\gamma(x,y)\leq\int_{\overline{\Omega}\times\overline{\Omega}}\overline{c}_\varphi(x,y)d\gamma(x,y)+\varepsilon.$$

If $\varphi$ is an arbitrary continuous function on $\overline{\Omega}$, without any strictly positive lower bound, one can consider the function $\varphi + \varepsilon_0 \geq \varepsilon_0 > 0$ and get

$$\int_{\overline{\Omega}} (\varphi + \varepsilon_0)(x) di_{Q_\varepsilon}(x) = \int_{\overline{\Omega}\times\overline{\Omega}} L_{\varphi+\varepsilon_0}(S_\varepsilon(x,y))d\gamma(x,y)$$
$$\leq \int_{\overline{\Omega}\times\overline{\Omega}} c_{\varphi+\varepsilon_0}(x,y)d\gamma(x,y) + \varepsilon.$$

Letting $\varepsilon_0$ tend to 0, since $i_{Q_\varepsilon}$ is a finite measure and $c_{\varphi+\varepsilon_0}$ uniformly converges to $c_\varphi = \overline{c}_\varphi$, then

$$\int_{\overline{\Omega}} \varphi(x) di_{Q_\varepsilon}(x) \leq \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\varphi(x,y)d\gamma(x,y) + \varepsilon.$$

Taking the inf over the set $\mathcal{Q}_H(\mu,\nu)$ on the left-hand side and using the fact that $\gamma$ and $\varepsilon$ are abitrary and (3.41) we get

$$\inf_{Q\in\mathcal{Q}_H(\mu,\nu)} \int_{\overline{\Omega}} \varphi(x) di_Q(x) = \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\varphi(x,y)d\gamma(x,y), \quad \forall\varphi\in C(\overline{\Omega}), \varphi\geq 0. \tag{3.43}$$

Now, let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of even mollifiers for the group structure, see [59]. Lemma 3.3.10 implies that, for any $n\in\mathbb{N}\setminus\{0\}$, the regularized problem

$$\inf_{Q\in\mathcal{Q}_H(\mu,\nu)} \int_{\mathbb{H}^n} G(\rho_n * i_Q(x))dx \tag{3.44}$$

admits a solution $\overline{Q}_n$.

Define $j_n := \rho_n * i_{\overline{Q}_n}$, $\varphi_n := G'(j_n)$, $\eta_n := \rho_n * \varphi_n$. From the Fubini's Theorem, changes of variables formula and the fact that $\rho_n$ is an even function, we get that

$$\int_{\mathbb{H}^n} G'(\rho_n * i_{\overline{Q}_n}(x))(\rho_n * i_{\overline{Q}_n}(x))dx = \int_{\mathbb{H}^n} \varphi_n(x)j_n(x)dx = \int_{\mathbb{H}^n} \eta_n(x)di_{\overline{Q}_n}(x).$$

Moreover, Lemma (3.3.10) implies that

$$\int_{\mathbb{H}^n} \eta_n(x)di_{\overline{Q}_n}(x) = \inf_{Q\in\mathcal{Q}_H(\mu,\nu)} \int_{\mathbb{H}^n} \eta_n(x)di_Q(x). \tag{3.45}$$

Combining (3.43) and (3.45), we then get:

$$\int_{\mathbb{H}^n} \eta_n(x)di_{\overline{Q}_n}(x) = \int_{\mathbb{H}^n} \varphi_n(x)j_n(x)dx = \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\eta_n}(x,y)d\gamma(x,y). \tag{3.46}$$

From the minimality of $\overline{Q}_n$, the convexity of $G$, the Fubini's Theorem and the fact that $G(0) = 0$, it follows that

$$\int_{\mathbb{H}^n} G(j_n(x))dx \leq \int_{\mathbb{H}^n} G(\rho_n * i_{\overline{Q}}(x))dx \leq \int_{\mathbb{H}^n} \rho_n * G(i_{\overline{Q}})(x)dx < +\infty \quad (3.47)$$

and hence that $j_n$ is bounded in $L^p$. Passing to subsequences, we may assume that

$$j_n \rightharpoonup j \text{ in } L^p,\ \varphi_n \rightharpoonup \varphi \text{ in } L^q,\ \eta_n \rightharpoonup \varphi \text{ in } L^q. \quad (3.48)$$

Moreover, as in the proof of Lemma 3.3.10, we can get a uniform bound on $(i_{\overline{Q}_n}(\overline{\Omega}))_{n\in\mathbb{N}}$ because the total mass of $i_{\overline{Q}_n}$ is the same of $j_n$ and $j_n$ is bounded in $L^p$, and hence in $L^1$. Hence, we may assume that

$$\overline{Q}_n \rightharpoonup Q \text{ in } \mathcal{M}_+(C([0,1],\overline{\Omega})), \quad i_{\overline{Q}_n} \rightharpoonup i \text{ in } \mathcal{M}_+(\overline{\Omega}). \quad (3.49)$$

It follows that $i = j\mathcal{L}^{2n+1}$ and Lemma 3.2.2 implies $j\mathcal{L}^{2n+1} \geq i_Q$. Using the monotonicity of $G$, the weak lower semicontinuity of convex functions and (3.47) we get

$$\int_\Omega G(i_Q(x))dx \leq \int_\Omega G(j(x))dx \leq \liminf_{n\to+\infty} \int_{\mathbb{H}^n} G(j_n(x))dx \leq \int_\Omega G(i_{\overline{Q}}(x))dx. \quad (3.50)$$

Since $\overline{Q}$ is optimal and the strictly convexity of $G$ implies the uniqueness of minimum, it follows that

$$i_{\overline{Q}} = i_Q = j \in L^p(\Omega) \text{ and } \liminf_{n\to+\infty} \int_{\mathbb{H}^n} G(j_n(x))dx = \int_{\mathbb{H}^n} G(i_{\overline{Q}}(x))dx. \quad (3.51)$$

From (3.51), from $j_n \rightharpoonup j$ in $L^p(\Omega)$ and the fact that $\overline{\varphi} = G'(i_{\overline{Q}}) \in L^q(\Omega)$ it follows that, up to some subsequence,

$$G(j_n) - G(i_{\overline{Q}}) - \overline{\varphi}(j_n - i_{\overline{Q}}) \to 0 \text{ a.e. and in } L^1$$

and sine $G$ is strictly convex, we get that $j_n$ converges a.e. to $i_{\overline{Q}}$. This implies that $\varphi_n$ converges a.e. to $\overline{\varphi} = G'(i_{\overline{Q}})$ and that $\varphi = \overline{\varphi}$. It follows from Fatou's

Lemma and (3.46), that:

$$\begin{aligned}\int_\Omega \overline{\varphi}(x) i_{\overline{Q}}(x)dx &= \int_{\mathbb{H}^n} G'(i_{\overline{Q}}(x)) i_{\overline{Q}}(x)dx \\ &\leq \liminf_{n\to+\infty} \int_{\mathbb{H}^n} \varphi_n(x) j_n(x)dx \\ &= \liminf_{n\to+\infty} \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\eta_n}(x,y)d\gamma(x,y).\end{aligned}$$

Since $\eta_n \rightharpoonup \overline{\varphi}$ in $L^q(\Omega)$, using Lemma 3.3.4 we get that $c_{\eta_n}$ uniformly converges to $c \leq \overline{c}_{\overline{\varphi}}$. On sets of finite measure the uniform convergence implies $L^1$ convergence, hence

$$\int_\Omega \overline{\varphi}(x) i_{\overline{Q}}(x)dx \leq \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\varphi}}(x,y)d\gamma(x,y).$$

The thesis follows from the last inequality together with (3.34) and (3.40). □

**Theorem 3.3.12.** *Let $q > N$ and $G$ strictly convex. Then $\overline{Q} \in \mathcal{Q}^p_H(\mu,\nu)$ solves* (3.33) *if, and only if, $\overline{\gamma} := (e_0,e_1)_\# \overline{Q} \in \Pi(\mu,\nu)$ solves the Monge-Kantorovich problem*

$$\inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\varphi}}(x,y)d\gamma(x,y), \tag{3.52}$$

*with $\overline{\varphi} := G'(i_{\overline{Q}}) \in L^q(\Omega)$, and*

$$\overline{Q}\left(\{\sigma \in H : L_{\overline{\varphi}}(\sigma) = \overline{c}_{\overline{\varphi}}(\sigma(0),\sigma(1))\}\right) = 1. \tag{3.53}$$

*Proof.* Let $\overline{Q} \in \mathcal{Q}^p_H(\mu,\nu)$ that solves (3.33). Let $\overline{\varphi} := G'(i_{\overline{Q}})$ and $\overline{\gamma} := (e_0,e_1)_\# \overline{Q} \in \Pi(\mu,\nu)$, then from the definition of push-forward measure, Lemma 3.3.7 and Proposition 3.3.11 it follows that

$$\begin{aligned}\int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\varphi}}(x,y)d\overline{\gamma}(x,y) &= \int_{C([0,1],\overline{\Omega})} \overline{c}_{\overline{\varphi}}(\sigma(0),\sigma(1))d\overline{Q}(\sigma) \\ &\leq \int_{C([0,1],\overline{\Omega})} L_{\overline{\varphi}}(\sigma)d\overline{Q}(\sigma) = \int_\Omega \overline{\varphi}(x) i_{\overline{Q}}(x)dx \\ &= \inf_{\gamma\in\Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\varphi}}(x,y)d\gamma(x,y)\end{aligned}$$

which proves that $\overline{\gamma}$ solves (3.52). It also follows that all the above inequalities are equalities and then (3.53) follows from (3.28).

Conversely, let $\overline{Q} \in \mathcal{Q}^p_H(\mu,\nu)$ be a horizontal traffic plan satisfying the conditions (3.52) and (3.53). First of all, (3.27) and (3.53) imply

$$\int_\Omega \overline{\varphi}(x) i_{\overline{Q}}(x)dx = \int_{C([0,1],\overline{\Omega})} L_{\overline{\varphi}}(\sigma)d\overline{Q}(\sigma) = \int_{C([0,1],\overline{\Omega})} \overline{c}_{\overline{\varphi}}(\sigma(0),\sigma(1))d\overline{Q}(\sigma) =$$
$$= \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\varphi}}(x,y)d\overline{\gamma}(x,y).$$

On the other hand, for any $Q \in \mathcal{Q}^p_H(\mu,\nu)$, if $\gamma := (e_0,e_1)_\# Q \in \Pi(\mu,\nu)$ it holds that

$$\int_\Omega \overline{\varphi}(x) i_Q(x)dx = \int_{C([0,1],\overline{\Omega})} L_{\overline{\varphi}}(\sigma)dQ(\sigma) \geq \int_{C([0,1],\overline{\Omega})} \overline{c}_{\overline{\varphi}}(\sigma(0),\sigma(1))dQ(\sigma) =$$
$$= \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\varphi}}(x,y)d\gamma(x,y),$$

and hence

$$\int_\Omega \overline{\varphi}(x) i_{\overline{Q}}(x)dx \leq \int_\Omega \overline{\varphi}(x) i_Q(x)dx, \quad \forall Q \in \mathcal{Q}^p_H(\mu,\nu),$$

since $\overline{\gamma}$ solves (3.52). Then Proposition 3.3.3 implies that $\overline{Q}$ solves (3.33). □

*Remark* 14. All the arguments in this Chapter still work if the function $G : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+$ also depends on $x \in \overline{\Omega}$. It is enough to replace the convexity hypotheses on $G$ with the convexity for the function $i \mapsto G(x,i)$ for any $x \in \overline{\Omega}$; moreover, one has to suppose that

1. $G(x,0) = 0$, for any $x \in \overline{\Omega}$;

2. there exist a function $h \in L^1(\Omega)$ and two constants $a, b > 0$ such that

$$ai^p \leq G(x,i) \leq bi^p + h(x),$$

for any $(x,i) \in \overline{\Omega} \times \mathbb{R}_+$.

### 3.3.5 Existence of Wardrop equilibria

The aim of this subsection is to relate the previous results with the existence of equilibrium configurations for the congested optimal transport problem in $\mathbb{H}^n$. These equilibria are known as *Wardrop equilibria*.

Let us consider a congestion function

$$g : \mathbb{R}_+ \to \mathbb{R}_+$$

as in (3.9). Let us suppose that $g$ is continuous, strictly increasing and there exist non-negative constants $a$ and $b$ such that

$$ai^{p-1} \leq g(i) \leq bi^{p-1} + 1$$

for all $i \in \mathbb{R}_+$ and some $p \in \left(1, \frac{N}{N-1}\right)$.

Recall that, given $Q \in \mathcal{Q}^p_H(\mu, \nu)$, we denote by $\varphi_Q := g \circ i_Q \in L^q(\Omega)$ and by $\overline{c}_{\varphi_Q}$ the function defined as (3.21).

**Definition 3.3.** A *Wardrop equilibrium* is a horizontal traffic plan $Q \in \mathcal{Q}^p_H(\mu, \nu)$ such that

1. $Q\big(\left\{\sigma \in H : L_{\varphi_Q}(\sigma) = \overline{c}_{\varphi_Q}(\sigma(0), \sigma(1))\right\}\big) = 1$;
2. $\gamma_Q := (e_0, e_1)_\# Q \in \Pi(\mu, \nu)$ solves the Monge-Kantorovich problem
$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\varphi_Q}(x, y) d\gamma(x, y).$$

The first condition in Definition 3.3 is an equilibrium condition of Wardrop type: given such a configuration $Q$ no agent is interested in changing its path, since everyone is paying the least. This equilibrium condition can be seen as a particular case of *Nash equilibrium*, in a game where the players are the agents and their goal is to minimize travelling costs. On the other hand, the second condition means that the transport plan $\gamma_Q$ is optimal for a Monge-Kantorovic problem associated with a cost depending on $Q$ itself. Hence, Definition 3.3 can be viewed as a generalization of the notion of Wardrop equilibrium to the case where no transport plan is given a priori.

Theorems 3.3.8 and 3.3.12 guarantee the existence of Wardrop equilibria.

**Theorem 3.3.13.** *If $\mathcal{Q}^p_H(\mu, \nu) \neq \emptyset$, there exists a Wardrop equilibrium.*

*Moreover $\overline{Q} \in \mathcal{Q}^p_H(\mu, \nu)$ is a Wardrop equilibrium if, and only if, it solves*

$$\inf_{Q \in \mathcal{Q}^p_H(\mu,\nu)} \int_\Omega G(i_Q(x)) dx, \tag{3.54}$$

*where* $G : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable function such that* $G'(i) := g(i),\ \forall i \in \mathbb{R}_+$ *and* $G(0) = 0$.

*Remark* 15. If not only the marginal $\mu$ and $\nu$ are given, but also the transport plan $\overline{\gamma} \in \Pi(\mu, \nu)$ is, equilibria consist in the set of $Q$'s that satisfies the first condition of definition 3.3. As before, if

$$\mathcal{Q}^p_H(\gamma) := \Bigg\{ Q \in \mathcal{P}(C([0,1], \overline{\Omega})) : Q(H) = 1,\ (e_0, e_1)_\sharp Q = \gamma \\ \int_{C([0,1],\overline{\Omega})} l_{SR}(\sigma) dQ(\sigma) < +\infty \text{ and } i_Q \in L^q(\Omega) \Bigg\} \neq \emptyset$$

then all the previous arguments can be adapted to this new situation and one can get the existence of equilibria; moreover, $\overline{Q}$ is a Wardrop equilibrium if, and only if, it solves

$$\inf_{Q \in \mathcal{Q}^p_H(\gamma)} \int_\Omega G(i_Q(x)) dx.$$

# Chapter 4

# Equivalent formulations of congested optimal transport problem in $\mathbb{H}^n$

In this chapter we want to model the transport activities of Chapter 3 through horizontal vector fields.

In order to do this we consider a regular bounded domain $\Omega$ in $\mathbb{H}^n$, with $C^{1,1}$ boundary, and $\mu, \nu \in \mathcal{P}(\overline{\Omega})$ two probability measures over the closure of $\Omega$. Moreover, we consider a total cost function

$$\mathcal{G} : \mathbb{R}^{2n} \to \mathbb{R}_+,$$

of the form

$$\mathcal{G} := G \circ |\cdot|_H, \tag{4.1}$$

where the function $G$ is as in Chapter 3.

Given a horizontal vector field $\mathbf{w} : \overline{\Omega} \to H\overline{\Omega}$, the quantity $\mathcal{G}(\mathbf{w}(x))$ gives the total cost to let an amount of mass $|\mathbf{w}(x)|_H$ to transit with direction $\frac{\mathbf{w}(x)}{|\mathbf{w}(x)|_H}$ through the point $x$. In this way the transportation process associated with $\mathbf{w}$ has total cost

$$\int_\Omega \mathcal{G}(\mathbf{w}(x))dx. \tag{4.2}$$

Moreover the vector field $\mathbf{w}$ turns out to satisfy a divergence-type constraint

$$\operatorname{div}_H \mathbf{w} = \mu - \nu, \tag{4.3}$$

for a suitable notion of divergence.

Since we are dealing with a cost function $\mathcal{G}$ with $p$-growth, we will be interested in minimizing (4.2) over the set of $p$-summable horizontal vector fields, subjected to the constraint (4.3)

$$\inf_{\mathbf{w}\in L^p(\Omega,H\Omega)} \left\{ \int_\Omega \mathcal{G}(\mathbf{w}(x))dx : \operatorname{div}_H \mathbf{w} = \mu - \nu \right\}. \tag{$\mathcal{B}$}$$

First, we show under which assumptions this problem admits solutions. Second, under the same assumptions we will show that ($\mathcal{B}$) admits the following dual formulation

$$\max_{\varphi\in HW^{1,q}(\Omega)} \left\{ -\int_\Omega \varphi d(\mu-\nu) - \int_\Omega \mathcal{G}^*(\nabla_H \varphi(x))dx \right\}, \tag{$\mathcal{D}$}$$

where $\mathcal{G}^*$ is the Legendre transform of $\mathcal{G}$. If in addition $G$ is strictly convex, then $\mathcal{G}^*$ is $C^1(\mathbb{R}^{2n})$ and the unique solution $\mathbf{w}$ to the problem ($\mathcal{B}$) is of the form

$$\mathbf{w} = D\mathcal{G}^*(\nabla_H \varphi),$$

where $D$ denotes the Euclidean gradient and $\varphi \in HW^{1,q}(\Omega)$ is a solution to ($\mathcal{D}$). Moreover, we will show that the problem ($\mathcal{B}$) is equivalent to the problem

$$\inf_{Q\in\mathcal{Q}^p_H(\mu,\nu)} \int_\Omega G(i_Q(x))dx, \tag{$\mathcal{W}$}$$

defined in (3.33).

The main theorem of this Chapter is the following one.

**Theorem 4.0.1.** *If*

$$\mathcal{Q}^p_H(\mu,\nu) \neq \emptyset,$$

*then* ($\mathcal{W}$), ($\mathcal{B}$) *and* ($\mathcal{D}$) *admit solutions and*

$$(\mathcal{W}) = (\mathcal{B}) = (\mathcal{D}).$$

*Moreover, any solution to* $(\mathcal{W})$ *corresponds to a solution to* $(\mathcal{B})$*; vice versa, one can find a solution to* $(\mathcal{W})$ *starting from a solution to* $(\mathcal{B})$*; if* $(\mathbf{w}, \varphi)$ *is a pair of optimizers for* $(\mathcal{B})$ *and* $(\mathcal{D})$*, they are linked by the relation*

$$\nabla_H \varphi = D\mathcal{G}(\mathbf{w}).$$

In the limit case $p = 1$, $(\mathcal{B})$ reads as

$$\inf_{\mathbf{w} \in L^1(\Omega, H\Omega)} \left\{ \int_\Omega |\mathbf{w}(x)| dx : \operatorname{div}_H \mathbf{w} = \mu - \nu \right\}, \tag{4.4}$$

which looks like the problem (BP) in Subsection 2.3.3.

As observed in [87, Subsection 4.2.2] this problem is not well-posed a priori: due to the non reflexivity of $L^1$, there may not exist any $L^1$ horizontal vector field minimizing the $L^1$ norm under the divergence constraint. This is why in Subsection 2.3.3 we stated the problem (4.4) in the space of the compactly supported vector measures whose horizontal divergence is the signed Radon measure $\mu - \nu$. We proved in Theorem 2.3.2 that its dual reformulation is

$$\sup_{u \in HW^{1,\infty}} \left\{ \int_\Omega u d(\mu - \nu) : \|\nabla_H u\|_\infty \leq 1 \right\}$$

and its Lagrangian reformulation is

$$\inf_{Q \in \mathcal{Q}_H(\mu,\nu)} \int_{C([0,1],\overline{\Omega})} l_H(\sigma) dQ(\sigma).$$

The equivalence between these three problems is the core of Chapter 2: we studied the equivalence on the whole $\mathbb{H}^n$, because otherwise the geodesically convexity hypothesis on $\Omega$ would be necessary: as we already observed in Remark 9, non-trivial geodesically convex subsets of $\mathbb{H}^n$ do not exist.

## 4.1 Vector horizontal traffic intensities

Let $\Omega$ be a regular bounded domain in $\mathbb{H}^n$, with $C^{1,1}$ boundary, and $\mu, \nu \in \mathcal{P}(\overline{\Omega})$.

As for transport plans $\gamma \in \Pi(\mu,\nu)$, one can associate with any horizontal traffic plan $Q \in \mathcal{Q}_H(\mu,\nu)$ both a scalar and a vector measures. The scalar measure is the traffic intensity $i_Q \in \mathcal{M}_+\left(\overline{\Omega}\right)$ we introduced in Definition 3.2. As for the vector measure we can give the following definition.

**Definition 4.1** (Vector horizontal traffic intensity)**.** Let $Q \in \mathcal{Q}_H(\mu,\nu)$ be a horizontal traffic plan admissible between $\mu$ and $\nu$. One can associate with $Q$ the finite vector Radon measure $\mathbf{w}_Q \in \mathcal{M}(\overline{\Omega}, H\overline{\Omega})$ defined as

$$\int_{\overline{\Omega}} \phi(x)\cdot d\mathbf{w}_Q = \int_{C([0,1],\overline{\Omega})} \left(\int_0^1 \langle \phi(\sigma(t)), \dot\sigma(t)\rangle_H \, dt\right) dQ(\sigma),$$

for any continuous horizontal vector field $\phi \in C(\overline{\Omega}, H\overline{\Omega})$. We will call this measure *vector horizontal traffic intensity induced by Q*.

Even in this case, the *variation measure* $|\mathbf{w}_Q|$ satisfies

$$|\mathbf{w}_Q| \le i_Q \tag{4.5}$$

for any $Q \in Q_H(\mu,\nu)$. Indeed, given $Q \in \mathcal{Q}_H(\mu,\nu)$, $A \subseteq \overline{\Omega}$ Borel set and $\phi \in C_c(\overline{\Omega}, H\overline{\Omega})$, such that supp $\phi \subseteq A$, and $\|\phi\|_\infty \le 1$, it follows

$$\int_{\overline{\Omega}} \phi\cdot d\mathbf{w}_Q \le \left|\int_{\overline{\Omega}} \phi\cdot d\mathbf{w}_Q\right| \le \int_{\overline{\Omega}} |\phi|_H di_Q \le i_Q(A).$$

Hence, taking the sup among all the admissible $\phi$ on the left hand side and having in mind (1.21), we get the desired result. The finitness of the vector measure $\mathbf{w}_Q$ immediately follows from the definition of horizontal traffic plan: indeed it holds

$$\|\mathbf{w}_Q\|_{\mathcal{M}\left(\overline{\Omega}, H\overline{\Omega}\right)} \le \int_{C([0,1],\overline{\Omega})} l_{SR}(\sigma) dQ(\sigma) < +\infty,$$

for any $Q \in \mathcal{Q}_H(\mu,\nu)$.

*Remark* 16. In general the previous inequality is not an equality because curves of $Q$ may produce cancellations: this is due to the fact that $\mathbf{w}_Q$ takes into account the orientation of the curves, while $i_Q$ does not. Following [32, Example 4.1], we can consider the two measures

$$\mu := \delta_x \text{ and } \nu := \delta_y, \text{ for some } x \in \mathbb{H}^n \text{ and some } y \in \mathbb{H}^n \setminus L_x$$

and a bounded domain $\Omega$, with regular $C^{1,1}$ boundary, such that $x, y \in \Omega$ and $\sigma([0,1]) \subset \Omega$. Then, the horizontal traffic plan

$$Q := \frac{1}{2}\delta_\sigma + \frac{1}{2}\delta_{\tilde{\sigma}} \in \mathcal{Q}_H(\mu, \nu),$$

where $\sigma : [0,1] \to \mathbb{H}^n$ is the geodesic between $x$ and $y$, as in Theorem 1.11, and $\tilde{\sigma}(t) := \sigma(-t)$, satisfies

$$i_Q = \mathcal{H}^1 \llcorner_{\sigma([0,1])},$$

while

$$\mathbf{w}_Q \equiv 0.$$

As for the vector horizontal transport density, see Definition 2.3, one can recover the signed Radon measure $\mu - \nu$, by testing the vector measure $\mathbf{w}_Q$ against the horizontal gradient $\phi = \nabla_H \varphi$ of some $\varphi \in C^\infty(\overline{\Omega})$. More explicitly, let $\varphi \in C^\infty(\overline{\Omega})$

$$\begin{aligned}
\int_{\overline{\Omega}} \nabla_H \varphi(x) \cdot d\mathbf{w}_Q &= \int_{C([0,1],\overline{\Omega})} \left( \int_0^1 \langle \nabla_H \varphi(\sigma(t)), \dot{\sigma}(t) \rangle_H \, dt \right) dQ(\sigma) \\
&= \int_{C([0,1],\overline{\Omega})} \left( \int_0^1 \frac{d}{dt} \left[ \varphi(\sigma(t)) \right] dt \right) dQ(\sigma) \\
&= \int_{C([0,1],\overline{\Omega})} \left[ \varphi(\sigma(1)) - \varphi(\sigma(0)) \right] dQ(\sigma) \\
&= \int_{\overline{\Omega}} \varphi d \left( (e_1)_\# Q - (e_0)_\# Q \right) = -\int_{\overline{\Omega}} \varphi d(\mu - \nu). \qquad (4.6)
\end{aligned}$$

Hence,

$$\int_{\overline{\Omega}} \nabla_H \varphi(x) \cdot d\mathbf{w}_Q = -\int_{\overline{\Omega}} \varphi d(\mu - \nu), \quad \forall \varphi \in C^\infty(\overline{\Omega}).$$

According to Subsection 2.3.3, this means that for any horizontal traffic plan $Q \in \mathcal{Q}_H(\mu, \nu)$, the divergence of the vector measure $\mathbf{w}_Q$ is the signed Radon measure $\mu - \nu$

$$\mathrm{div}_H \mathbf{w}_Q = \mu - \nu.$$

Following [88], if in addition $\mathbf{w} \in L^p(\Omega, H\Omega)$ for some $p \in (1, \infty)$, we can define the distributional horizontal divergence as a functional

$$\mathrm{div}_H \mathbf{w} : HW^{1,q}(\Omega) \to \mathbb{R}$$

defined by the rule

$$\langle \mathrm{div}_H \mathbf{w}, \varphi \rangle = -\int_\Omega \langle \mathbf{w}, \nabla_H \varphi \rangle_H dx, \quad \forall \varphi \in HW^{1,q}(\Omega),$$

where $q = \frac{p}{p-1}$. This is an element of $(HW^{1,q}(\Omega))'$ since

$$|\langle \mathrm{div}_H \mathbf{w}, \varphi \rangle| \le \|\mathbf{w}\|_{L^p(\Omega, H\Omega)} \|\varphi\|_{HW^{1,q}(\Omega)}, \quad \forall \varphi \in HW^{1,q}(\Omega).$$

Moreover $\|\mathrm{div}_H \mathbf{w}\|_{(HW^{1,q}(\Omega))'} \le \|\mathbf{w}\|_{L^p(\Omega, H\Omega)}$.

**Lemma 4.1.1.** *Let* $p \in (1, \infty)$. *If*

$$\mathcal{Q}_H^p(\mu, \nu) := \{Q \in \mathcal{Q}_H(\mu, \nu) : i_Q \in L^p(\Omega)\} \neq \emptyset,$$

*then*

$$\{\mathbf{w} \in L^p(\Omega, H\Omega) : \mathrm{div}_H \mathbf{w} = \mu - \nu\} \neq \emptyset. \tag{4.7}$$

*Moreover* $\mu - \nu \in (HW^{1,q}(\Omega))'$.

*Proof.* Let $Q \in \mathcal{Q}_H^p(\mu, \nu)$. Then, (4.5) and (4.6) imply that $\mathbf{w}_Q \in L^p(\Omega, H\Omega)$ and $\mathrm{div}_H \mathbf{w}_Q = \mu - \nu$. Moreover

$$\begin{aligned}
+\infty > \|\mathbf{w}_Q\|_{L^p(\Omega,H\Omega)} \\
&= \sup\left\{\left|\int_\Omega \langle \phi, \mathbf{w}_Q \rangle_H \, dx\right| : \phi \in L^q(\Omega, H\Omega), \|\phi\|_{L^q(\Omega,H\Omega)} \le 1\right\} \\
&\ge \sup\left\{\left|\int_\Omega \langle \nabla_H \varphi, \mathbf{w}_Q \rangle_H \, dx\right| : \varphi \in HW^{1,q}(\Omega), \|\varphi\|_{HW^{1,q}(\Omega)} \le 1\right\} \\
&= \sup\left\{\left|\int_\Omega \varphi \, d(\mu - \nu)\right| : \varphi \in HW^{1,q}(\Omega), \|\varphi\|_{HW^{1,q}(\Omega)} \le 1\right\} \\
&= \|\mu - \nu\|_{(HW^{1,q}(\Omega))'}.
\end{aligned}$$

□

*Remark* 17. Let us remark that the non emptiness assumption on $\mathcal{Q}_H^p(\mu, \nu)$ fails for $p \ge \frac{N}{N-1}$ when $\mu - \nu$ is a discrete measure: indeed, in this case

$$\mu - \nu \in \left(HW^{1,q}(\Omega)\right)' \Leftrightarrow q > N \Leftrightarrow p < \frac{N}{N-1}.$$

See [61, Theorem 1.11].

## 4.2 The Beckmann-type problem in $(HW^{1,q})'_\diamond(\Omega)$

The aim of this section is to study the Beckmann-type problem ($\mathcal{B}$) in its natural functional analytic setting, following [30] and [88]: we show that ($\mathcal{B}$) is well-posed if and only if the right-hand side of (4.3) belongs to the dual of some Sobolev space, whose elements are not measures in general.

In the rest of the section we will deal with a particular closed subspace $(HW^{1,q})'_\diamond(\Omega)$ of the space $(HW^{1,q}(\Omega))'$,

$$(HW^{1,q})'_\diamond(\Omega) := \left\{ f \in (HW^{1,q}(\Omega))' : \langle f, 1 \rangle = 0 \right\}.$$

Let $f \in (HW^{1,q})'_\diamond(\Omega)$ then its norm

$$\|f\|_{(HW^{1,q})'_\diamond(\Omega)} := \|f\|_{(HW^{1,q}(\Omega))'}.$$

If $\mathbf{w} \in L^p(\Omega, H\Omega)$, its distributional horizontal divergence $\mathrm{div}_H \mathbf{w} \in (HW^{1,q}(\Omega))'$. Moreover it is trivial to prove that $\langle \mathrm{div}_H \mathbf{w}, 1 \rangle = 0$, hence

$$\mathrm{div}_H \mathbf{w} \in (HW^{1,q})'_\diamond(\Omega).$$

### 4.2.1 Characterization of the space $(HW^{1,q})'_\diamond(\Omega)$

In the spirit of [88, Section 2], if $\Omega \subset \mathbb{H}^n$ is a regular bounded domain with $C^{1,1}$ boundary, then (1.20) holds on $\Omega$ and one can characterize the space $(HW^{1,q})'_\diamond(\Omega)$.

**Proposition 4.2.1.** *Let $\Omega \subset \mathbb{H}^n$ be a regular bounded domain with $C^{1,1}$ boundary. Given $f \in (HW^{1,q})'_\diamond(\Omega)$, then there exists a vector field $\mathbf{w} \in L^p(\Omega, H\Omega)$ such that*

$$\mathrm{div}_H \mathbf{w} = f.$$

*Moreover* $\|\mathbf{w}\|_{L^p(\Omega, H\Omega)} \le c(n, q, \Omega) \|f\|_{(HW^{1,q}(\Omega))'}$.

*Proof.* Consider the following minimization problem

$$\min_{\varphi \in HW^{1,q}(\Omega)} \mathcal{F}(\varphi),$$

where

$$\varphi \mapsto \mathcal{F}(\varphi) := \frac{1}{q}\int_\Omega |\nabla_H \varphi|_H^q dx + \langle f, \varphi\rangle.$$

This problem admits at least a solution. Indeed, we restrict the minimization to the subspace

$$HW^{1,q}_\diamond(\Omega) := \left\{\varphi \in HW^{1,q}(\Omega) : \int_\Omega \varphi(x)dx = 0\right\}, \tag{4.8}$$

which is a convex subset and closed w.r.t. weak topology of $HW^{1,q}(\Omega)$. The functional $\mathcal{F}$ is lower semicontinuous w.r.t. the weak topology of $HW^{1,q}(\Omega)$ and it is coercive on $HW^{1,q}_\diamond(\Omega)$: indeed, if $\varphi \in HW^{1,q}_\diamond(\Omega)$, then from (1.20) it follows the existence of a constant $c = c(n, q, \Omega) > 0$ such that

$$\begin{aligned}|\mathcal{F}(\varphi)| &\geq \frac{1}{q}\|\nabla_H\varphi\|^q_{L^q(\Omega)} - (c+1)\|f\|_{(HW^{1,q}(\Omega))'}\|\nabla_H\varphi\|_{L^q(\Omega)} \\ &= \|\nabla_H\varphi\|_{L^q(\Omega)}\left(\frac{1}{q}\|\nabla_H\varphi\|^{q-1}_{L^q(\Omega)} - (c+1)\|f\|_{(HW^{1,q}(\Omega))'}\right).\end{aligned}$$

Hence, the existence of minimizers follows.

Computing Euler Lagrange equations, a solution $\varphi$ of the problem above turns out to satisfy

$$-\int_\Omega \langle |\nabla_H\varphi|^{q-2}\nabla_H\varphi, \nabla_H\psi\rangle_H dx = \langle f, \psi\rangle, \quad \forall \psi \in HW^{1,q}.$$

This means that there exists $\mathbf{w} = |\nabla_H\varphi|^{q-2}\nabla_H\varphi \in L^p(\Omega, H\Omega)$ such that $\mathrm{div}_H\mathbf{w} = f$. Moreover, testing $f$ against $\varphi$ we get

$$\begin{aligned}\|\mathbf{w}\|^p_{L^p(\Omega,H\Omega)} &= \int_\Omega |\mathbf{w}|^p_H dx = \int_\Omega |\nabla_H\varphi|^q_H dx = -\langle f, \varphi\rangle \\ &\leq \|f\|_{(HW^{1,q}(\Omega))'}\|\varphi\|_{HW^{1,q}(\Omega)} \leq c\|f\|_{(HW^{1,q}(\Omega))'}\|\nabla_H\varphi\|_{L^q(\Omega,H\Omega)} \\ &= c\|f\|_{(HW^{1,q}(\Omega))'}\|\mathbf{w}\|^{p-1}_{L^p(\Omega,H\Omega)},\end{aligned}$$

as desired. □

### 4.2.2 The Beckmann-type problem

Following [88] we show how the Beckmann-type problem ($\mathcal{B}$) reads if the right hand side of (4.3) belongs to $(HW^{1,q}(\Omega))'$.

Let

$$\mathcal{G} : \mathbb{R}^{2n} \to \mathbb{R}_+$$

be a convex function such that

$$\mathcal{G}(0) = 0, \tag{4.9}$$

and it has $p$-growth

$$a|w|^p \le \mathcal{G}(w) \le b|w|^p + 1, \tag{4.10}$$

for some $p \in (1, \infty)$ and some positive constants $a, b$.

Given a horizontal vector field $\mathbf{w} : \overline{\Omega} \to H\overline{\Omega}$, we identify the horizontal vector field $\mathbf{w} = \sum_{i=1}^{2n} \mathbf{w}_i X_i$ with its canonical coordinates w.r.t. the moving frame $\{X_1, \dots, X_{2n}\}$

$$\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_{2n}) : \Omega \to \mathbb{R}^{2n}.$$

We are interested in the Beckmann-type problem

$$\inf_{\mathbf{w} \in L^p(\Omega, H\Omega)} \left\{ \int_\Omega \mathcal{G}(\mathbf{w}(x))dx : \operatorname{div}_H \mathbf{w} = f \right\}, \tag{$\mathcal{B}'$}$$

where for the moment $f$ is any element of $(HW^{1,q}(\Omega))'$, with $q = \frac{p}{p-1}$. One can prove that the previous problem is well-posed if and only if $f \in (HW^{1,q})'_\diamond(\Omega)$.

**Theorem 4.2.2.** *The problem* $(\mathcal{B}')$ *admits a minimizer, with finite value, if and only if* $f \in (HW^{1,q})'_\diamond(\Omega)$.

*Proof.* Let $f \in (HW^{1,q})'_\diamond(\Omega)$, then Proposition 4.2.1 implies that there exists at least one vector field $\mathbf{w} \in L^p(\Omega, H\Omega)$ such that $\operatorname{div}_H \mathbf{w} = f$: hence (4.10) implies that $(\mathcal{B}') < +\infty$. Let $(\mathbf{w}_n)_{n\in\mathbb{N}} \subseteq L^p(\Omega, H\Omega)$ be a minimizing sequence. From (4.10) it follows that this sequence is bounded in $L^p(\Omega, H\Omega)$: hence, up to a subsequence, it is weakly convergent to some $\tilde{\mathbf{w}} \in L^p(\Omega, H\Omega)$. This vector field is still admissible: indeed, by the weak convergence it follows that

$$-\int_\Omega \langle \tilde{\mathbf{w}}, \nabla_H \varphi \rangle_H \, dx = -\lim_{n\to\infty} \int_\Omega \langle \mathbf{w}_n, \nabla_H \varphi \rangle_H \, dx = \langle f, \varphi \rangle, \quad \forall \varphi \in HW^{1,q}(\Omega).$$

Since $\mathcal{G}$ is convex, the integral functional is convex, as well, and lower semicontinuous; this implies the lower semicontinuity of the integral functional w.r.t. the weak convergence in $L^p(\Omega, H\Omega)$, hence

$$\int_\Omega \mathcal{G}(\tilde{\mathbf{w}})dx \leq \liminf_{n\to+\infty} \int_\Omega \mathcal{G}(\mathbf{w}_n)dx = (\mathcal{B}') < +\infty.$$

This proves that $\tilde{\mathbf{w}}$ is a minimum.

Conversely, if $f \notin (HW^{1,q})'_\diamond(\Omega)$ there are no admissible vector fields, hence the problem is not well-posed. □

Let us just remark that , if we suppose in addition that $\mathcal{G}$ is strictly convex, then $(\mathcal{B}')$ admits a unique minimizer.

**Corollary 4.2.3.** *Let $p \in (1, \infty)$. If*

$$\mathcal{Q}_H^p(\mu, \nu) \neq \emptyset,$$

*then $\mu - \nu \in \left(HW^{1,q}\right)'_\diamond(\Omega)$. Moreover $(\mathcal{B}')< +\infty$ and it admits solutions.*

*Proof.* Lemma 4.1.1 and a simple computation imply that $\mu-\nu \in \left(HW^{1,q}\right)'_\diamond(\Omega)$. The existence of solutions follows from Theorem 4.2.2. □

## 4.3 Dual formulation of the Beckmann-type problem in $\mathbb{H}^n$

The aim of this section is to prove that the Beckmann-type problem $(\mathcal{B}')$ admits the dual formulation

$$\sup_{\varphi\in HW^{1,q}(\Omega)} \left\{ - \langle f, \varphi\rangle - \int_\Omega \mathcal{G}^*(\nabla_H \varphi(x))dx \right\}, \qquad (\mathcal{D}')$$

where $f \in (HW^{1,q}(\Omega))'$ and $\mathcal{G}^*$ is the Legendre transform of the convex function $\mathcal{G}$, see Theorem 4.3.1 for its definition. For the proof of this result, we follow the strategy in [30]. We recall the following important theorem in convex analysis.

**Theorem 4.3.1** (Convex duality). *[54, Proposition 5] Let $\mathcal{F} : Y \to \mathbb{R}$ a convex and lower semicontinuous functional on a reflexive Banach space $Y$. Let $X$ be another reflexive Banach space and $A : X \to Y$ a bounded linear operator, with adjoint operator $A' : Y' \to X'$. Then we have*

$$\sup_{x\in X}\langle x', x\rangle - \mathcal{F}(Ax) = \inf_{y'\in Y'} \{\mathcal{F}^*(y') : A'y' = x'\}, \quad x' \in X', \tag{4.11}$$

*where $\mathcal{F}^* : Y' \to \mathbb{R}$ denotes the Legendre-Fenchel transform of $\mathcal{F}$.*

*Moreover, if the supremum in (4.11) is attained at some $x_0 \in X$, then infimum in (4.11) is attained at some $y'_0 \in Y'$ such that*

$$y'_0 \in \partial\mathcal{F}(Ax_0), \tag{4.12}$$

*where $\partial\mathcal{F}$ denotes the subdifferential.*

**Theorem 4.3.2.** *If $f \in (HW^{1,q})'_\diamond(\Omega)$, then the problem $(\mathcal{D}')$ admits a solution and*

$$\min_{\mathbf{w}\in L^p(\Omega,H\Omega)} \left\{\int_\Omega \mathcal{G}(\mathbf{w}(x))dx : \mathrm{div}_H\mathbf{w} = f\right\}$$
$$= \max_{\varphi\in HW^{1,q}(\Omega)} \left\{-\int_\Omega \mathcal{G}^*(\nabla_H\varphi(x))dx - \langle f, \varphi\rangle\right\}. \tag{4.13}$$

*Moreover, if $\mathbf{w}_0 \in L^p(\Omega, H\Omega)$ in an optimizer for $(\mathcal{B}')$ and $\varphi_0 \in HW^{1,q}(\Omega)$ is an optimizer for $(\mathcal{D}')$, then we have the following primal-dual optimality condition*

$$\mathbf{w}_0 \in \partial\mathcal{G}^*(\nabla_H\varphi_0) \text{ a.e. in } \Omega. \tag{4.14}$$

*Proof.* First we observe that the existence of a minimizer for the problem $(\mathcal{B}')$ follows from Theorem 4.2.2. Thanks to the Direct method in calculus of variations also $(\mathcal{D}')$ admits solutions belonging the space $HW^{1,q}_\diamond(\Omega)$ of Sobolev functions with zero mean. First of all (4.10) implies the following growth condition on $\mathcal{G}^*$ holds

$$\frac{1}{qb^q}|z|^q - 1 \le \mathcal{G}^*(z) \le \frac{1}{qa^q}|z|^q. \tag{4.15}$$

Moreover, the convexity of $\mathcal{G}^*$ implies that the functional

$$HW^{1,q}(\Omega) \ni \varphi \mapsto \tilde{\mathcal{F}}(\varphi) = \int_\Omega \mathcal{G}^*(\nabla_H \varphi(x))dx + \langle f, \varphi \rangle.$$

is lower semicontinuous with respect to the weak topology of $HW^{1,q}(\Omega)$; moreover it is coercive on $HW^{1,q}_\diamond(\Omega)$: let $\varphi \in HW^{1,q}_\diamond(\Omega)$ from (4.15) and (1.20)

$$\begin{aligned}
\tilde{\mathcal{F}}(\varphi) &\geq \frac{1}{qb^{q-1}} \|\nabla_H \varphi\|^q_{L^q(\Omega,H\Omega)} - \|f\|_{(HW^{1,q}(\Omega))'} \|\varphi\|_{HW^{1,q}(\Omega)} - \mathcal{L}^{2n+1}(\Omega) \\
&\geq \frac{1}{qb^{q-1}} \|\nabla_H \varphi\|^q_{L^q(\Omega,H\Omega)} - (c+1)\|f\|_{(HW^{1,q}(\Omega))'} \|\nabla_H \varphi\|_{L^q(\Omega,H\Omega)} - \mathcal{L}^{2n+1}(\Omega) \\
&\geq \|\nabla_H \varphi\|_{L^q(\Omega,H\Omega)} \left( \frac{1}{qb^{q-1}} \|\nabla_H \varphi\|^{q-1}_{L^q(\Omega,H\Omega)} - (c+1)\|f\|_{(HW^{1,q}(\Omega))'} \right) - \mathcal{L}^{2n+1}(\Omega).
\end{aligned}$$

where $c = c(n, q, \Omega)$ is the constant appearing in (1.20). The existence of solutions follows from the fact that minimizing $\tilde{\mathcal{F}}$ or maximizing $-\tilde{\mathcal{F}}$ is the same.

Let now $X = HW^{1,q}(\Omega)$, $Y = L^q(\Omega, H\Omega)$ and let us consider the operator

$$A : X \to Y, \quad A(\varphi) = \nabla_H \varphi, \quad \forall \varphi \in X.$$

This operator is linear; moreover it is bounded since

$$\|A(\varphi)\|_Y = \|\nabla_H \varphi\|_{L^p(\Omega,H\Omega} \leq \|\varphi\|_{HW^{1,q}(\Omega)}.$$

We denote by $\mathcal{F} : Y \to \mathbb{R}$ the functional

$$\mathcal{F}(\phi) := \int_\Omega \mathcal{G}^*(\phi(x))dx.$$

By the convexity and the lower semicontinuity of $\mathcal{G}$ it follows that $\mathcal{F}^* : L^p(\Omega, H\Omega) \to \mathbb{R}$

$$\mathcal{F}^*(\mathbf{w}) = \int_\Omega \mathcal{G}^{**}(\mathbf{w}(x))dx = \int_\Omega \mathcal{G}(\mathbf{w}(x))dx.$$

We only miss to compute $A' : Y' \to X'$. We define the operator $\Psi : L^p(\Omega, H\Omega) \to (HW^{1,q}(\Omega))'$, defined by the following role

$$\langle \Psi(\mathbf{w}), \varphi \rangle = -\int_\Omega \langle \mathbf{w}, \nabla_H \varphi \rangle_H dx, \quad \forall \varphi \in HW^{1,q}(\Omega).$$

We observe that $\Psi$ is a linear operator whose image is contained in $(HW^{1,q})'_\diamond(\Omega)$ by construction and by Lemma 4.2.1. Moreover if we take $\varphi \in HW^{1,p}(\Omega)$ and $\mathbf{w} \in L^p(\Omega, H\mathbb{R}^n)$, then

$$\langle A(\varphi), \mathbf{w}\rangle = \int_\Omega \langle \mathbf{w}, \nabla_H \varphi\rangle_H dx = \langle -\Psi(\mathbf{w}), \varphi\rangle.$$

This proves that $-\Psi = A' : L^p(\Omega, H\Omega) \to (HW^{1,q})'_\diamond(\Omega) \subset (HW^{1,q}(\Omega))'$.

The rest of the thesis follows from the primal-dual optimality condition (4.12). □

*Remark* 18. Let us just remark that, in the Euclidean setting, another proof of (4.13) is available in [88]. The latter proof still works in $\mathbb{H}^n$ and we will use it in Chapter 5 for the proof of the analogous result in the orthotropic case, see Theorem 5.1.2.

*Remark* 19. If the function $\mathcal{G}^* \in C^1(\mathbb{R}^{2n})$, then the subgradient set $\partial\mathcal{G}^*$ consists of a unique element $D\mathcal{G}^*$. If $\varphi_0 \in HW^{1,q}(\Omega)$ is a solution to $(\mathcal{D}')$, then it solves the following Euler Lagrange equation

$$\mathrm{div}_H(D\mathcal{G}^*(\nabla_H\varphi)) = f, \quad \text{in } \Omega.$$

Moreover, if $\mathbf{w}_0$ is a minimizer for $(\mathcal{B}')$ and $\varphi_0$ is a maximizer for $(\mathcal{D}')$, then

$$\mathbf{w}_0(x) = D\mathcal{G}^*(\nabla_H\varphi_0(x)) = \sum_{j=1}^{2n} D_j\mathcal{G}^*(\nabla_H\varphi_0(x))X_j, \text{ for } \mathcal{L}^{2n+1} - a.e.x \in \Omega.$$

**Corollary 4.3.3.** *Let us suppose that*

$$\mathcal{Q}^p_H(\mu,\nu) \neq \emptyset$$

*and $\mathcal{G}$ is strictly convex. Then,*

$$\min_{\mathbf{w}\in L^p(\Omega,H\Omega)} \left\{ \int_\Omega \mathcal{G}(\mathbf{w}(x))dx : \mathrm{div}_H\mathbf{w} = \mu - \nu \right\}$$
$$= \max_{\varphi\in HW^{1,q}(\Omega)} \left\{ -\int_\Omega \varphi d(\mu-\nu) - \int_\Omega \mathcal{G}^*(\nabla_H\varphi(x))dx \right\}.$$

*Moreover, if* $\mathbf{w}_0 \in L^p(\Omega, H\Omega)$ *is the unique minimizer of* $(\mathcal{B})$ *and* $\varphi_0 \in HW^{1,q}(\Omega)$ *is an optimizer for* $(\mathcal{D})$*, we have that*

$$\mathbf{w}_0 = D\mathcal{G}^*(\nabla_H \varphi) \ a.e. \ in \ \Omega,$$

*where* $\varphi_0$ *is a weak solution to*

$$\mathrm{div}_H(D\mathcal{G}^*(\nabla_H \varphi)) = \mu - \nu, \quad in \ \Omega.$$

*Proof.* The strict convexity and the superlinearity of $\mathcal{G}$ imply that $\mathcal{G}^* \in C^1(\mathbb{R}^{2n})$. The thesis follows from Remark 19. □

### 4.3.1 Beckmann potential

Following [33], a *Beckmann potential* in $\mathbb{H}^n$ is a function $\varphi_0 \in HW^{1,q}(\Omega)$ satisfying $(\mathcal{D})$. Let us denote by $d_{\varphi_0}$ the sub-Riemannian distance associated with $|\nabla_H \varphi_0|_H$: given $x, y \in \Omega$,

$$d_{\varphi_0}(x,y) = \inf \left\{ \int_0^1 |\nabla_H \varphi(\sigma(t))|_H |\dot{\sigma}(t)|_H dt : \sigma \in H, \sigma(0) = x, \sigma(1) = y \right\}.$$

Since

$$\int_0^1 |\nabla_H \varphi_0(\sigma(t))|_H |\dot{\sigma}(t)|_H dt \geq \int_0^1 \langle \nabla_H \varphi_0(\sigma(t)), \dot{\sigma}(t) \rangle_H dt = \varphi_0(\sigma(1)) - \varphi_0(\sigma(0)),$$

we get

$$d_{\varphi_0}(x,y) \geq \varphi_0(y) - \varphi_0(x),$$

and the equality holds if and only if $\sigma$ is an integral curve of $\nabla_H \varphi_0$ connecting $x$ and $y$. This means that $\varphi_0$ acts as a Kantorovich potential for the optimal transport problem

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} d_{\varphi_0}(x,y) d\gamma(x,y) : \gamma \in \Pi(\mu,\nu) \right\}$$

associated with the metric $d_{\varphi_0}$ induced by $\varphi_0$ itself.

Let $\mathcal{G} = G \circ |\cdot|_H$, where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is as in Chapter 3. The condition (4.14) implies that the distance $d_{\varphi_0}$ is induced by

$$|\nabla_H \varphi_0(x)|_H = |D\mathcal{G}(\mathbf{w}_0(x))|_H = G'(|\mathbf{w}_0(x)|_H),$$

where $\mathbf{w}_0$ is the solution to $(\mathcal{B})$ (remember that this equality must be understood as an equality a.e., even because $\mathcal{G}$ is not differentiable in 0), in the same way in which the equilibrium metric in Wardrop's problem is induced by $G'(i_{Q_0}(x))$ for an optimal $Q_0 \in \mathcal{Q}^p_H(\mu,\nu)$. This suggests a connection between this two problems and that one can prove the existence of Wardrop equilibria by looking at the problem $(\mathcal{B})$. The aim of the next Section is to investigate the connection between this two problems.

## 4.4 Beckmann-type problem and congested optimal transport problem in $\mathbb{H}^n$

Following [30], in this section we investigate the relation between the two problems $(\mathcal{B})$ and $(\mathcal{W})$. Moreover we will show how to pass from a solution to $(\mathcal{W})$ to a solution to $(\mathcal{B})$, and vice versa.

### 4.4.1 The equivalence between the two problems

Let us consider the function $\mathcal{G} = G \circ |\cdot|_H$ and the two optimization problems $(\mathcal{W})$ and $(\mathcal{B})$. We know that for any $Q \in \mathcal{Q}^p_H(\mu,\nu)$, it holds that

$$|\mathbf{w}_Q| \leq i_Q,$$

as measures; hence $\mathbf{w}_Q \in L^p(\Omega, H\Omega)$, see Lemma 4.1.1. Since the function $G$ is non decreasing and $\mathbf{w}_Q$ is admissible for $(\mathcal{B})$, it follows that

$$(\mathcal{B}) \leq (\mathcal{W}).$$

Hence, the equivalence between the two problems is reduced to the converse inequality. In order to do this we have to pass through the well-known *superposition principle* in Riemannian setting.

**Superposition principle in $\mathbb{H}^n$ in Riemannian approximation**

Let $\epsilon > 0$ and let us consider a vector field $\mathbf{w}_\epsilon \in L^p(\Omega, T\Omega)$, where $T\Omega$ is the restriction to the domain $\Omega$ of the tangent bundle generated by the vector fields $\{X_i^\epsilon\}_{i=1}^{2n+1}$, see Subsection 1.1.6. We define its distributional *$\epsilon$-divergence* by the rule

$$\langle \mathrm{div}_\epsilon \mathbf{w}_\epsilon, \varphi \rangle = -\int_\Omega b_\epsilon \left( \mathbf{w}_\epsilon, \nabla_\epsilon \varphi \right) dx, \quad \forall \varphi \in W_\epsilon^{1,q}(\Omega). \tag{4.16}$$

It follows that $\mathrm{div}_\epsilon \mathbf{w}_\epsilon \in (W_\epsilon^{1,q}(\Omega))'$.

If instead we consider a horizontal vector field $\mathbf{w} \in L^p(\Omega, H\Omega)$, in particular $\mathbf{w} \in L^p(\Omega, T\Omega)$; hence, its distributional divergence $\mathrm{div}_\epsilon \mathbf{w}$ in the sense of (4.16) is well-defined.

On the other hand, since $\|\varphi\|_{HW^{1,q}(\Omega)} \leq \|\varphi\|_{W_\epsilon^{1,q}(\Omega)}$ we get

$$W_\epsilon^{1,q}(\Omega) \hookrightarrow HW^{1,q}(\Omega),$$

and $\mathrm{div}_H \mathbf{w} \in (W_\epsilon^{1,q}(\Omega))'$.

Hence, given $\varphi \in W_\epsilon^{1,q}(\Omega)$,

$$\langle \mathrm{div}_\epsilon \mathbf{w}, \varphi \rangle = -\int_\Omega b_\epsilon \left( \mathbf{w}, \nabla_\epsilon \varphi \right) dx = -\int_\Omega \langle \mathbf{w}, \nabla_H \varphi \rangle_H dx = \langle \mathrm{div}_H \mathbf{w}, \varphi \rangle.$$

This means that, if $\mathbf{w} \in L^p(\Omega, H\Omega)$, then

$$\mathrm{div}_\epsilon \mathbf{w} = \mathrm{div}_H \mathbf{w}_{|W_\epsilon^{1,q}(\Omega)}. \tag{4.17}$$

We write down the *Superposition principle* for the Riemannian manifolds $(\mathbb{H}^n, b_\epsilon)$.

**Theorem 4.4.1.** *[15, Theorem 5.8] Let $V : [0,1] \times \mathbb{H}^n \to T\mathbb{H}^n$ be a Borel time dependent vector fields. For every probability measure $\lambda$ on $[0,1] \times \mathbb{H}^n$ that solves*

$$\partial_t \lambda + \mathrm{div}_\epsilon (\lambda V) = 0$$

*in the sense of distributions and satisfies*

$$\int_{[0,1]\times\mathbb{H}^n} |V_t(x)|_\epsilon d\lambda_t(x) < +\infty,$$

*there exists a probability measure* $Q \in \mathcal{P}(C([0,1], \mathbb{R}^{2n+1}))$ *such that* $\lambda_t = (e_t)_\# Q, \forall t \in [0,1]$ *and* $Q$*-a.e.* $\sigma \in C([0,1], \mathbb{R}^{2n+1})$ *is an absolutely continuous integral curve of the vector field* $V$*.*

**Theorem 4.4.2.** *Let us suppose that*

$$\mathcal{Q}_H^p(\mu, \nu) \neq \emptyset.$$

*If* $\mu, \nu \in L^p(\Omega)$ *and they are bounded by below then*

$$(\mathcal{B}) = (\mathcal{W}).$$

*Moreover,* $Q \in \mathcal{Q}_H^p(\mu, \nu)$ *solves* $(\mathcal{W}) \iff \mathbf{w}_Q$ *solves* $(\mathcal{B})$ *and* $|\mathbf{w}_Q|_H = i_Q$*.*

*Proof.* The non-emptiness of the set $\mathcal{Q}_H^p(\mu, \nu)$ implies the finiteness of the infima of both problems and the existence of minimizers, otherwise there is nothing to prove.

Now let us take a minimizer $\mathbf{w}$ of $(\mathcal{B})$ and let us consider the time-depending horizontal vector field

$$\hat{\mathbf{w}}_t(x) = \frac{\mathbf{w}(x)}{(1-t)\mu(x) + t\nu(x)}, \tag{4.18}$$

where we use the fact that $\mu$ and $\nu$ are $p$-summable functions bounded by below.

Since $\mathbf{w} \in L^p(\Omega, H\Omega)$, in particular $\mathbf{w} \in L^p(\Omega, T\Omega)$ and (4.17) implies that $\mathrm{div}_\epsilon \mathbf{w} = \mu - \nu$. Then the linear interpolating curve $\overline{\mu}_t = (1-t)\mu + t\nu$ is a positive distributional solution to the following initial value problem for the Riemannian continuity equation

$$\begin{cases} \partial_t \lambda + \mathrm{div}_\epsilon(\hat{\mathbf{w}}\lambda) = 0, \\ \lambda_0 = \mu. \end{cases}$$

Indeed,

$$\partial_t \overline{\mu}_t + \mathrm{div}_\epsilon(\hat{\mathbf{w}}_t \overline{\mu}_t) = \mu - \nu + \mathrm{div}_\epsilon \mathbf{w} = 0.$$

In the spirit of [65, Proposition 3.1], we can apply Theorem 4.4.1 in the setting of the continuity equation on the Riemannian manifold $(\mathbb{H}, b_\epsilon)$ as far as

$$\int_0^1 \int_\Omega |\hat{\mathbf{w}}_t(x)|_\epsilon d\overline{\mu}_t(x)dt = \int_\Omega |\mathbf{w}(x)|_H dx < +\infty;$$

let us just notice that this holds true because $\mathbf{w}$ is admissible for $(\mathcal{B})$, hence it is in $L^p(\Omega, H\Omega)$ for $p > 1$. Then, we can conclude that there exists a probability measure $Q \in \mathcal{P}(C([0,1], \overline{\Omega}))$, such that $Q(AC([0,1], \overline{\Omega})) = 1$ and

$$\overline{\mu}_t = (e_t)_\# Q$$

such that $Q$-a.e. curve $\sigma$ is an integral curve of $\hat{\mathbf{w}}_t$. Since $\hat{\mathbf{w}}_t$ is a horizontal vector field, then Q-a.e. $\sigma$ is a horizontal curve and $Q \in \mathcal{Q}_H(\mu, \nu)$. Moreover

$$\begin{aligned}\int_{\overline{\Omega}} \varphi(x) di_Q(x) &= \int_0^1 \int_{C([0,1],\overline{\Omega})} \varphi(\sigma(t))|\dot{\sigma}(t)|_H dQ(\sigma)dt = \\ &= \int_0^1 \int_{\overline{\Omega}} \varphi(x)|\hat{\mathbf{w}}_t(x)|_H d\mu_t(x)dt = \int_0^1 \int_{\overline{\Omega}} \varphi(x)|\mathbf{w}(x)|_H dxdt,\end{aligned}$$

for every $\varphi \in C(\overline{\Omega})$. This implies that $i_Q = |\mathbf{w}|_H$ and thus $Q \in \mathcal{Q}_H^p(\mu, \nu)$. The monotonicity of $G$ implies that

$$\int_\Omega G(i_Q(x))dx = \int_\Omega G(|\mathbf{w}(x)|_H)dx = (\mathcal{B}) \leq (\mathcal{W}),$$

hence $Q$ solves $(\mathcal{W})$ and the previous inequality is an equality.

The rest of the thesis follows by noticing that, if $Q \in \mathcal{Q}_H^p(\mu, \nu)$, then $|\mathbf{w}_Q|_H \leq i_Q$ a.e. in $\Omega$ and $\mathbf{w}_Q$ is admissible for $(\mathcal{B})$; hence the monotonicity of $G$ implies that

$$(\mathcal{B}) \leq \int_\Omega \mathcal{G}(|\mathbf{w}_Q(x)|_H)dx \leq \int_\Omega G(i_Q(x))dx.$$

□

*Remark* 20. Let us just remark that the convexity assumption on the function $G : \mathbb{R}_+ \to \mathbb{R}_+$ is made to get the existence of minimizers for both problems but it plays no role in the proof of the equality

$$(\mathcal{B}) = (\mathcal{W}).$$

*Remark* 21. In the proof of the Theorem 4.4.2 we started from a solution to $(\mathcal{B})$ and we get the existence of a solution to $(\mathcal{W})$, by using the Superposition Principle. In [29] the authors provide a different proof of the previous result in the Euclidean setting: this proof is based on Moser's flow argument, see [51] and [73], together with a regularization procedure and it doesn't require any assumption on $\mu$ and $\nu$. It also works in the Heisenberg setting and we will use it to prove the analogous result in the orthotropic case, see Theorem 5.1.5.

*Remark* 22. All the arguments in this Chapter still work if the function $\mathcal{G} : \overline{\Omega} \times \mathbb{R}^{2n} \to \mathbb{R}_+$ also depends on $x \in \overline{\Omega}$. It is enough to suppose that

1. $w \mapsto \mathcal{G}(x, w)$ is convex for any $x \in \overline{\Omega}$;
2. $\mathcal{G}(x, 0) = 0$, for any $x \in \overline{\Omega}$;
3. there exist a function $h \in L^1(\Omega)$ and two constants $a, b > 0$ such that

$$a|w|^p \leq \mathcal{G}(x, w) \leq b|w|^p + h(x),$$

   for any $(x, w) \in \overline{\Omega} \times \mathbb{R}^{2n}$.

# Chapter 5

# Congested optimal transport in $\mathbb{H}^n$ with orthotropic cost

The first part of this chapter is devoted to the study of the equivalence between the three problems $(\mathcal{W})$, $(\mathcal{B})$ and $(\mathcal{D})$, when the cost function has $p$-growth in each direction. In this case, in the Euclidean setting, the problem $(\mathcal{W})$ is known as the *orthotropic congested optimal transport problem*: it arises from the discrete model on networks introduced in [94] and [14], when the network becomes very dense. See [9] for more details about this topic.

In the second part we will study the Lipschitz regularity for solutions of a homogeneous quasi-linear equation, arising from this new formulation of the problem.

## 5.1 The three equivalent formulations

As in the previous chapters, we consider a regular bounded domain $\Omega \subset \mathbb{H}^n$, with $C^{1,1}$ boundary.

First, we deal with the orthotropic Beckmann-type problem and its dual formulation. Second, we treat the orthotropic congested optimal transport problem and its relation with the Beckmann-type problem.

### 5.1.1 Beckmann-type problem and dual formulation

Let u consider a non-decreasing and convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

1. $G(0) = 0$;

2. there exist $a, b \in \mathbb{R}_+$ and $p \in (1, \infty)$ such that

$$ai^p \leq G(i) \leq bi^p + 1, \quad \forall i \in \mathbb{R}_+. \tag{5.1}$$

In the remainder of the chapter we will consider a function $\mathcal{G} : \mathbb{R}^{2n} \to \mathbb{R}_+$ of the form

$$\mathcal{G}(w) = \sum_{i=1}^{2n} G(|w_i|), \quad w \in \mathbb{R}^{2n}. \tag{5.2}$$

Let us just observe that the convexity of the function $G : \mathbb{R}_+ \to \mathbb{R}_+$ implies the convexity of the function $\mathcal{G} : \mathbb{R}^{2n} \to \mathbb{R}_+$. Moreover (5.1) implies that

$$\mathcal{G}(w) \geq a \sum_{i=1}^{2n} |w_i|^p \geq \overline{a} \left( \sum_{i=1}^{2n} (w_i)^2 \right)^{\frac{p}{2}} = \overline{a}|w|^p, \tag{5.3}$$

for some constant $\overline{a} = \overline{a}(n)$, which in turn implies coercivity of the integral functional

$$L^p(\Omega, H\Omega) \ni \mathbf{w} \mapsto \int_\Omega \sum_{i=1}^{2n} G(|\mathbf{w}_i(x)|) dx, \tag{5.4}$$

where $\mathbf{w} = \sum_{i=1}^{2n} \mathbf{w}_i X_i = (\mathbf{w}_1, \ldots, \mathbf{w}_{2n}) \in L^p(\Omega, H\Omega)$. Similarly

$$\mathcal{G}(w) \leq \sum_{i=1}^{2n} (b|w_i|^p + 1) = b \sum_{i=1}^{2n} |w_i|^p + 2n \leq \overline{b} \left( |w|^p + 1 \right), \tag{5.5}$$

for some constant $\overline{b} = \overline{b}(n)$.

Let $f \in (HW^{1,q}(\Omega))'$, the *Beckmann-type problem* reads as

$$\inf_{\mathbf{w} \in L^p(\Omega, H\Omega)} \left\{ \int_\Omega \sum_{i=1}^{2n} G(|\mathbf{w}_i(x)|) dx : \mathrm{div}_H \mathbf{w} = f \right\}, \tag{5.6}$$

The coercivity of the functional (5.4) implies a bound on minimizing sequences, hence, following the proof of Theorem 4.2.2, one can get the existence of solutions.

**Theorem 5.1.1.** *The problem* (5.6) *admits solutions, with finite value, if and only if* $f \in (HW^{1,q})'_{\diamond}(\Omega)$.

### Dual Formulation

This problem admits the dual formulation

$$\sup_{\varphi \in HW^{1,q}(\Omega)} \left\{ -\langle f, \varphi \rangle - \int_\Omega \sum_{i=1}^{2n} G^*(|X_i \varphi(x)|) dx \right\}, \tag{5.7}$$

where $G^*$ is the Legendre transform of $G$: in particular it is a convex function satisfying

$$\frac{1}{qb^q} s^q - 1 \le G^*(s) \le \frac{1}{qa^q} s^q, \quad \forall s \in \mathbb{R}_+,$$

where $q = \frac{p}{p-1}$.

As anticipated in Remark 18, in the next theorem we follow [88] to prove that

$$(5.7) = (5.6);$$

the rest of the thesis follows as in Theorem 4.3.2 and Corollary 4.3.3.

**Theorem 5.1.2.** *If* $f \in (HW^{1,q})'_{\diamond}(\Omega)$, *then*

$$\min_{\mathbf{w} \in L^p(\Omega, H\Omega)} \left\{ \int_\Omega \sum_{i=1}^{2n} G(|\mathbf{w}_i(x)|) dx : \mathrm{div}_H \mathbf{w} = f \right\}$$
$$= \max_{\varphi \in HW^{1,q}(\Omega)} \left\{ -\langle f, \varphi \rangle - \int_\Omega \sum_{i=1}^{2n} G^*(|X_i \varphi(x)|) dx \right\}.$$

*If in addition* $G : \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly convex, then the unique solution* $\mathbf{w}_0$ *is of the form*

$$\mathbf{w}_0 = D \left( \sum_{i=1}^{2n} G^*(|X_i \varphi_0(x)|) \right),$$

*where* $\varphi_0$ *is a solution to* (5.7) *an it solves*

$$\mathrm{div}_H \left( D \left( \sum_{i=1}^{2n} G^*(|X_i \varphi(x)|) \right) \right) = f, \quad \text{in } \Omega. \tag{5.8}$$

*Proof.* Let consider the functional

$$\mathcal{F} : (HW^{1,q}(\Omega))' \to \mathbb{R}$$

defined as

$$\mathcal{F}(h) := \min_{\mathbf{w}\in L^p(\Omega,H\Omega)} \left\{ \int_\Omega \sum_{i=1}^{2n} G(|\mathbf{w}_i(x)|)dx : \mathrm{div}_H \mathbf{w} = f + h \right\}.$$

If $h \in (HW^{1,q})'_\diamond(\Omega)$, then $\mathcal{F}(h)$ is well-defined and real-valued thanks to (5.5), the fact that we are minimizing over the set $L^p(\Omega, H\Omega)$ and Proposition 4.2.1.

Let us compute $\mathcal{F}^* : HW^{1,q}(\Omega) \to \mathbb{R}$

$$\begin{aligned}
\mathcal{F}^*(\varphi) &= \sup\left\{\langle h,\varphi\rangle - \mathcal{F}(h) : h \in (HW^{1,q}(\Omega))'\right\}\\
&= \sup\left\{\langle h,\varphi\rangle - \int_\Omega \sum_{i=1}^{2n} G(|\mathbf{w}_i(x)|)dx : h, \mathbf{w} \text{ such that } \mathrm{div}_H\mathbf{w} = f+h\right\}\\
&= \sup_{\mathbf{w}}\left\{-\int_\Omega \langle \mathbf{w}, \nabla_H\varphi\rangle_H dx - \langle f,\varphi\rangle - \int_\Omega \sum_{i=1}^{2n} G(|\mathbf{w}_i(x)|)dx\right\}\\
&= -\langle f,\varphi\rangle + \int_\Omega \sum_{i=1}^{2n} G^*(|X_i\varphi(x)|)dx.
\end{aligned}$$

By definition $\mathcal{F}^{**}(0) = \sup_{\varphi\in HW^{1,q}(\Omega)}\{-\mathcal{F}^*(\varphi)\}$; moreover one can restrict the minimization to the space $HW^{1,q}_\diamond(\Omega)$ of Sobolev functions with zero mean, hence $\sup -\mathcal{F}^* < +\infty$. By taking the sup on $-\varphi$ instead of $\varphi$ we also have

$$\mathcal{F}^{**}(0) = \sup_{\varphi\in HW^{1,q}(\Omega)} \left\{-\langle f,\varphi\rangle - \int_\Omega \sum_{i=1}^{2n} G^*(|X_i\varphi(x)|)dx\right\}.$$

If $\mathcal{F}$ is convex and l.s.c., then $\mathcal{F}^{**}(0) = \mathcal{F}(0)$ and the thesis follows.

Let us prove convexity: let $h_0, h_1 \in (HW^{1,q})'_\diamond(\Omega)$ and set $h_t := (1-t)h_0 + th_1$. Let $\mathbf{w}_0$ and $\mathbf{w}_1$ optimal in the definition of $\mathcal{F}(h_0)$ and $\mathcal{F}(h_1)$, i.e. $\int_\Omega \sum_{i=1}^{2n} G(|\mathbf{w}_i|)dx = \mathcal{F}(h_i)$ and $\mathrm{div}_H\mathbf{w}_i = f + h_i$. Let $\mathbf{w}_t := (1-t)\mathbf{w}_0 + t\mathbf{w}_1$. Of course we have $\mathrm{div}_H\mathbf{w}_t = f + h_t$ and, by convexity of $\mathcal{G}$ we have

$$\begin{aligned}
\mathcal{F}(h_t) \le \int_\Omega \mathcal{G}(\mathbf{w}_t(x))dx \le (1-t)\int_\Omega \mathcal{G}(\mathbf{w}_0(x))dx + t\int_\Omega \mathcal{G}(\mathbf{w}_1(x))dx =\\
= (1-t)\mathcal{F}(h_0) + t\mathcal{F}(h_1).
\end{aligned}$$

Let us prove semicontinuity: let $h_n \to h$ in $(HW^{1,q})'(\Omega)$. We can suppose that $\mathcal{F}(h_n) \leq C < +\infty$, $\forall n \in \mathbb{N}$, otherwise there is nothing to prove. Hence $h_n \in (HW^{1,q})'_\diamond(\Omega)$, $\forall n \in \mathbb{N}$. Take the corresponding optimal $\mathbf{w}_n$, $\forall n \in \mathbb{N}$. We can take a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathcal{F}(h_{n_k}) = \liminf_{n\to\infty} \mathcal{F}(h_n).$$

Moreover,

$$\overline{a}\|\mathbf{w}_n\|^p_{L^p(\Omega,H\Omega)} \leq \int_\Omega \mathcal{G}(\mathbf{w}_n)dx = \mathcal{F}(h_n) \leq C, \quad \forall n \in \mathbb{N},$$

where we have used (5.3) in the first inequality. Hence, up to an extra subsequence extraction, we can assume that $\mathbf{w}_{n_k} \rightharpoonup \mathbf{w}$ in $L^p(\Omega, H\Omega)$. It holds that $\mathrm{div}_H \mathbf{w} = f + h$ and, since $\mathcal{G}$ is convex, the integral functional is lower semicontinuous w.r.t. the weak convergence in $L^p(\Omega, H\Omega)$, hence

$$\begin{aligned}\mathcal{F}(h) \leq \int_\Omega \mathcal{G}(\mathbf{w}(x))dx \leq \liminf_{k\to\infty} \int_\Omega \mathcal{G}(\mathbf{w}_{n_k}(x))dx \\ = \lim_{k\to\infty} \mathcal{F}(h_{n_k}) = \liminf_{n\to\infty} \mathcal{F}(h_n).\end{aligned}$$

□

### 5.1.2 Orthotropic congested optimal transport problem in $\mathbb{H}^n$

As in Chapter 3 we consider $\mu, \nu \in \mathcal{P}(\overline{\Omega})$. Remember that, given $Q \in \mathcal{Q}_H(\mu,\nu)$, we can define a vector measure $\mathbf{w}_Q \in \mathcal{M}(\overline{\Omega}, H\overline{\Omega})$ defined by

$$\int_{\overline{\Omega}} \phi(x) \cdot d\mathbf{w}_Q(x) = \int_{C([0,1],\overline{\Omega})} \left( \int_0^1 \langle \phi(\sigma(t)), \dot{\sigma}(t)\rangle_H \, dt \right) dQ(\sigma),$$

for any $\phi \in C(\overline{\Omega}, H\overline{\Omega})$.

We can think the measure as a vector $\mathbf{w}_Q = \left(\mathbf{w}_Q^1, \ldots, \mathbf{w}_Q^{2n}\right) \in \mathcal{M}(\overline{\Omega}, H\overline{\Omega})$, where $\mathbf{w}_Q^i \in \mathcal{M}(\overline{\Omega})$ is given by

$$\int_{\overline{\Omega}} \varphi(x) d\mathbf{w}_Q^i(x) = \int_{C([0,1],\overline{\Omega})} \left( \int_0^1 \varphi(\sigma(t))\dot{\sigma}_i(t)dt \right) dQ(\sigma),$$

for any $\varphi \in C(\overline{\Omega})$. Let us also define the following non-negative finite Radon measures $m^i_Q \in \mathcal{M}_+(\overline{\Omega})$, $i = 1, \ldots, 2n$:

$$H \ni \sigma \mapsto \int_{\overline{\Omega}} \varphi(x) dm^i_Q(x) := \int_{C([0,1],\overline{\Omega})} \mathcal{L}^i_\varphi(\sigma) dQ(\sigma),$$

where

$$\mathcal{L}^i_\varphi(\sigma) := \int_0^1 \varphi(\sigma(t)) |\dot{\sigma}_i(t)| dt,$$

for any $\varphi \in C(\overline{\Omega})$.

For any $\phi = \sum_{i=1}^{2n} \phi_i X_i = (\phi_1, \ldots, \phi_{2n}) \in C(\overline{\Omega}, H\overline{\Omega})$, we denote by

$$H \ni \sigma \mapsto \mathcal{L}_\phi(\sigma) := \sum_{i=1}^{2n} \mathcal{L}^i_{\phi_i}(\sigma)$$

and by $m_Q$ the vector measure $m_Q = (m^1_Q, \ldots, m^{2n}_Q) \in \mathcal{M}_+\left(\overline{\Omega}, H\overline{\Omega}\right)$,

$$\int_{\overline{\Omega}} \phi(x) \cdot dm_Q(x) = \int_{\overline{\Omega}} \mathcal{L}_\phi(\sigma) dQ(\sigma).$$

In particular, for any Borel set $A \subseteq \overline{\Omega}$ and any $\varphi \in C_c(\overline{\Omega})$ such that $\operatorname{supp} \varphi \subseteq A$ and $\|\varphi\|_\infty \leq 1$ it holds that

$$\begin{aligned}\left| \int_{\overline{\Omega}} \varphi(x) d\mathbf{w}^i_Q(x) \right| &\leq \int_{C([0,1],\overline{\Omega})} \left( \int_0^1 |\varphi(\sigma(t))|\, |\dot{\sigma}_i(t)| dt \right) dQ(\sigma) \\ &= \int_{\overline{\Omega}} |\varphi(x)| dm^i_Q(x) \leq m^i_Q(A),\end{aligned}$$

hence

$$\mathbf{w}^i_Q \leq |\mathbf{w}^i_Q| \leq m^i_Q, \quad \forall i = 1, \ldots, 2n$$

as measures. Similarly, one can prove that

$$\mathbf{w}_Q \leq |\mathbf{w}_Q| \leq \sum_{i=1}^{2n} m^i_Q,$$

as measures.

Let us denote by

$$\overline{\mathcal{Q}}^p_H(\mu, \nu) := \left\{ Q \in \mathcal{Q}_H(\mu, \nu) : m^i_Q \in L^p(\Omega), \forall i = 1, \ldots, 2n \right\}.$$

Let us just remark that, by definition, given $\varphi \in C(\overline{\Omega}, \mathbb{R}_+)$

$$\begin{aligned}\int_{\overline{\Omega}} \varphi dm_Q^j &= \int_{C([0,1],\overline{\Omega})} \left( \int_0^1 \varphi(\sigma(t)) |\dot{\sigma}_j(t)| dt \right) dQ(\sigma) \\ &\leq \int_{C([0,1],\overline{\Omega})} \left( \int_0^1 \varphi(\sigma(t)) |\dot{\sigma}(t)|_H dt \right) dQ(\sigma) \left( = \int_{\overline{\Omega}} \varphi(x) di_Q(x) \right) \\ &\leq \int_{C([0,1],\overline{\Omega})} \left( \int_0^1 \varphi(\sigma(t)) \left( \sum_{j=1}^{2n} |\dot{\sigma}_j(t)| \right) dt \right) dQ(\sigma) = \sum_{j=1}^{2n} \int_{\overline{\Omega}} \varphi(x) dm_Q^j(x).\end{aligned}$$

Hence

$$m_Q^j \leq i_Q \leq \sum_{j=1}^{2n} m_Q^j, \quad \forall i = 1, \dots, 2n$$

as measures. This implies that

$$\mathcal{Q}_H^p(\mu,\nu) \neq \emptyset \Leftrightarrow \overline{\mathcal{Q}}_H^p(\mu,\nu) \neq \emptyset. \tag{5.9}$$

Moreover, if $Q \in \overline{\mathcal{Q}}_H^p(\mu,\nu)$ then $\mathbf{w}_Q \in L^p(\Omega, H\Omega)$ and for any $i = 1, \dots, 2n$

$$|\mathbf{w}_Q^i| \leq m_Q^i \text{ a.e. in } \Omega, \tag{5.10}$$

and

$$|\mathbf{w}_Q| \leq \sum_{i=1}^{2n} m_Q^i \text{ a.e. in } \Omega.$$

The *orthotropic congested optimal transport in* $\mathbb{H}^n$ reads as

$$\inf_{Q \in \overline{\mathcal{Q}}_H^p(\mu,\nu)} \int_\Omega \sum_{i=1}^{2n} G(m_Q^i(x)) dx. \tag{5.11}$$

### Existence of solutions

Arguing as in Chapter 3 and [43] one can get existence of solutions.

**Theorem 5.1.3.** *Let us suppose that*

$$\overline{\mathcal{Q}}_H^p(\mu,\nu) \neq \emptyset.$$

*Then, the problem* (5.11) *admits solutions.*

*Sketch of the proof.* The proof goes exactly as the proof of Theorem 3.3.8. Here we highlight the main steps.

Let $(Q_n)_{n\in\mathbb{N}} \subset \overline{Q}^p_H(\mu,\nu)$ be a minimizing sequence. Since the measures $m^i_Q$ are invariant by reparametrization, for any $i = 1,\dots,2n$, one may assume that $Q_n = \tilde{Q}_n$, for any $n \in \mathbb{N}$. Let us consider the corresponding sequences $(m_{Q_n})_{n\in\mathbb{N}}$, for any $i = 1,\dots,2n$: since $(Q_n)_{n\in\mathbb{N}}$ is a minimizing sequence, (5.1) implies the boundedness in $L^p(\Omega)$ of the sequences $(m^i_{Q_n})_{n\in\mathbb{N}}$, for any $i = 1,\dots,2n$. Hence, up to subsequences, we may assume that $m^i_{Q_n} \rightharpoonup m^i \in L^p(\Omega)$. Since $(m^i_{Q_n})_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$, it is also bounded in $L^1(\Omega)$, for any $i = 1,\dots,2n$. As for the proofs of Lemma 3.2.1 and Lemma 3.2.2, Ascoli-Arzelà and Prokhorov's theorems imply the existence of a measure $Q \in \mathcal{P}(C([0,1],\overline{\Omega}))$ such that $Q_n \rightharpoonup Q$ in $\mathcal{M}(C([0,1],\mathbb{R}^{2n+1}))$ and it is concentrated on $H$. Moreover $m^i_Q \le m^i$ for any $i = 1,\dots,2n$. Hence $Q \in \overline{Q}^p_H(\mu,\nu)$ and thanks to the lower semi continuity of the integral functional w.r.t. the weak convergence in $L^p(\Omega)$

$$\begin{aligned}\int_\Omega \sum_{i=1}^{2n} G(m^i_Q)dx &\le \int_\Omega \sum_{i=1}^{2n} G(m^i)dx \\ &\le \liminf_{n\to\infty} \int_\Omega \sum_{i=1}^{2n} G(m^i_{Q_n})dx = \inf_{Q\in\overline{\mathcal{Q}}_H(\mu,\nu)} \int_\Omega \sum_{i=1}^{2n} G(m^i_Q)dx.\end{aligned}$$

□

### Characterization of solutions

Arguing as in Chapter 3 and [43] one can characterize the solutions to the problem (5.11).

If $\phi = \sum_{i=1}^{2n} \phi_i X_i = (\phi_1,\dots,\phi_{2n}) \in C(\overline{\Omega}, H\overline{\Omega})$, $\phi_i \ge 0$, for any $i = 1,\dots,2n$, we denote by

$$c_\phi(x,y) := \inf\left\{ \sum_{i=1}^{2n} \mathcal{L}^i_{\phi_i}(\sigma) : \sigma \in H^{x,y} \right\}, \quad \forall (x,y) \in \overline{\Omega}\times\overline{\Omega},$$

compare with (3.17).

Let $p \in \left(1, \frac{N}{N-1}\right)$, i.e. $q > N$, then as in Lemma 3.3.3

$$|c_\phi(x_1, y) - c_\phi(x_2, y)| \leq C\|\phi\|_{L^q(\Omega, H\Omega)} d_{SR}(x_1, x_2)^\alpha, \quad \forall (x_1, y), (x_2, y) \in \Omega \times \Omega,$$

and

$$|c_\phi(x, y_1) - c_\phi(x, y_2)| \leq C\|\phi\|_{L^q(\Omega, H\Omega)} d_{SR}(y_1, y_2)^\alpha, \quad \forall (x, y_1), (x, y_2) \in \Omega \times \Omega.$$

for some constant $C > 0$, where $\alpha := 1 - \frac{N}{q}$.

Hence, if $(\phi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^q(\Omega, H\Omega)$, then the sequence $(c_{\phi_n})_{n\in\mathbb{N}}$ admits a subsequence that converges in $C(\overline{\Omega} \times \overline{\Omega})$.

For any horizontal vector field $\phi \in L^q(\Omega, H\Omega)$ such that $\phi_i \geq 0$, for any $i = 1, \ldots, 2n$, one can define

$$\overline{c}_\phi(x, y) = \sup\{c(x, y) : c \in \mathcal{C}(\phi)\}, \quad \forall (x, y) \in \overline{\Omega} \times \overline{\Omega},$$

where

$$\mathcal{C}(\phi) := \left\{ \lim_{n\to\infty} c_{\phi_n} \text{ in } C(\overline{\Omega} \times \overline{\Omega}) : (\phi_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega}, H\overline{\Omega}), \phi_{i,n} \geq 0, \forall i, \forall n \right.$$
$$\left. \phi_n \to \phi \text{ in } L^q(\Omega, H\Omega) \right\}.$$

In particular, if $\phi \in C(\overline{\Omega}, H\overline{\Omega})$, $\phi_i \geq 0$ for any $i = 1, \ldots, 2n$, then following the proof of Proposition 3.3.5 one can prove that

$$c_\phi = \overline{c}_\phi.$$

Let now consider $Q \in \overline{\mathcal{Q}}^p_H(\mu, \nu)$. Then, for any sequence $(\phi_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega}, H\overline{\Omega})$, $\phi_{i,n} \geq 0$, for any $i = 1, \ldots, 2n$ and $\forall n \in \mathbb{N}$, which converges to $\phi$ in $L^q(\Omega, H\Omega)$, following the proof of Lemma 3.3.7, one can prove that:

(i) $(\mathcal{L}_{\phi_n})_{n\in\mathbb{N}}$ converges strongly in $L^1(C([0,1], \overline{\Omega}), Q)$ to some limit, independent of the approximating sequence $(\phi_n)_{n\in\mathbb{N}}$, which we denote by $\mathcal{L}_\phi$.

(ii) $\int_\Omega \phi(x) \cdot m_Q(x) dx = \int_{C([0,1],\overline{\Omega})} \mathcal{L}_\phi(\sigma) dQ(\sigma)$;

(iii) $\mathcal{L}_\phi(\sigma) \geq \overline{c}_\phi(\sigma(0), \sigma(1))$, for $Q$-a.e. $\sigma \in H$.

Let us suppose that $G : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $G(0) = 0$ and it satisfies (5.1) for some $p \in \left(1, \frac{N}{N-1}\right)$. Let us suppose that in addition $G$ is strictly convex and differentiable, with

$$0 \leq G'(i) \leq ci^{p-1} + 1, \quad \forall i \geq 0,$$

for some constant $c > 0$.

**Theorem 5.1.4.** *A horizontal traffic plan $Q \in \overline{\mathcal{Q}}^p_H(\mu, \nu) \neq \emptyset$ is optimal if, and only if,*

1. $\gamma_Q := (e_0, e_1)_\# Q \in \Pi(\mu, \nu)$ *solves*

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\phi_Q}(x, y) d\gamma(x, y),$$

*where* $L^q(\Omega, H\Omega) \ni \phi_Q := \sum_{i=1}^{2n} G'(m^i_Q) X_i = \left(G'(m^1_Q), \ldots, G'(m^{2n}_Q)\right)$;

2. $Q\left(\left\{\sigma \in H : \mathcal{L}(\sigma) = \overline{c}_{\phi_Q}(\sigma(0), \sigma(1)\right\}\right) = 1$.

*Proof.* See Proposition 3.3.11 and Theorem 3.3.12 for the proof. □

**Equivalence between Orthotropic Congested OT and the Beckmann-type problem**

As anticipated in Remark 21, following [29] and [85] we will use the Dacorogna-Moser construction plus a regularization procedure to prove the equivalence between the problems (5.11) and (5.6).

**Theorem 5.1.5.** *If $\overline{\mathcal{Q}}^p_H(\mu, \nu) \neq \emptyset$, then*

$$\min_{Q \in \overline{\mathcal{Q}}^p_H(\mu,\nu)} \int_\Omega \sum_{i=1}^{2n} G(m^i_Q(x)) dx$$
$$= \min_{\mathbf{w} \in L^p(\Omega, H\Omega)} \left\{ \int_\Omega \sum_{i=1}^{2n} G(|\mathbf{w}_i(x)|) dx : \operatorname{div}_H \mathbf{w} = \mu - \nu \right\}.$$

*Moreover, $Q \in \overline{\mathcal{Q}}^p_H(\mu, \nu)$ solves (5.11) $\Longleftrightarrow$ $\mathbf{w}_Q$ solves (5.6) and $|\mathbf{w}^i_Q| = m^i_Q, \forall i = 1, \ldots, 2n$.*

*Proof.* The non emptiness of the set $\overline{\mathcal{Q}}^p_H(\mu,\nu)$ implies that (5.11) $< +\infty$, (5.6) $< +\infty$ and the existence of minimizers, otherwise there is nothing to prove.

The inequality (5.10) and the monotonicity of $G : \mathbb{R}_+ \to \mathbb{R}_+$ imply

$$\min_{Q\in\overline{\mathcal{Q}}^p_H(\mu,\nu)} \int_\Omega \sum_{j=1}^{2n} G(m^j_Q(x))dx$$
$$\geq \min_{\mathbf{w}\in L^p(\Omega,H\Omega)} \left\{ \int_\Omega \sum_{j=1}^{2n} G(|\mathbf{w}_j(x)|)dx : \mathrm{div}_H \mathbf{w} = \mu - \nu \right\}.$$

Let us take a solution $\mathbf{w} = \sum_{j=1}^{2n} \mathbf{w}_j X_j \in L^p(\Omega, H\Omega)$ to (5.6) and let us find a traffic plan $Q \in \overline{\mathcal{Q}}^p_H(\mu,\nu)$ such that $m^j_Q \leq |\mathbf{w}_j|$ for any $j = 1,\dots,2n$, as measures. Such a traffic plan will be a solution to (5.11).

We extend $\mathbf{w}$ by 0 outside $\Omega$ and we consider the horizontal vector field $\mathbf{w}^\varepsilon = \sum_{j=1}^{2n} \mathbf{w}^\varepsilon_j X_j \in C^\infty_0(\Omega_\varepsilon, H\Omega_\varepsilon)$, where

$$\mathbf{w}^\varepsilon_j(x) := \rho_\varepsilon * \mathbf{w}_j(x), \quad \forall j = 1,\dots,2n,$$

$\rho_\varepsilon$ is a mollifier for the group structure of $\mathbb{H}^n$ and $\Omega_\varepsilon := \Omega \cdot B(0,\varepsilon)$. A simple computation implies that

$$\mathrm{div}_H \mathbf{w}^\varepsilon = \mu^\varepsilon - \nu^\varepsilon,$$

where we are supposing that both $\mu$ and $\nu$ are extended by 0 outside $\Omega$ with

$$\mu^\varepsilon = \rho_\varepsilon * \mu + \varepsilon, \quad \nu^\varepsilon = \rho_\varepsilon * \nu + \varepsilon \in C^\infty.$$

Let us introduce the non-autonomous horizontal vector field

$$\hat{\mathbf{w}}^\varepsilon(t,x) := \frac{\mathbf{w}^\varepsilon(x)}{\overline{\mu}^\varepsilon_t(x)}, \quad \forall (t,x) \in [0,1] \times \overline{\Omega}_\varepsilon,$$

where

$$\overline{\mu}^\varepsilon_t(x) := (1-t)\mu^\varepsilon(x) + t\nu^\varepsilon(x) > \varepsilon > 0, \quad (t,x) \in [0,1], \times\overline{\Omega}_\varepsilon. \qquad (5.12)$$

Let us notice that $\hat{\mathbf{w}}^\varepsilon(t,\cdot) \in C^\infty_0(\Omega_\varepsilon, H\Omega_\varepsilon)$, for any $t \in [0,1]$, and, for any $\epsilon > 0$, it holds that

$$\mu^\varepsilon - \nu^\varepsilon = \mathrm{div}_H \mathbf{w}^\varepsilon = \mathrm{div}_\epsilon \mathbf{w}^\varepsilon,$$

where $\operatorname{div}_\epsilon$ denotes the $\epsilon$-divergence, for a fixed $\epsilon > 0$, see (4.16).

In particular, this implies that the curve $\overline{\mu}_t^\varepsilon$ satisfies the following initial value problem for the Riemannian continuity equation

$$\begin{cases} \partial_t \lambda + \operatorname{div}_\epsilon(\hat{\mathbf{w}}^\varepsilon \lambda) = 0, \\ \lambda_0 = \mu^\varepsilon. \end{cases} \tag{5.13}$$

Since $\hat{\mathbf{w}}^\varepsilon$ is smooth, the unique solution to (5.13) is given by $(\Psi^\varepsilon(t,\cdot))_\# \mu^\varepsilon$, where

$$\Psi_\varepsilon : [0,1] \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$$

is the flow of the vector field $\hat{\mathbf{w}}^\varepsilon$ in the Riemannian sense,

$$\begin{cases} \frac{d}{dt}\Psi^\varepsilon(t,x) = \hat{\mathbf{w}}^\varepsilon(t, \Psi_\varepsilon(t,x)), \\ \Psi^\varepsilon(0,x) = x, \end{cases}$$

with $\Psi^\varepsilon\big|_{[0,1]\times\mathbb{R}^{2n+1}\setminus(\overline{\Omega}_\varepsilon)} = \mathrm{id}$. Hence

$$(\Psi^\varepsilon(t,\cdot))_\# \mu^\varepsilon = \overline{\mu}_t^\varepsilon, \quad \forall t \in [0,1].$$

If we denote by

$$\Phi^\varepsilon : \overline{\Omega}_\varepsilon \to C([0,1], \overline{\Omega}_\varepsilon),$$

the trajectory map, i.e. $\Phi^\varepsilon(x) := \Psi^\varepsilon(\cdot, x)$, then the measure

$$Q_\varepsilon := (\Phi^\varepsilon)_\# \mu^\varepsilon \in \mathcal{M}_+(C([0,1], \overline{\Omega}_\varepsilon)),$$

satisfies

$$(e_t)_\# Q_\varepsilon = (e_t \circ \Phi^\varepsilon)_\# \mu^\varepsilon = (\Psi^\varepsilon(t,\cdot))_\# \mu^\varepsilon = \overline{\mu}_t^\varepsilon, \quad \forall t \in [0,1];$$

in particular,

$$(e_0)_\# Q_\varepsilon = \mu^\varepsilon \quad (e_1)_\# Q_\varepsilon = \nu^\varepsilon. \tag{5.14}$$

By construction $Q_\varepsilon$ is supported on the integral curves of the horizontal vector field $\hat{\mathbf{w}}^\varepsilon$ and the definition of $Q_\varepsilon = \Phi^\varepsilon_\# \mu^\varepsilon$ and a change of variables imply

$$\int_{C([0,1],\overline{\Omega}_\varepsilon)} l_{SR}(\sigma)dQ_\varepsilon(\sigma) = \int_{\overline{\Omega}_\varepsilon}\left(\int_0^1 |\hat{\mathbf{w}}^\varepsilon(t,\Psi^\varepsilon(t,x))|_H dt\right)\mu^\varepsilon(x)dx$$
$$= \int_0^1\left(\int_{\overline{\Omega}_\varepsilon} |\hat{\mathbf{w}}^\varepsilon(t,y)|_H d\Psi^\varepsilon(t,\cdot)_\#\mu^\varepsilon(y)\right)dt = \int_0^1\left(\int_{\overline{\Omega}_\varepsilon}|\hat{\mathbf{w}}^\varepsilon(t,y)|_H\overline{\mu}^\varepsilon_t(y)dy\right)dt$$
$$= \int_{\overline{\Omega}_\varepsilon}|\mathbf{w}^\varepsilon(y)|_H dy \le \|\mathbf{w}\|_{L^1(\Omega,H\Omega)} \le \mathcal{L}^{2n+1}(\Omega)^{1/q}\|\mathbf{w}\|_{L^p(\Omega,H\Omega)} < +\infty,$$

because $\mathbf{w}$ is a solution to (5.6).

Hence, one can define the measures $m^j_{Q_\varepsilon} \in \mathcal{M}_+\left(\overline{\Omega}_\varepsilon\right)$ associated with $Q_\varepsilon$ as

$$\int_{\overline{\Omega}_\varepsilon}\varphi(x)dm^j_{Q_\varepsilon}(x) := \int_{C([0,1],\overline{\Omega}_\varepsilon)}\left(\int_0^1\varphi(\sigma(t))|\dot{\sigma}_i(t)|dt\right)dQ_\varepsilon(\sigma),$$

for any $\varphi \in C(\overline{\Omega}_\varepsilon)$. Moreover, the definition of $Q_\varepsilon$ and a change of variables imply

$$\int_{\overline{\Omega}_\varepsilon}\varphi(x)dm^j_{Q_\varepsilon}(x) = \int_{\overline{\Omega}_\varepsilon}\left(\int_0^1\varphi(\Psi^\varepsilon(t,x))|\hat{\mathbf{w}}^\varepsilon_j(t,\Psi^\varepsilon(t,x))|_H dt\right)\mu^\varepsilon(x)dx$$
$$= \int_0^1\left(\int_{\overline{\Omega}_\varepsilon}\varphi(y)|\hat{\mathbf{w}}^\varepsilon_j(t,y)|_H\overline{\mu}^\varepsilon_t(y)dy\right)dt = \int_{\overline{\Omega}_\varepsilon}\varphi(y)|\mathbf{w}^\varepsilon_j(y)|_H dy, \quad (5.15)$$

for any $\varphi \in C(\overline{\Omega}_\varepsilon)$; hence $m^j_{Q_\varepsilon} = |\mathbf{w}^\varepsilon_j| \in C^\infty_0(\Omega_\varepsilon)$ and again

$$\|m^j_{Q_\varepsilon}\|_{L^1(\Omega_\varepsilon)} = \|\mathbf{w}^\varepsilon_j\|_{L^1(\Omega_\varepsilon)} \le \mathcal{L}^{2n+1}(\Omega)^{1/q}\|\mathbf{w}\|_{L^p(\Omega,H\Omega)} < +\infty. \quad (5.16)$$

Analogously we may define and compute the measure $\mathbf{w}_{Q_\varepsilon} \in \mathcal{M}(\overline{\Omega}_\varepsilon, H\overline{\Omega}_\varepsilon)$ associated with $Q_\varepsilon$: given $\phi \in C(\overline{\Omega}_\varepsilon, H\overline{\Omega}_\varepsilon)$

$$\int_{\overline{\Omega}_\varepsilon}\phi(x)\cdot d\mathbf{w}_{Q_\varepsilon}(x) := \int_{\overline{\Omega}_\varepsilon}\left(\int_0^1\langle\phi(\Psi^\varepsilon(t,x)),\hat{\mathbf{w}}^\varepsilon(t,\Psi^\varepsilon(t,x))\rangle_H dt\right)\mu^\varepsilon(x)dx$$
$$= \int_{\overline{\Omega}_\varepsilon}\langle\phi(y),\mathbf{w}^\varepsilon(y)\rangle_H dy,$$

hence $\mathbf{w}_{Q_\varepsilon} = \mathbf{w}^\varepsilon \in C^\infty_0(\Omega_\varepsilon, H\Omega_\varepsilon)$.

We may consider $(Q_\varepsilon)_{\varepsilon>0}$ as a sequence of measures in $C([0,1],\Omega')$, which may not be probability measures, for some compact set $\Omega' \subset \mathbb{H}^n$, such that $\Omega \subset \Omega_\varepsilon \subset \Omega'$, for any $\varepsilon$. Since both $m_{Q_\varepsilon}$ and $\mathbf{w}_{Q_\varepsilon}$ are invariant by reparametrization, we may suppose that $Q_\varepsilon$ is supported on curves parametrized with constant speed, for any $\varepsilon$. Hence, the sequence $(Q_\varepsilon)_{\varepsilon>0} \subset \mathcal{M}_+(C([0,1],\Omega'))$ is tight. Indeed, the sets $\{\sigma \in H([0,1],\Omega') : |\dot\sigma|_H \le K\}$, for any $K>0$ are compact by Ascoli-Arzelà Theorem. Moreover

$$Q_\varepsilon\left(\{\sigma \in H([0,1],\Omega') : |\dot\sigma|_H > K\}\right) = Q_\varepsilon\left(\{\sigma \in H([0,1],\Omega') : l_{SR}(\sigma) > K\}\right)$$
$$\le \frac{1}{K}\int_{\Omega'} i_{Q_\varepsilon}(x)dx \le \frac{1}{K}\sum_{j=1}^{2n}\int_{\Omega'} m^j_{Q_\varepsilon}(x)dx \le \frac{(2n)\mathcal{L}^{2n+1}(\Omega)^{1/q}}{K}\|\mathbf{w}\|_{L^p(\Omega,H\Omega)},$$

thanks to (5.16). Hence, $(Q_\varepsilon)_{\varepsilon>0}$ is tight, and thus admits a subsequence that weakly converges to some $Q \in \mathcal{M}_+(C([0,1],\Omega'))$. It is obvious that $Q$ is concentrated on curves valued in $\overline{\Omega}$. In particular

$$Q \in \mathcal{P}(C([0,1],\overline{\Omega})), \tag{5.17}$$

since by definition it holds that

$$Q_\varepsilon(C([0,1],\Omega')) = \int_{\Omega'} \mu^\varepsilon(x)dx = \int_{\Omega_\varepsilon} \mu^\varepsilon(x)dx = 1+\varepsilon.$$

Letting $\varepsilon$ tend to 0 we have the desired result.

Moreover, following the same proof of [43, Lemma 2.8], one can prove that $Q$ is concentrated on $H([0,1],\Omega')$; hence, letting $\varepsilon$ tend to 0 and using (5.17), it follows that in particular $Q$ is concentrated on $H$.

Since $Q_\varepsilon \rightharpoonup Q$, then (5.14) passes to the limit, hence

$$Q \in \mathcal{Q}_H(\mu,\nu).$$

In the end, $m^j_{Q_\varepsilon} = |\mathbf{w}^\varepsilon_j|$ converges to $|\mathbf{w}_j|$ in $L^p(\Omega')$, hence it converges to the same limit in $\mathcal{M}_+(\Omega')$. Then, for any $\varphi \in C(\mathbb{H}^n)$,

$$\begin{aligned}\int_{\Omega'} \varphi(x)|\mathbf{w}_j(x)|dx &= \lim_{\varepsilon\to 0}\int_{\Omega'} \varphi(x)|\mathbf{w}^\varepsilon_j(x)|dx \\ &= \lim_{\varepsilon\to 0}\int_{C([0,1],\Omega')}\left(\int_0^1 \varphi(\sigma(t))|\dot\sigma_j(t)|dt\right)dQ_\varepsilon(\sigma) \\ &\ge \int_{C([0,1],\Omega')}\left(\int_0^1 \varphi(\sigma(t))|\dot\sigma_j(t)|dt\right)dQ(\sigma) = \int_{\Omega'}\varphi(x)dm^j_Q(x),\end{aligned}$$

by the lower semicontinuity of the map $Q \mapsto \int \left(\int \varphi(\sigma(t))|\dot\sigma(t)|_H dt\right) dQ(\sigma)$ with respect to the weak convergence of measures. In particular this means that

$$m_Q^j \le |\mathbf{w}_j| \in L^p(\Omega), \quad \forall j = 1, \dots, 2n,$$

hence

$$Q \in \overline{\mathcal{Q}}_H^p(\mu, \nu).$$

By using the monotonicity of $G$

$$\int_\Omega \sum_{j=1}^{2n} G(m_Q^j(x))dx \le \int_\Omega \sum_{j=1}^{2n} G(|\mathbf{w}_j(x)|)dx = (5.6).$$

Hence, taking the infimum over the set $\overline{\mathcal{Q}}_H^p(\mu, \nu)$ on the left hand-side, we get (5.11) $\le$ (5.6) and hence the equality between the two minimal values follows.

The rest of the thesis follows by noticing that, if $Q \in \overline{\mathcal{Q}}_H^p(\mu, \nu)$, then $|\mathbf{w}_Q^j| \le m_Q^j$ a.e. in $\Omega$, $\forall j = 1, \dots, 2n$, and $\mathbf{w}_Q$ is admissible for (5.6); hence the monotonicity of $G$ implies

$$(5.6) \le \int_\Omega \sum_{j=1}^{2n} G(|\mathbf{w}_Q^j(x)|)dx \le \int_\Omega \sum_{j=1}^{2n} G(m_Q^j(x))dx.$$

□

*Remark* 23. The previous argument shows how to construct a minimizing sequence for the problem $(\mathcal{W})$, starting from a regularized solution $\mathbf{w}$ to $(\mathcal{B})$. In the Euclidean setting this regularization procedure is not needed if the vector field $\mathbf{w}$ and $\mu, \nu$ are smooth enough: this is why one can define the flow in the classical sense, if $\mathbf{w}$ is Lipschitz, or at least in the weaker DiPerna-Lion sense, if $\mathbf{w}$ is in some Sobolev space, see [30, Section 3.3]. In the Heisenberg setting this regularization procedure is necessary, since the notion of flow is not well-posed, even if $\sigma \in \mathrm{Lip}(\overline{\Omega}, d_{SR})$, see [68].

**Corollary 5.1.6.** *Let $p \in (1, \infty)$ and $q = \frac{p}{p-1}$, then*

$$\mathcal{Q}_H^p(\mu, \nu) \neq \emptyset \iff \overline{\mathcal{Q}}_H^p(\mu, \nu) \neq \emptyset \iff \mu - \nu \in \left(HW^{1,q}\right)'_\diamond(\Omega).$$

*Proof.* Thanks to Lemma 4.1.1 and (5.9) it is enough to prove that, if $\mu-\nu \in (HW^{1,q})'_\diamond(\Omega)$, then $\overline{\mathcal{Q}}^p_H(\mu,\nu) \neq \emptyset$. This follows from the proof of Theorem 5.1.5: indeed, starting from a solution $\mathbf{w} \in L^p(\Omega, H\Omega)$ to (5.6) one can find a sequence $(Q_\varepsilon)_{\varepsilon>0}$ of measures weakly converging to a $Q \in \mathcal{Q}^p_H(\mu,\nu)$. □

## 5.2 Local Lipschitz regularity for solutions to a pseudo $q$-Laplacian type equations in $\mathbb{H}^n$

In this section we study the local Lipschitz regularity for solutions to an equation of type (5.8), in the homogeneous case.

Let us consider $1<p<2$ and the function $\mathcal{G}:\mathbb{R}^{2n}\to\mathbb{R}_+$,

$$\mathcal{G}(w)=\sum_{i=1}^n \frac{((w_i)^2+(w_{i+n})^2)^{p/2}}{p}.$$

Hence, the function $\mathcal{G}^*\in C^2(\mathbb{R}^{2n})$ is

$$\mathcal{G}^*(z)=\sum_{i=1}^n \frac{((z_i)^2+(z_{i+n})^2)^{q/2}}{q},$$

where $q=\frac{p}{p-1}>2$.

Let us consider the equation

$$\operatorname{div}_H(D\mathcal{G}^*(\nabla_H u))=0, \quad \text{in } \Omega. \tag{5.18}$$

The gradient $D\mathcal{G}^*$ of the function $\mathcal{G}^*$ is

$$D_i\mathcal{G}^*(z)=\left(z_i^2+z_{i+n}^2\right)^{\frac{q-2}{2}} z_i,$$
$$D_{i+n}\mathcal{G}^*(z)=\left(z_i^2+z_{i+n}^2\right)^{\frac{q-2}{2}} z_{i+n},$$

for $i=1,\dots,n$. Hence, the equation (5.18) reads as

$$\sum_{i=1}^n \Bigg[X_i\left(\left(|X_iu|^2+|X_{n+i}u|^2\right)^{\frac{q-2}{2}} X_iu\right) + X_{n+i}\left(\left(|X_iu|^2+|X_{n+i}u|^2\right)^{\frac{q-2}{2}} X_{n+i}u\right)\Bigg]=0,$$

The eigenvalues of the Hessian matrix $D^2\mathcal{G}^*$ are

$$\lambda_i(z) := (z_i^2 + z_{i+n}^2)^{(q-2)/2},$$
$$\lambda_{i+n}(z) := (q-1)(z_i^2 + z_{i+n}^2)^{(q-2)/2} = (q-1)\lambda_i(z),$$

for $i = 1, \ldots, n$, hence the following growth condition holds

$$\sum_{i=1}^{n} \lambda_i(z)(\xi_i^2 + \xi_{n+i}^2) \le \langle D^2\mathcal{G}^*(z)\xi, \xi\rangle \le (q-1)\sum_{i=1}^{n} \lambda_i(z)(\xi_i^2 + \xi_{n+i}^2),$$

for any $\xi \in \mathbb{R}^{2n}$.

**Definition 5.1** (Weak Solution)**.** We say that a function $u \in HW^{1,q}(\Omega)$ is a weak solution to the equation (5.18) if

$$\sum_{i=1}^{2n} \int_\Omega D_i\mathcal{G}^*(\nabla_H u)X_i\psi dx = 0,$$

for any $\psi \in C_c^\infty(\Omega)$.

Let us remark that the $q$-Laplace equation degenerates at $z = 0$, while equation (5.18) is much more degenerate, since it degenerates in the bigger set

$$\bigcup_{i=1}^{n} \left\{z_i^2 + z_{i+n}^2 = 0\right\}.$$

Let us spend few words about the restriction $q > 2$. If $q = 2$, then equation (5.18) is the sub-Laplacian $\Delta_H u := \sum_{i=1}^{2n} X_i^2 u = 0$ in $\Omega$, for which the smoothness of solutions directly follows from [63]. If instead $1 < q < 2$, then (5.18) has the $q$-Laplacian-type structure: indeed, in this case $q-2 < 0$ and $\lambda_i(z) \ge |z|^{q-2}$, $\forall i = 1, \ldots, n$, hence

$$\langle D^2\mathcal{G}^*(z)\xi, \xi\rangle \ge n|z|^{q-2}|\xi|^2,$$

while of course

$$|D\mathcal{G}^*(z)| \le |z|^{q-1}.$$

In this case the optimal regularity is the Hölder continuity for the horizontal gradient of solutions, see [75].

### 5.2.1 Riemannian approximation scheme

In the spirit of [38], [40] and [41], we introduce the Riemannian approximation procedure in order to get a priori estimates for weak solutions of (5.18).

Let us recall that $q > 2$. Then, for any $\delta \in (0,1)$ we denote by

$$\lambda_{i,\delta}(z) := \left(\delta + z_i^2 + z_{i+n}^2\right)^{\frac{q-2}{2}},$$

and

$$A_{i,\delta}(z) := \lambda_{i,\delta}(z) z_i$$
$$A_{i+n,\delta}(z) := \lambda_{i,\delta}(z) z_{i+n}.$$

for any $i \in \{1,\dots,n\}$.

It follows that

- $\lambda_{i,\delta}(z) \to \lambda_i(z)$ uniformly for $z \in \mathbb{R}^{2n}$, as $\delta \to 0$.
- $A_{i,\delta}(z) \to D_i \mathcal{G}^*(z)$ uniformly on compact subsets of $\mathbb{R}^{2n}$, as $\delta \to 0$.

Now, for any $\epsilon > 0$ and for any $z = \sum_{i=1}^{2n+1} z_i X_i^\epsilon \in \mathbb{R}^{2n+1}$ we denote by $z_H = \sum_{i=1}^{2n} z_i X_i^\epsilon = \sum_{i=1}^{2n} z_i X_i \in \mathbb{R}^{2n}$. Moreover, let us denote by

$$\lambda_{i,\delta}^\epsilon(z) := (\delta^2 + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-2)/2}, \tag{5.19}$$

for any $i \in \{1,\dots,n\}$ and by

$$\lambda_{2n+1,\delta}^\epsilon(z) := \sum_{i=1}^{n} \lambda_{i,\delta}^\epsilon(z).$$

Let us consider now $A_\delta^\epsilon : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ defined component-wise by

$$\begin{aligned} A_{i,\delta}^\epsilon(z) &:= \lambda_{i,\delta}^\epsilon(z) z_i, \\ A_{i+n,\delta}^\epsilon(z) &:= \lambda_{i,\delta}^\epsilon(z) z_{i+n}, \\ A_{2n+1,\delta}^\epsilon(z) &:= \lambda_{2n+1,\delta}^\epsilon(z) z_{2n+1} = \sum_{j=1}^{n} \lambda_{j,\delta}^\epsilon(z) z_{2n+1}. \end{aligned} \tag{5.20}$$

for any $i \in \{1\dots,n\}$. If we now denote $z^\epsilon = \sum_{i=1}^{2n} z_i X_i^\epsilon + \epsilon z_{2n+1} X_{2n+1}^\epsilon$, then

$$\lambda_{i,\delta}^\epsilon(z^\epsilon) \xrightarrow[\epsilon \to 0^+]{} \lambda_{i,\delta}(z_H),$$

for any $i \in \{1, \dots, n\}$, hence

$$A^{\epsilon}_{i,\delta}(z^{\epsilon}) \xrightarrow[\delta,\epsilon \to 0^+]{} D_i\mathcal{G}^*(z_H), \forall i \in \{1, \dots, 2n\},$$

while for $i = 2n + 1$ one has

$$A^{\epsilon}_{2n+1,\delta}(z^{\epsilon}) = \epsilon\lambda^{\epsilon}_{2n+1,\delta}(z^{\epsilon})z_{2n+1} \xrightarrow[\epsilon \to 0^+]{} 0.$$

Let us take $j \in \{1, \dots, 2n + 1\}$ and $i \in \{1, \dots, n\}$ and let us compute

$$\begin{aligned}\partial_j A^{\epsilon}_{i,\delta}(z) &= (q-2)(\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-4)/2} z_i \\ &\quad (z_i\delta_{j,i} + z_{i+n}\delta_{j,i+n} + z_{2n+1}\delta_{j,2n+1}) + (\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-2)/2}\delta_{j,i},\end{aligned}$$

and

$$\begin{aligned}\partial_j A^{\epsilon}_{i+n,\delta}(z) &= (q-2)(\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-4)/2} z_{i+n} \\ &\quad (z_i\delta_{j,i} + z_{i+n}\delta_{j,i+n} + z_{2n+1}\delta_{j,2n+1}) + (\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-2)/2}\delta_{j,i+n}.\end{aligned}$$

If $i = 2n+1$, let us denote by $A^{\epsilon}_{2n+1,i,\delta}(z) := (\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-2)/2} z_{2n+1}$, hence

$$A^{\epsilon}_{2n+1,\delta}(z) = \sum_{i=1}^{n} A^{\epsilon}_{2n+1,i,\delta}(z),$$

and

$$\begin{aligned}\partial_j A^{\epsilon}_{2n+1,\delta}(z) &= \sum_{i=1}^{n} \partial_j A^{\epsilon}_{2n+1,i,\delta}(z) \\ = \sum_{i=1}^{n} \Bigg[ &(q-2)(\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-4)/2} z_{2n+1} \left(\delta_{j,i}z_i + \delta_{j,i+n}z_{i+n} + \delta_{j,2n+1}z_{2n+1}\right) \\ &+ (\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-2)/2}\delta_{j,2n+1} \Bigg].\end{aligned}$$

We compute

$$\begin{aligned}\partial_j A^{\epsilon}_{i,\delta}(z)\xi_i\xi_j + \partial_j A^{\epsilon}_{i+n,\delta}(z)\xi_{i+n}\xi_j &= (q-2)(\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-4)/2} \\ &\quad (z_i\xi_i\xi_j + z_{i+n}\xi_{i+n}\xi_j)\left(z_i\delta_{j,i} + z_{i+n}\delta_{j,i+n} + z_{2n+1}\delta_{j,2n+1}\right) \\ &\quad + (\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-2)/2}\left(\delta_{j,i}\xi_i\xi_j + \delta_{j,i+n}\xi_{i+n}\xi_j\right),\end{aligned}$$

and

$$\begin{aligned}\partial_j A^\epsilon_{2n+1,\delta}(z)\xi_{2n+1}\xi_j &= \sum_{i=1}^n \partial_j A^\epsilon_{2n+1,i,\delta}(z)\xi_{2n+1}\xi_j \\ &= \sum_{i=1}^n \Big[(q-2)(\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-4)/2} \\ &\quad z_{2n+1}\xi_{2n+1}\xi_j(z_i\delta_{j,i} + z_{i+n}\delta_{j,i+n} + z_{2n+1}\delta_{j,2n+1}) \\ &\qquad + (\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-2)/2}\delta_{j,2n+1}\xi_{2n+1}\xi_j\Big].\end{aligned}$$

Hence, for any $\xi \in \mathbb{R}^{2n+1}$

$$\begin{aligned}&\sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(z)\xi_i\xi_j \\ &= \sum_{j=1}^{2n+1}\sum_{i=1}^{n}\left(\partial_j A^\epsilon_{i,\delta}(z)\xi_i\xi_j + \partial_j A^\epsilon_{i+n,\delta}(z)\xi_{i+n}\xi_j\right) + \sum_{j=1}^{2n+1}\partial_j A^\epsilon_{2n+1,\delta}(z)\xi_{2n+1}\xi_j \\ &= \sum_{i=1}^n\Big[(q-2)(\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-4)/2}(z_i\xi_i + z_{i+n}\xi_{i+n} + z_{2n+1}\xi_{2n+1})^2 \\ &\qquad + (\delta + z_i^2 + z_{i+n}^2 + z_{2n+1}^2)^{(q-2)/2}(\xi_i^2 + \xi_{i+n}^2 + \xi_{2n+1}^2)\Big] \\ &\qquad\qquad \geq \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(z)(\xi_i^2 + \xi_{i+n}^2 + \xi_{2n+1}^2).\end{aligned}$$

Hence we can conclude that $A^\epsilon_\delta = \left(A^\epsilon_{1,\delta}, \ldots, A^\epsilon_{2n+1,\delta}\right) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ satisfies the following structure condition

$$\begin{aligned}\sum_{i=1}^n \lambda^\epsilon_{i,\delta}(z)(\xi_i^2 + \xi_{n+i}^2 + \xi_{2n+1}^2) &\leq \sum_{i,j=1}^{2n+1}\partial_j A^\epsilon_{i,\delta}(z)\xi_i\xi_j \\ &\leq L\sum_{i=1}^n \lambda^\epsilon_{i,\delta}(z)(\xi_i^2 + \xi_{n+i}^2 + \xi_{2n+1}^2), \quad (5.21)\end{aligned}$$

for any $\xi \in \mathbb{R}^{2n+1}$, where $L = L(n,q) > 1$ is a constant.

We now consider the equation

$$\sum_{i=1}^{2n+1} X_i^\epsilon(A^\epsilon_{i,\delta}(\nabla_\epsilon u)) = 0. \quad (5.22)$$

**Definition 5.2** ($\epsilon$-Weak Solution)**.** We say that a function $u^\epsilon \in W^{1,q}_\epsilon(\Omega)$, see (1.19), is a weak solution to the equation (5.22) if

$$\sum_{i=1}^{2n+1} \int_\Omega A^\epsilon_{i,\delta}(\nabla_\epsilon u^\epsilon) X^\epsilon_i \psi dx = 0,$$

for any $\psi \in C^\infty_c(\Omega)$.

From the Euclidean and Riemannian elliptic theory it follows that every $\epsilon$-weak solution $u^\epsilon$ is smooth, that is $u^\epsilon \in C^\infty(\Omega)$.

### Caccioppoli type inequalities for the first derivatives of solutions

The aim of this subsection is to get higher regularity estimates for weak solutions $u^\epsilon$ to (5.22) that are stable in $\epsilon$ and $\delta$.

Through the whole subsection, with an abuse of notation, we will drop the indexes $\epsilon$ and we will denote by $u$ a weak solution to (5.22).

**Lemma 5.2.1.** *[Caccioppoli for $\nabla_\epsilon u$] There exists a constant $c = c(q, n, L) > 0$, independent of $\epsilon$ and $\delta$, such that, for every weak solution $u \in W^{1,q}_\epsilon(\Omega)$ to (5.22), for every $\beta \geq 0$ and for every $\eta \in C^\infty_c(\Omega)$ one has*

$$\begin{aligned}
&\int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \\
&\qquad \sum_{i=1}^{n} \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(|\nabla_\epsilon X^\epsilon_i u|^2 + |\nabla_\epsilon X^\epsilon_{i+n} u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1} u|^2\right) dx \leq \\
&\qquad c \int_\Omega (|\nabla_\epsilon \eta|^2 + \eta |X_{2n+1}\eta|) \sum_{i=1}^{n} \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta+2}{2}} dx \\
&\qquad\qquad + c(\beta+2)^4 \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \sum_{i=1}^{n} \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) |X_{2n+1} u|^2 dx.
\end{aligned}$$

*Proof.* We use $\psi = X^\epsilon_l(\eta^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} X^\epsilon_l u)$ as test function, with $l \in$

$\{1,\dots,n\}$ and $\beta\geq 0$, in the weak formulation

$$\begin{aligned}
0=&\int_\Omega \sum_{i=1}^{2n+1} A^\epsilon_{i,\delta}(\nabla_\epsilon u)X^\epsilon_i\psi dx\\
&=\int_\Omega \sum_{i=1}^{2n+1} A^\epsilon_{i,\delta}(\nabla_\epsilon u)X^\epsilon_i(X^\epsilon_l(\eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}X^\epsilon_l u))dx=\\
&-\int_\Omega \sum_{i=1}^{2n+1} X^\epsilon_l(A^\epsilon_{i,\delta}(\nabla_\epsilon u))X^\epsilon_i(\eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}X^\epsilon_l u)dx\\
&-\int_\Omega A^\epsilon_{n+l,\delta}(\nabla_\epsilon u)X_{2n+1}(\eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}X^\epsilon_l u)dx=\\
&-\int_\Omega \eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}\sum_{i,j=1}^{2n+1}\partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u)X^\epsilon_l X^\epsilon_j u X^\epsilon_l X^\epsilon_i u dx\\
&+\int_\Omega \eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}\sum_{j=1}^{2n+1}\partial_j A^\epsilon_{n+l,\delta}(\nabla_\epsilon u)X^\epsilon_l X^\epsilon_j u X_{2n+1} u dx\\
&-\int_\Omega X^\epsilon_l u\sum_{i,j=1}^{2n+1}\partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u)X^\epsilon_l X^\epsilon_j u X^\epsilon_i(\eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}})dx\\
&\qquad-\int_\Omega A^\epsilon_{n+l,\delta}(\nabla_\epsilon u)X_{2n+1}(\eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}X^\epsilon_l u)dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_\Omega \eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}\sum_{i,j=1}^{2n+1}\partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u)X^\epsilon_l X^\epsilon_j u X^\epsilon_l X^\epsilon_i u dx\\
&\quad+\int_\Omega X^\epsilon_l u\eta^2\sum_{i,j=1}^{2n+1}\partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u)X^\epsilon_l X^\epsilon_j u X^\epsilon_i((\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}})dx=\\
&\quad-2\int_\Omega \eta X^\epsilon_l u\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}\sum_{i,j=1}^{2n+1}\partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u)X^\epsilon_l X^\epsilon_j u X^\epsilon_i\eta dx\\
&\quad+\int_\Omega \eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}\sum_{j=1}^{2n+1}\partial_j A^\epsilon_{n+l,\delta}(\nabla_\epsilon u)X^\epsilon_l X^\epsilon_j u X_{2n+1} u dx\\
&\quad-\int_\Omega A^\epsilon_{n+l,\delta}(\nabla_\epsilon u)X_{2n+1}(\eta^2\left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}}X^\epsilon_l u)dx=I^l_1+I^l_2+I^l_3.
\end{aligned}$$

If $l \in \{n+1, \dots, 2n\}$

$$\begin{aligned}
&\int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_l X^\epsilon_j u X^\epsilon_l X^\epsilon_i u dx \\
&\quad + \int_\Omega X^\epsilon_l u \eta^2 \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_l X^\epsilon_j u X^\epsilon_i (\left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}) dx = \\
&\quad - 2\int_\Omega \eta X^\epsilon_l u \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_l X^\epsilon_j u X^\epsilon_i \eta dx \\
&\quad - \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{j=1}^{2n+1} \partial_j A^\epsilon_{l-n,\delta}(\nabla_\epsilon u) X^\epsilon_l X^\epsilon_j u X_{2n+1} u dx \\
&\quad + \int_\Omega A^\epsilon_{l-n,\delta}(\nabla_\epsilon u) X_{2n+1}(\eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} X^\epsilon_l u) dx = I^l_1 + I^l_2 + I^l_3.
\end{aligned}$$

In the end, if $l = 2n+1$, then $[X^\epsilon_{2n+1}, X^\epsilon_i] = 0, \forall i = 1, \dots, 2n+1$ implies that $I^{2n+1}_2 = 0$ and $I^{2n+1}_3 = 0$. Hence

$$\begin{aligned}
&\int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_{2n+1} X^\epsilon_j u X^\epsilon_{2n+1} X^\epsilon_i u dx \\
&\quad + \int_\Omega X^\epsilon_{2n+1} u \eta^2 \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_{2n+1} X^\epsilon_j u X^\epsilon_i (\left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}) dx = \\
&- 2\int_\Omega \eta X^\epsilon_{2n+1} u \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_{2n+1} X^\epsilon_j u X^\epsilon_i \eta dx =: I^{2n+1}_1.
\end{aligned}$$

Let us consider the second integral in the left hand side, for any $l = 1, \dots, 2n+$

1:

$$\int_\Omega X_l^\epsilon u\eta^2 \sum_{i,j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_i^\epsilon \left(\left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}\right) dx$$

$$= \beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i,j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u \sum_{k=1}^{2n+1} X_i^\epsilon X_k^\epsilon u X_k^\epsilon u dx$$

$$= \beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i=1}^{n} \sum_{j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u \sum_{k=1}^{2n+1} X_i^\epsilon X_k^\epsilon u X_k^\epsilon u dx$$

$$+\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i=n+1}^{2n} \sum_{j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u \sum_{k=1}^{2n+1} X_i^\epsilon X_k^\epsilon u X_k^\epsilon u dx$$

$$+\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{j=1}^{2n+1} \partial_j A_{2n+1,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u \sum_{k=1}^{2n+1} X_{2n+1}^\epsilon X_k^\epsilon u X_k^\epsilon u dx$$

where we just separated the summation in $i$. Since the only non-trivial bracket-relations between $X_i$ and $X_k$, for any $i, k = 1\dots, 2n+1$, are

$$\begin{cases} [X_i^\epsilon, X_k^\epsilon] = X_{2n+1}, \text{ if } k = i+n \text{ and } i \in \{1, \dots, n\}, \\ [X_i^\epsilon, X_k^\epsilon] = -X_{2n+1}, \text{ if } k = i-n \text{ and } i \in \{n+1, \dots, 2n\}, \end{cases}$$

we get

$$=\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i=1}^{n} \sum_{j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u \sum_{k=1}^{2n+1} X_k^\epsilon X_i^\epsilon u X_k^\epsilon u dx$$

$$+\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i=1}^{n} \sum_{j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u X_{2n+1} u X_{i+n}^\epsilon u dx$$

$$+\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i=n+1}^{2n} \sum_{j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u \sum_{k=1}^{2n+1} X_k^\epsilon X_i^\epsilon u X_k^\epsilon u dx$$

$$-\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i=n+1}^{2n} \sum_{j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u X_{2n+1} u X_{i-n}^\epsilon u dx$$

$$+\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{j=1}^{2n+1} \partial_j A_{2n+1,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u \sum_{k=1}^{2n+1} X_k^\epsilon X_{2n+1}^\epsilon u X_k^\epsilon u dx,$$

which in a more compact form is equal to

$$
\begin{aligned}
=&\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i,j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u \sum_{k=1}^{2n+1} X_k^\epsilon X_i^\epsilon u X_k^\epsilon u dx \\
+&\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} X_{2n+1} u X_l^\epsilon u \langle J A_\delta^\epsilon(\nabla_\epsilon u) X_l^\epsilon \nabla_\epsilon u, \\
&\left(X_{n+1}^\epsilon u, \ldots, X_{2n}^\epsilon u, -X_1^\epsilon u, \ldots, -X_n^\epsilon u, 0\right)\rangle dx,
\end{aligned}
$$

where $JA_\delta^\epsilon = \left(\partial_j A_{i,\delta}^\epsilon\right)_{ij}$ denotes the Jacobian matrix of $A_\delta^\epsilon$ in $\mathbb{R}^{2n+1}$. Denoting by $I_4^l$ the last integral and summing the above equations for all $l$ from 1 to $2n+1$ we get

$$
\begin{aligned}
&\int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i,j,l=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon X_i^\epsilon u dx \\
&+ \beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{i,j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) \langle \nabla_\epsilon X_j^\epsilon, \nabla_\epsilon u\rangle \langle \nabla_\epsilon X_i^\epsilon u, \nabla_\epsilon u\rangle dx \\
&= \sum_{l=1}^{2n+1} \left(I_1^l + I_2^l + I_3^l - I_4^l\right).
\end{aligned}
$$

The second integral in the left hand side is always positive, hence by (5.21)

$$
\begin{aligned}
(LHS) \geq &\sum_{l=1}^{2n+1} \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \\
&\sum_{i=1}^{n} \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left((X_l^\epsilon X_i^\epsilon u)^2 + (X_l^\epsilon X_{n+i}^\epsilon u)^2 + (X_l^\epsilon X_{2n+1}^\epsilon u)^2\right) dx \\
&= \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^{n} \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \\
&\qquad \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx.
\end{aligned}
$$

By the structure condition (5.21) and Young's inequality it follows that

$$|I_1^l| \leq 2\int_\Omega \eta \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+1}{2}} \sum_{i,j=1}^{2n+1} \partial_j A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_i^\epsilon \eta dx$$

$$\leq c\tau \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}$$

$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx$$

$$+ \frac{c}{\tau}\int_\Omega \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) |\nabla_\epsilon \eta|^2 dx,$$

for all $l \in \{1, \ldots, 2n+1\}$, where $c = c(q, n, L) > 0$. For $\tau$ small enough the first term can be reabsorbed in the $(LHS)$.

Let us estimate $I_2^l$, for $l \in \{1, \ldots, 2n\}$: again by (5.21) and Young's inequality we have

$$|I_2^l| \leq c\tau \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}$$

$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx$$

$$+ \frac{c}{\tau}\int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) |X_{2n+1} u|^2 dx.$$

As before, for $\tau$ small enough the first term can be reabsorbed in the $(LHS)$. In order to estimate $I_3^l$, for all $l \in \{1, \ldots, 2n\}$, let us compute

$$X_{2n+1}(\eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} X_l^\epsilon u) = 2\eta X_{2n+1}\eta \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} X_l^\epsilon u$$

$$+\beta \eta^2 X_l^\epsilon u \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} \sum_{k=1}^{2n+1} X_k^\epsilon u X_k^\epsilon X_{2n+1} u + \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} X_l^\epsilon X_{2n+1} u.$$

If $l \in \{1,\dots,n\}$

$$\begin{aligned}
I_3^l &= -2\int_\Omega \eta X_{2n+1}\eta A^\epsilon_{n+l,\delta}(\nabla_\epsilon u)\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} X_l^\epsilon u dx\\
&- \int_\Omega \eta^2\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} A^\epsilon_{n+l,\delta}(\nabla_\epsilon u) X_l^\epsilon X_{2n+1} u dx\\
&- \beta\int_\Omega \eta^2 X_l^\epsilon u\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} A^\epsilon_{n+l,\delta}(\nabla_\epsilon u)\sum_{k=1}^{2n+1} X_k^\epsilon u X_k^\epsilon X_{2n+1} u dx\\
&=: I^l_{3,1}+I^l_{3,2}+I^l_{3,3},
\end{aligned}$$

otherwise $l \in \{n+1,\dots,2n\}$

$$\begin{aligned}
I_3^l &= 2\int_\Omega \eta X_{2n+1}\eta A^\epsilon_{l-n,\delta}(\nabla_\epsilon u)\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} X_l^\epsilon u dx\\
&+ \int_\Omega \eta^2\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} A^\epsilon_{l-n,\delta}(\nabla_\epsilon u) X_l^\epsilon X_{2n+1} u dx\\
&+ \beta\int_\Omega \eta^2 X_l^\epsilon u\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-2}{2}} A^\epsilon_{l-n,\delta}(\nabla_\epsilon u)\sum_{k=1}^{2n+1} X_k^\epsilon u X_k^\epsilon X_{2n+1} u dx\\
&=: I^l_{3,1}+I^l_{3,2}+I^l_{3,3},
\end{aligned}$$

By (5.20) it follows that

$$\begin{aligned}
|I^l_{3,1}| &\le c\int_\Omega \eta|X_{2n+1}\eta|\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}\lambda^\epsilon_{l,\delta}(\nabla_\epsilon u)|X_l^\epsilon u||X^\epsilon_{n+l}u|dx\\
&\le c\int_\Omega \eta|X_{2n+1}\eta|\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}}\sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)dx,
\end{aligned}$$

for any $l \in \{1,\dots,n\}$; analogously if $l \in \{n+1,\dots,2n\}$

$$|I^l_{3,1}| \le c\int_\Omega \eta|X_{2n+1}\eta|\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}}\sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)dx.$$

Hence

$$|I^l_{3,1}| \le c\int_\Omega \eta|X_{2n+1}\eta|\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}}\sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)dx,$$

for any $l \in \{1,\dots,2n\}$, where $c = c(q,n,L)$.

In order to estimate $I^l_{3,2}$, we integrate by parts and we compute the derivative of the integrand function: hence for $l \in \{1, \dots, n\}$

$$\begin{aligned} I^l_{3,2} = &\int_\Omega X^\epsilon_l(\eta^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} A^\epsilon_{n+l,\delta}(\nabla_\epsilon u))X_{2n+1}u dx \\ &= \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \sum_{j=1}^{2n+1} \partial_j A^\epsilon_{n+l,\delta}(\nabla_\epsilon u) X^\epsilon_l X^\epsilon_j u X_{2n+1} u dx \\ &\quad + 2\int_\Omega \eta X^\epsilon_l \eta A^\epsilon_{n+l,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} X_{2n+1} u dx \\ &+ \beta \int_\Omega \eta^2 A^\epsilon_{n+l,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta-2}{2}} \sum_{k=1}^{2n+1} X^\epsilon_l X^\epsilon_k u X^\epsilon_k u X_{2n+1} u dx. \quad (5.23) \end{aligned}$$

The first integral in (5.23) can estimated as $I^l_2$, hence we will estimate the last two of them: again by using (5.20) and Young's inequality, the second term in (5.23) can be bounded by

$$\begin{aligned} c&\int_\Omega \eta |X^\epsilon_l \eta| \lambda^\epsilon_{l,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} |X^\epsilon_{n+l} u||X_{2n+1} u| dx \\ &\le c\int_\Omega \eta |\nabla_\epsilon \eta| \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta+1}{2}} |X_{2n+1} u| dx \\ &\le c\int_\Omega \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta+2}{2}} |\nabla_\epsilon \eta|^2 \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) dx \\ &\qquad + c\int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) |X_{2n+1} u|^2 dx. \end{aligned}$$

for some constant $c = c(n, q, L)$. While for the third one we use $[X^\epsilon_l, X^\epsilon_k] = X_{2n+1}$ if $k = l + n$, then

$$\begin{aligned} \beta &\int_\Omega \eta^2 A^\epsilon_{n+l,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta-2}{2}} \sum_{k=1}^{2n+1} X^\epsilon_l X^\epsilon_k u X^\epsilon_k u X_{2n+1} u dx \\ &= \beta \int_\Omega \eta^2 A^\epsilon_{n+l,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta-2}{2}} \sum_{k=1}^{2n+1} X^\epsilon_k X^\epsilon_l u X^\epsilon_k u X_{2n+1} u dx \\ &\quad + \beta \int_\Omega \eta^2 A^\epsilon_{n+l,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta-2}{2}} X^\epsilon_{n+l} u |X_{2n+1} u|^2 dx \quad (5.24) \end{aligned}$$

(by (5.20) and Young's inequality)

$$
\begin{aligned}
&\leq c\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \lambda^\epsilon_{l,\delta}(\nabla_\epsilon u)|\nabla_\epsilon X^\epsilon_l u||X_{2n+1}u|dx \\
&\qquad + c\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)|X_{2n+1}u|^2 dx \\
&\qquad \leq c\tau \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \lambda^\epsilon_{l,\delta}(\nabla_\epsilon u)|\nabla_\epsilon X^\epsilon_l u|^2 dx \\
&\qquad + \frac{c\beta^2}{\tau} \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)|X_{2n+1}u|^2 dx \\
&\qquad + c \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)|X_{2n+1}u|^2 dx \\
&\qquad \leq c\tau \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \lambda^\epsilon_{l,\delta}(\nabla_\epsilon u) \\
&\qquad \left(|\nabla_\epsilon X^\epsilon_l u|^2 + |\nabla_\epsilon X^\epsilon_{n+l} u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1} u|^2\right) dx \\
&\qquad + c\left(1 + \frac{\beta^2}{\tau}\right) \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)|X_{2n+1}u|^2 dx,
\end{aligned}
$$

where again $c = c(q, n, L)$.

If $l \in \{n+1, \ldots, 2n\}$ the same bounds hold for $I^l_{3,2}$, hence:

$$
\begin{aligned}
|\sum_{l=1}^{2n} I^l_{3,2}| &\leq c\tau \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{l=1}^n \lambda^\epsilon_{l,\delta}(\nabla_\epsilon u) \\
&\qquad \left(|\nabla_\epsilon X^\epsilon_l u|^2 + |\nabla_\epsilon X^\epsilon_{n+l} u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1} u|^2\right) dx \\
&\qquad + c\left(1 + \frac{\beta^2}{\tau}\right) \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)|X_{2n+1}u|^2 dx.
\end{aligned}
$$

$I^l_{3,3}$ can be estimated as (5.24).

In the end, for $I_4^l$ it holds

$$|I_4^l| \le \beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta-1}{2}} |X_{2n+1}u| \Bigg( \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)$$

$$\left((X_l^\epsilon X_i^\epsilon u)^2 + (X_l^\epsilon X_{n+i}^\epsilon u)^2 + (X_l^\epsilon X_{2n+1}^\epsilon u)^2\right) \Bigg)^{\frac{1}{2}} \left( \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)|\nabla_\epsilon u|_\epsilon^2 \right)^{\frac{1}{2}} dx \le$$

$$c\tau \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}$$

$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left((\nabla_\epsilon X_i^\epsilon u)^2 + (\nabla_\epsilon X_{n+i}^\epsilon u)^2 + (\nabla_\epsilon X_{2n+1}^\epsilon u)^2\right) dx$$

$$+ \frac{c\beta^2}{\tau} \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)|X_{2n+1}u|^2 dx,$$

and the thesis follows. □

In order to handle the term

$$\int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)|X_{2n+1}u|^2 dx,$$

we adapt to our case the mixed Caccioppoli-type inequality in Lemma [95, Lemma 3.5]. The first step is to prove the following Caccioppoli-type inequality for $X_{2n+1}^\epsilon u$.

**Lemma 5.2.2** (Caccioppoli for $X_{2n+1}u$)**.** *There exists a constant $c = c(q, n, L) > 0$, independent of $\epsilon$ and $\delta$, such that, for every weak solution $u \in W_\epsilon^{1,q}(\Omega)$ to* (5.22)*, for every $\beta \ge 0$ and for every $\eta \in C_c^\infty(\Omega)$ one has*

$$\int_\Omega \eta^2 |X_{2n+1}u|^\beta$$

$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \Big( (X_i^\epsilon X_{2n+1}u)^2 + (X_{n+i}^\epsilon X_{2n+1}u)^2 + (X_{2n+1}^\epsilon X_{2n+1}u)^2 \Big) dx$$

$$\le \frac{c}{(\beta+1)^2} \int_\Omega |\nabla_\epsilon \eta|^2 |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)|X_{2n+1}u|^2 dx.$$

*Proof.* Let us consider $\psi = X_{2n+1}(\eta^2|X_{2n+1}u|^\beta X_{2n+1}u)$ as test function in the weak formulation.

$$0 = \sum_{i=1}^{2n+1} \int_\Omega A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_i (X_{2n+1}(\eta^2|X_{2n+1}u|^\beta X_{2n+1}u))dx.$$

Since the vector field $X_{2n+1}$ commutes with $X^\epsilon_i$, for any $i = 1, \dots, 2n+1$, an integration by parts gives

$$\int_\Omega \eta^2|X_{2n+1}u|^\beta \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_j X_{2n+1}u X^\epsilon_i X_{2n+1}u dx$$
$$= - \int_\Omega X_{2n+1}u \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_j X_{2n+1}u X^\epsilon_i (\eta^2|X_{2n+1}u|^\beta)dx.$$

By derivating on the right hand side, one get

$$\int_\Omega \eta^2|X_{2n+1}u|^\beta \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_j X_{2n+1}u X^\epsilon_i X_{2n+1}u dx$$
$$\leq \frac{2}{(\beta+1)} \int_\Omega \eta|X_{2n+1}u|^{\beta+1} \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_j X_{2n+1}u X^\epsilon_i \eta dx.$$

By the structure condition (5.21) it follows that the left hand side

$$(LHS) \geq \int_\Omega \eta^2|X_{2n+1}u|^\beta \sum_{i=1}^{n} \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)$$
$$\Big((X^\epsilon_i X_{2n+1}u)^2 + (X^\epsilon_{n+i} X_{2n+1}u)^2 + (X^\epsilon_{2n+1} X_{2n+1}u)^2\Big)dx$$

As for the right hand side, again by the structure condition (5.21)

$$(RHS) \leq \tau \int_\Omega \eta^2|X_{2n+1}u|^\beta \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_j X_{2n+1}u X^\epsilon_i X_{2n+1}u dx$$
$$+ \frac{c_\tau}{(\beta+1)^2} \int_\Omega |X_{2n+1}u|^{\beta+2} \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X^\epsilon_j \eta X^\epsilon_i \eta dx = I_1 + I_2.$$

$I_1$ can be reabsorbed in the left hand side, for $\tau$ small enough while

$$I_2 \leq \frac{c_\tau L}{(\beta+1)^2}\int_\Omega |X_{2n+1}u|^{\beta+2}|\nabla_\epsilon \eta|^2 \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)dx.$$

□

The mixed Caccioppoli-type inequality in this case reads as:

**Lemma 5.2.3.** *There exists a constant $c = c(q, n, L) > 0$, independent of $\epsilon$ and $\delta$, such that, for every weak solution $u \in W^{1,q}_\epsilon(\Omega)$ to (5.22), for every $\alpha, \beta \geq 2$ and for every $\eta \in C^\infty_c(\Omega)$ one has*

$$\int_\Omega \eta^{\alpha+2}|X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\Big(|\nabla_\epsilon X_i^\epsilon u|^2+|\nabla_\epsilon X_{i+n}^\epsilon u|^2+|\nabla_\epsilon X_{2n+1}^\epsilon u|^2\Big)dx \leq$$
$$\leq c_0 \int_\Omega \eta^\alpha \left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)|X_{2n+1}u|^{\beta-2}$$
$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\left(|\nabla_\epsilon X_i^\epsilon u|^2+|\nabla_\epsilon X_{i+n}^\epsilon u|^2+|\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right)dx,$$

*where $c_0 = c(q, n, L)((\alpha+2)^2 + (\beta+1)^2)\|\nabla_\epsilon \eta\|^2_\infty$.*

*If $\alpha = \beta$, then*

$$\int_\Omega \eta^{\beta+2}|X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\Big(|\nabla_\epsilon X_i^\epsilon u|^2+|\nabla_\epsilon X_{i+n}^\epsilon u|^2+|\nabla_\epsilon X_{2n+1}^\epsilon u|^2\Big)dx \leq$$
$$\leq c_0 \int_\Omega \eta^\beta \left(\delta+|\nabla_\epsilon u|^2_\epsilon\right)|X_{2n+1}u|^{\beta-2}$$
$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\left(|\nabla_\epsilon X_i^\epsilon u|^2+|\nabla_\epsilon X_{i+n}^\epsilon u|^2+|\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right)dx,$$

*with $c_0 = 2c(\beta+2)^2\|\nabla_\epsilon \eta\|^2_\infty$.*

*Proof.* Let $\eta \in C^\infty_c(\Omega)$ be a non negative cut-off function, $l \in \{1, \dots, 2n+1\}$ and consider $\psi = X_l^\epsilon(\eta^{\alpha+2}|X_{2n+1}u|^\beta X_l^\epsilon u)$ as test function. Let us compute the derivative

$$X_i^\epsilon(\eta^{\alpha+2}|X_{2n+1}u|^\beta X_l^\epsilon u) = (\alpha+2)\eta^{\alpha+1}X_i^\epsilon \eta |X_{2n+1}u|^\beta X_l^\epsilon u$$
$$+ \beta\eta^{\alpha+2}|X_{2n+1}u|^{\beta-2}X_{2n+1}uX_l^\epsilon u X_i^\epsilon X_{2n+1}u + \eta^{\alpha+2}|X_{2n+1}u|^\beta X_i^\epsilon X_l^\epsilon u.$$

Let us suppose that $l \in \{1, \dots, n\}$, we obtain

$$\sum_{i=1}^{2n+1} \int_\Omega X_l^\epsilon (A_{i,\delta}^\epsilon(\nabla_\epsilon u)) X_l^\epsilon X_i^\epsilon u |X_{2n+1}u|^\beta \eta^{\alpha+2} dx =$$
$$= \int_\Omega X_l^\epsilon (A_{n+l,\delta}^\epsilon(\nabla_\epsilon u)) X_{2n+1}u |X_{2n+1}u|^\beta \eta^{\alpha+2} dx -$$
$$-(\alpha+2) \sum_{i=1}^{2n+1} \int_\Omega X_l^\epsilon (A_{i,\delta}^\epsilon(\nabla_\epsilon u)) X_l^\epsilon u |X_{2n+1}u|^\beta \eta^{\alpha+1} X_i^\epsilon \eta dx +$$
$$+ \int_\Omega X_{2n+1}(A_{n+l,\delta}^\epsilon(\nabla_\epsilon u)) |X_{2n+1}u|^\beta X_l^\epsilon u \eta^{\alpha+2} dx -$$
$$-\beta \sum_{i=1}^{2n+1} \int_\Omega X_l^\epsilon (A_{i,\delta}^\epsilon(\nabla_\epsilon u) X_i^\epsilon X_{2n+1}u X_l^\epsilon u |X_{2n+1}u|^{\beta-2} X_{2n+1}u \eta^{\alpha+2} dx$$
$$= I_1^l + I_2^l + I_3^l + I_4^l. \tag{5.25}$$

From the structure condition (5.21) it follows that

$$(LHS) \geq \int_\Omega \eta^{\alpha+2} |X_{2n+1}u|^\beta$$
$$\sum_{i=1}^{n} \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)((X_l^\epsilon X_i^\epsilon u)^2 + (X_l^\epsilon X_{n+i}^\epsilon u)^2 + (X_l^\epsilon X_{2n+1}^\epsilon u)^2) dx.$$

We will prove that

$$|I_k^l| \leq c\tau \int_\Omega \eta^{\alpha+2} |X_{2n+1}u|^\beta$$
$$\sum_{i=1}^{n} \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left( |\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{n+i}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2 \right) dx$$
$$+ \frac{c(\alpha+2)^2 \|\nabla_\epsilon \eta\|_\infty^2}{\tau} \int_\Omega \eta^\alpha \left( \delta + |\nabla_\epsilon u|_\epsilon^2 \right) |X_{2n+1}u|^{\beta-2}$$
$$\sum_{i=1}^{n} \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left( |\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon|^2 \right) dx, \tag{5.26}$$

for some $c = c(q, n, L) > 0$, for any $k = 1, 2, 3, 4$.

Let us start by estimating $I_4^l$: via the Cauchy Schwartz inequality and the structure condition (5.21) we have

$$|I_4^l| \le \frac{c\tau}{\|\nabla_\epsilon \eta\|_\infty^2} \int_\Omega \eta^{\alpha+4} |X_{2n+1}u|^\beta$$
$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left( (X_i^\epsilon X_{2n+1}u)^2 + (X_{n+i}^\epsilon X_{2n+1}u)^2 + (X_{2n+1}^\epsilon X_{2n+1}u)^2 \right) dx$$
$$+ \frac{c((\beta+1)^2 + (\alpha+2)^2)\|\nabla_\epsilon \eta\|_\infty^2}{\tau} \int_\Omega \eta^\alpha |X_{2n+1}u|^{\beta-2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)$$
$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left( (X_l^\epsilon X_i^\epsilon u)^2 + (X_l^\epsilon X_{n+i}^\epsilon u)^2 + (X_l^\epsilon u X_{2n+1}^\epsilon u)^2 \right) dx,$$

where $c = c(q, n, L) > 0$. By Lemma 5.2.2 the first term can be bounded by

$$\tau c \int_\Omega \eta^{\alpha+2} |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) |X_{2n+1}u|^2 dx. \tag{5.27}$$

Let us observe that, by definition

$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) |X_{2n+1}u|^2 = \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) |X_i^\epsilon X_{n+i}^\epsilon u - X_{n+i}^\epsilon X_i^\epsilon u|^2$$
$$\le \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left( |\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{n+i}^\epsilon u|^2 \right)$$
$$\le \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left( |\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{n+i}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2 \right), \tag{5.28}$$

where in the last inequality we simply added a positive term.

Hence, (5.27) can be bounded by

$$c\tau \int_\Omega \eta^{\alpha+2} |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)$$
$$\left( |\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{n+i}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2 \right) dx, \tag{5.29}$$

which will be reabsorbed in the left hand side for $\tau$ small enough.

Let us estimate $I_3^l$

$$I_3^l = \int_\Omega \sum_{j=1}^{2n} \partial_j A^\epsilon_{n+l,\delta}(\nabla_\epsilon u) X_j^\epsilon X_{2n+1}u |X_{2n+1}u|^\beta X_l^\epsilon u \eta^{\alpha+2} \leq$$

$$\leq \frac{c\tau}{\|\nabla_\epsilon \eta\|^2_\infty} \int_\Omega \eta^{\alpha+4} |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)$$
$$\left((X_i^\epsilon X_{2n+1}u)^2 + (X^\epsilon_{n+i}X_{2n+1}u)^2 + (X^\epsilon_{2n+1}X_{2n+1}u)^2\right) dx +$$
$$+ \frac{c\|\nabla_\epsilon \eta\|^2_\infty}{\tau} \int_\Omega \eta^\alpha |X_{2n+1}u|^\beta \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right) \lambda^\epsilon_{l,\delta}(\nabla_\epsilon u) dx,$$

where $c = c(q, n, L) > 0$. The first term can be handled by using Lemma 5.2.2 and (5.28), hence it can be bounded by (5.29); while for the second we use (5.28)

$$\frac{c\|\nabla_\epsilon \eta\|^2_\infty}{\tau} \int_\Omega \eta^\alpha |X_{2n+1}u|^{\beta-2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right) \lambda^\epsilon_{l,\delta}(\nabla_\epsilon u) |X_{2n+1}u|^2 dx \leq$$
$$\frac{c\|\nabla_\epsilon \eta\|^2_\infty}{\tau} \int_\Omega \eta^\alpha |X_{2n+1}u|^{\beta-2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right) \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) |X_{2n+1}u|^2 dx \leq$$
$$\frac{c(\alpha+2)^2\|\nabla_\epsilon \eta\|^2_\infty}{\tau} \int_\Omega \eta^\beta |X_{2n+1}u|^{\beta-2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)$$
$$\sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X^\epsilon_{i+n} u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1} u|^2\right) dx.$$

Let us estimate

$$I_2^l = -(\alpha+2) \int_\Omega |X_{2n+1}u|^\beta \eta^{\alpha+1} \sum_{i,j=1}^{2n+1} \partial_j A^\epsilon_{i,\delta}(\nabla_\epsilon u) X_l^\epsilon X_j^\epsilon u X_l^\epsilon u X_i^\epsilon \eta dx.$$

From (5.21), Cauchy-Schwartz's and Young's inequalities

$$|I_2^l| \leq \frac{c\tau}{\|\nabla_\epsilon \eta\|^2_\infty} \int_\Omega \eta^{\alpha+2} |X_{2n+1}u|^{\beta+2}$$
$$\sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)((X_i^\epsilon \eta)^2 + (X^\epsilon_{n+i}\eta)^2 + (X^\epsilon_{2n+1}\eta)^2) dx +$$
$$+ \frac{c(\alpha+2)^2\|\nabla_\epsilon \eta\|^2_\infty}{\tau} \int_\Omega \eta^\alpha \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right) |X_{2n+1}u|^{\beta-2}$$
$$\sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X^\epsilon_{i+n} u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1}|^2\right) dx.$$

By using (5.28) the first term can be bounded by

$$c\tau \int_\Omega \eta^{\alpha+2}|X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{n+i}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx.$$

In the end

$$\begin{aligned} I_1^l = &-(\beta+1)\int_\Omega \eta^{\alpha+2}|X_{2n+1}u|^\beta A_{n+l,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon X_{2n+1} u dx - \\ &-(\alpha+2)\int_\Omega \eta^{\alpha+1}|X_{2n+1}u|^\beta X_{2n+1}u A_{n+l,\delta}^\epsilon(\nabla_\epsilon u) X_l^\epsilon \eta dx = I_{1,1}^l + I_{1,2}^l. \end{aligned}$$

As for $I_{1,1}^l$, (5.20), the Young inequality, (5.28) and Lemma 5.2.2 imply that

$$\begin{aligned} |I_{1,1}^l| \leq &(\beta+1)\int_\Omega \eta^{\alpha+2}|X_{2n+1}u|^\beta |X_l^\epsilon X_{2n+1}u| \lambda_{l,\delta}^\epsilon(\nabla_\epsilon u)|X_{n+l}^\epsilon u| dx \leq \\ &\leq \frac{c\tau}{\|\nabla_\epsilon \eta\|_\infty^2}\int_\Omega \eta^{\alpha+4}|X_{2n+1}u|^\beta \lambda_{l,\delta}^\epsilon(\nabla_\epsilon u)(X_l^\epsilon X_{2n+1}u)^2 dx \\ &+ \frac{c(\beta+1)^2\|\nabla_\epsilon\eta\|_\infty^2}{\tau}\int_\Omega \eta^\alpha |X_{2n+1}u|^\beta \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)\lambda_{l,\delta}^\epsilon(\nabla_\epsilon u) dx \\ &\leq \frac{c\tau}{\|\nabla_\epsilon \eta\|_\infty^2}\int_\Omega \eta^{\alpha+4}|X_{2n+1}u|^\beta \\ &\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\left((X_i^\epsilon X_{2n+1}u)^2 + (X_{n+i}^\epsilon X_{2n+1}u)^2 + (X_{2n+1}^\epsilon X_{2n+1}u)^2\right) dx \\ &+ \frac{c(\beta+1)^2\|\nabla_\epsilon\eta\|_\infty^2}{\tau}\int_\Omega \eta^\alpha |X_{2n+1}u|^{\beta-2}\left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)|X_{2n+1}u|^2 dx \\ &\leq c\tau\int_\Omega \eta^{\alpha+2}|X_{2n+1}u|^\beta \\ &\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{n+i}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx \\ &+ \frac{c(\beta+1)^2\|\nabla_\epsilon\eta\|_\infty^2}{\tau}\int_\Omega \eta^\alpha \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)|X_{2n+1}u|^{\beta-2} \\ &\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon|^2\right) dx, \end{aligned}$$

while for $I_{1,2}^l$, (5.20), the Young inequality and (5.28) imply that

$$|I_{1,2}^l| \leq (\alpha+2)\int_\Omega \eta^{\alpha+1}\lambda_{l,\delta}^\epsilon(\nabla_\epsilon u)|\nabla_\epsilon u|_\epsilon|\nabla_\epsilon\eta||X_{2n+1}u|^{\beta+1}dx$$
$$\leq \frac{c\tau}{\|\nabla_\epsilon\eta\|_\infty^2}\int_\Omega \eta^{\alpha+2}\lambda_{l,\delta}^\epsilon(\nabla_\epsilon u)|\nabla_\epsilon\eta|^2|X_{2n+1}u|^{\beta+2}dx$$
$$+\frac{c(\alpha+2)^2\|\nabla_\epsilon\eta\|_\infty^2}{\tau}\int_\Omega \eta^\alpha|X_{2n+1}u|^\beta\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)\lambda_{l,\delta}^\epsilon(\nabla_\epsilon u)dx$$
$$\leq c\tau\int_\Omega \eta^{\alpha+2}|X_{2n+1}u|^\beta\sum_{i=1}^n\lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)|X_{2n+1}u|^2dx$$
$$+\frac{c(\alpha+2)^2\|\nabla_\epsilon\eta\|_\infty^2}{\tau}\int_\Omega \eta^\alpha|X_{2n+1}u|^{\beta-2}\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)\sum_{i=1}^n\lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)|X_{2n+1}u|^2dx$$
$$\leq c\tau\int_\Omega \eta^{\alpha+2}|X_{2n+1}u|^\beta$$
$$\sum_{i=1}^n\lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\left(|\nabla_\epsilon X_i^\epsilon u|^2+|\nabla_\epsilon X_{n+i}^\epsilon u|^2+|\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right)dx$$
$$+\frac{c(\alpha+2)^2\|\nabla_\epsilon\eta\|_\infty^2}{\tau}\int_\Omega \eta^\alpha\left(\delta+|\nabla_\epsilon u|_\epsilon^2\right)|X_{2n+1}u|^{\beta-2}$$
$$\sum_{i=1}^n\lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)\left(|\nabla_\epsilon X_i^\epsilon u|^2+|\nabla_\epsilon X_{i+n}^\epsilon u|^2+|\nabla_\epsilon X_{2n+1}^\epsilon|^2\right)dx.$$

The bound (5.26) holds true even if $l\in\{n+1,\dots,2n\}$.

If $l=2n+1$, the test function reads as $\psi=X_{2n+1}^\epsilon(\eta^{\alpha+2}|X_{2n+1}u|^\beta X_{2n+1}^\epsilon u)$: since $[X_{2n+1}^\epsilon,X_i^\epsilon]=0,\forall i=1,\dots,2n+1$, in (5.25) the terms $I_1^{2n+1}$ and $I_3^{2n+1}$ will disappear. Hence

$$\sum_{i=1}^{2n+1}\int_\Omega X_{2n+1}^\epsilon(A_{i,\delta}^\epsilon(\nabla_\epsilon u))X_{2n+1}^\epsilon X_i^\epsilon u|X_{2n+1}u|^\beta\eta^{\alpha+2}dx=$$
$$-(\alpha+2)\sum_{i=1}^{2n+1}\int_\Omega X_{2n+1}^\epsilon(A_{i,\delta}^\epsilon(\nabla_\epsilon u))X_{2n+1}^\epsilon u|X_{2n+1}u|^\beta\eta^{\alpha+1}X_i^\epsilon\eta dx+$$
$$-\beta\sum_{i=1}^{2n+1}\int_\Omega X_{2n+1}^\epsilon(A_{i,\delta}^\epsilon(\nabla_\epsilon u))X_i^\epsilon X_{2n+1}uX_{2n+1}^\epsilon u|X_{2n+1}u|^{\beta-2}X_{2n+1}u\eta^{\alpha+2}dx$$
$$=I_2^{2n+1}+I_4^{2n+1}.$$

For $I_2^{2n+1}$ and $I_4^{2n+1}$ the bound (5.26) holds. As for the left hand side, the structure condition (5.21) implies that

$$(LHS) \geq \int_\Omega \eta^{\beta+2} |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)$$
$$((X_{2n+1}^\epsilon X_i^\epsilon u)^2 + (X_{2n+1}^\epsilon X_{n+i}^\epsilon u)^2 + (X_{2n+1}^\epsilon X_{2n+1}^\epsilon u)^2) dx. \quad (5.30)$$

Hence, summing for $l = 1$ to $2n+1$ and taking $\tau = \frac{1}{2}$ then the thesis follows. □

**Corollary 5.2.4.** *There exists a constant $c = c(q, n, L) > 0$, independent of $\epsilon$ and $\delta$, such that, for every weak solution $u \in W_\epsilon^{1,q}(\Omega)$ to (5.22), for every $\alpha \geq \beta \geq 2$ and for every $\eta \in C_c^\infty(\Omega)$ one has*

$$\int_\Omega \eta^{\alpha+2} |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \Big(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\Big) dx \leq$$
$$\leq \frac{c^{\frac{\beta}{2}}}{\beta} ((\beta+1)^2 + (\alpha+2)^2)^{\frac{\beta}{2}} \|\nabla_\epsilon \eta\|_\infty^\beta \int_\Omega \eta^{\alpha+2-\beta} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}$$
$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \Big(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\Big) dx.$$

*If $\alpha = \beta$, then*

$$\int_\Omega \eta^{\beta+2} |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \Big(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\Big) dx \leq$$
$$\leq 2^{\frac{\beta}{2}} c^{\frac{\beta}{2}} (\beta+2)^\beta \|\nabla_\epsilon \eta\|_\infty^\beta \int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}$$
$$\sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \Big(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\Big) dx.$$

*Proof.* From Lemma 5.2.3 we know that

$$\int_\Omega \eta^{\alpha+2} |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \Big(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\Big) dx \leq$$
$$\leq c_0 \int_\Omega \eta^{\alpha-\beta+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right) |X_{2n+1}u|^{\beta-2} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u)$$
$$\left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx.$$

The statement follows by applying Hölder's inequality $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ of exponents $p = \frac{\beta}{\beta-2}$ and $p' = \frac{\beta}{2}$ to the right hand-side and representing $\eta^\alpha = \eta^{\frac{(\alpha+2)(\beta-2)}{\beta}} \eta^{\frac{2(\alpha+2-\beta)}{\beta}}$

$$c_0 \int_\Omega \eta^\alpha |X_{2n+1}u|^{\beta-2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right) \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)\Big(|\nabla_\epsilon X^\epsilon_i u|^2 + |\nabla_\epsilon X^\epsilon_{i+n}u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1}u|^2\Big)dx \leq$$

$$\frac{2c_0^{\frac{\beta}{2}}}{\tau^{\beta\frac{\beta-2}{2}}} \int_\Omega \eta^{\alpha+2-\beta} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)\Big(|\nabla_\epsilon X^\epsilon_i u|^2 + |\nabla_\epsilon X^\epsilon_{i+n}u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1}u|^2\Big)dx$$

$$+\tau \int_\Omega \eta^{\alpha+2} |X_{2n+1}u|^\beta \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u)\Big(|\nabla_\epsilon X^\epsilon_i u|^2 + |\nabla_\epsilon X^\epsilon_{i+n}u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1}u|^2\Big)dx.$$

Choosing $\tau = \frac{1}{2}$ the thesis follows. □

Now we are in position to get a uniform (in $\delta$ and $\epsilon$) Caccioppoli-type inequality for $\nabla_\epsilon u$, in which the term containing $X_{2n+1}u$ has disappeared.

**Theorem 5.2.5.** *There exists a constant $c = c(q, n, L) > 0$, independent of $\epsilon$ and $\delta$, such that, for every weak solution $u \in W^{1,q}_\epsilon(\Omega)$ to (5.22), for every $\alpha \geq 0$ and $\beta \geq 2$ and for every $\eta \in C^\infty_c(\Omega)$ one has*

$$\int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(|\nabla_\epsilon X^\epsilon_i u|^2 + |\nabla_\epsilon X^\epsilon_{i+n}u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1}u|^2\right) dx \leq$$

$$\leq c(\beta + \alpha + 2)^{10} K \int_\Omega \eta^\alpha \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta+2}{2}} dx,$$

*where $K = (\|\nabla_\epsilon \eta\|^2_\infty + \|\eta X_{2n+1}\eta\|_\infty)$ and $c = c(q, n, L) > 0$.*

*In particular, for $\alpha = 0$ we get*

$$\int_\Omega \eta^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx \leq$$
$$\leq c(\beta+2)^{10} K \int_{\operatorname{supp}(\eta)} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}} dx.$$

*Proof.* We apply Lemma 5.2.1 with $\eta = \eta_1^{\frac{\alpha+2}{2}}$

$$\int_\Omega \eta_1^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx$$
$$\leq c \left(\frac{\alpha+2}{2}\right)^2 K \int_\Omega \eta_1^\alpha \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}} dx$$
$$+ c(\beta+2)^4 \int_\Omega \eta_1^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) |X_{2n+1} u|^2 dx, \quad (5.31)$$

where $K = \|\eta_1\|_\infty^2 + \|\eta_1 X_{2n+1} \eta_1\|_\infty$. We apply Hölder's inequality $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ with exponents $\frac{\beta+2}{\beta}$ and $\frac{\beta+2}{2}$ to second term, noticing that $\eta_1^{\alpha+2} = \eta_1^{\frac{2(\alpha+\beta+2)}{\beta+2}} \eta_1^{\frac{\alpha\beta}{\beta+2}}$. Hence, calling

$$I = c(\beta+2)^4 \int_\Omega \eta_1^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) |X_{2n+1} u|^2 dx$$

and using (5.28), we obtain

$$I \leq \tau \int_{\Omega} \eta_1^{\alpha+\beta+2} |X_{2n+1}u|^{\beta} \sum_{i=1}^{n} \lambda_{i,\delta}^{\epsilon}(\nabla_\epsilon u) |X_{2n+1}u|^2 dx$$
$$+ \frac{1}{\tau^{\frac{2}{\beta}}} c(\beta+2)^{\frac{4(\beta+2)}{\beta}} \int_{\Omega} \eta_1^{\alpha} \sum_{i=1}^{n} \lambda_{i,\delta}^{\epsilon}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}} dx \leq$$
$$\tau \int_{\Omega} \eta_1^{\alpha+\beta+2} |X_{2n+1}u|^{\beta} \sum_{i=1}^{n} \lambda_{i,\delta}^{\epsilon}(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx$$
$$+ \frac{1}{\tau^{\frac{2}{\beta}}} c(\alpha+\beta+2)^{\frac{4(\beta+2)}{\beta}} \int_{\Omega} \eta_1^{\alpha} \sum_{i=1}^{n} \lambda_{i,\delta}^{\epsilon}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}} dx$$

(we continue by estimating the first term in the right hand side by using Corollary 5.2.4 with $\alpha_1 = \alpha + \beta \geq 2$)

$$\leq \tau c(\alpha+\beta+2)^{\beta} \|\nabla_\epsilon \eta_1\|^{\beta} \int_{\Omega} \eta_1^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}}$$
$$\sum_{i=1}^{n} \lambda_{i,\delta}^{\epsilon}(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{i+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx$$
$$+ \frac{1}{\tau^{\frac{2}{\beta}}} c(\alpha+\beta+2)^{\frac{4(\beta+2)}{\beta}} \int_{\Omega} \eta_1^{\alpha} \sum_{i=1}^{n} \lambda_{i,\delta}^{\epsilon}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}} dx.$$

where $c = c(q, n, L) > 0$. Plugging this estimate to (5.31) and calling

$$J = \int_{\Omega} \eta_1^{\alpha} \sum_{i=1}^{n} \lambda_{i,\delta}^{\epsilon}(\nabla_\epsilon u) \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+2}{2}} dx$$

we get

$$(LHS) \leq c\tau(\alpha+\beta+2)^{\beta} \|\nabla_\epsilon \eta_1\|_\infty^{\beta} (LHS)$$
$$+ \left( \frac{1}{\tau^{\frac{2}{\beta}}} c(\alpha+\beta+2)^{\frac{4(\beta+2)}{\beta}} + c(\alpha+\beta+2)^2 K \right) J.$$

Choosing $\tau = 1/\left(2c(\alpha+\beta+2)^{\beta} \|\nabla_\epsilon \eta\|_\infty^{\beta}\right)$ we obtain

$$(LHS) \leq \overline{c}(\alpha+\beta+2)^{2+\frac{4(\beta+2)}{\beta}} KJ$$

and the thesis follows. □

**Theorem 5.2.6** ($L^\infty$ estimate for $\nabla_\epsilon u$). *Let $2 < q < \infty$ and $u \in W^{1,q}_\epsilon(\Omega)$ be a weak solution of* (5.22). *Then, for any ball $B_\epsilon(0, r)$ such that $B_\epsilon(0, 2r) \subset \Omega$ it holds that*

$$\|\nabla_\epsilon u\|_{L^\infty(B_\epsilon(0,r))} \leq c \left( ⨍_{B_\epsilon(0,2r)} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}},$$

*where $c = c(q, n, L) > 0$.*

*Proof.* For any $i = 1 \ldots, 2n + 1$ let us consider the function

$$\eta \left(\delta + |X^\epsilon_i u|^2\right)^{\frac{q+\beta}{4}}$$

and let us compute

$$\nabla_\epsilon \left(\eta \left(\delta + |X^\epsilon_i u|^2\right)^{\frac{q+\beta}{4}}\right) = \nabla_\epsilon \eta \left(\delta + |X^\epsilon_i u|^2\right)^{\frac{q+\beta}{4}} + \left(\frac{q+\beta}{2}\right) \eta \left(\delta + |X^\epsilon_i u^2|\right)^{\frac{q+\beta-4}{4}} X^\epsilon_i u \nabla_\epsilon X^\epsilon_i u, \quad (5.32)$$

hence

$$\left|\nabla_\epsilon \left(\eta \left(\delta + |X^\epsilon_i u|^2\right)^{\frac{q+\beta}{4}}\right)\right|^2 \leq 2|\nabla_\epsilon \eta|^2 \left(\delta + |X^\epsilon_i u|^2\right)^{\frac{q+\beta}{2}} + \frac{1}{2} (q+\beta)^2 \eta^2 \left(\delta + |X^\epsilon_i u|^2\right)^{\frac{q+\beta-2}{2}} |\nabla_\epsilon X^\epsilon_i u|^2. \quad (5.33)$$

From (5.19) it follows that

$$\left(\delta + |X^\epsilon_i u|^2\right)^{\frac{q-2}{2}} \leq \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \leq \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q-2}{2}}, \quad (5.34)$$

for any $i = 1, \ldots, 2n + 1$, then

$$\begin{aligned}(5.33) &\leq 2|\nabla_\epsilon \eta|^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta}{2}} + \frac{1}{2} (q+\beta)^2 \eta^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) |\nabla_\epsilon X^\epsilon_i u|^2 \\ &\leq 2|\nabla_\epsilon \eta|^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta}{2}} \\ &+ \frac{1}{2} (q+\beta)^2 \eta^2 \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(|\nabla_\epsilon X^\epsilon_i u|^2 + |\nabla_\epsilon X^\epsilon_{n+i} u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1} u|^2\right).\end{aligned}$$

Hence, integrating and summing up the previous inequalities for all $i = 1, \ldots, 2n+1$ we get

$$\sum_{i=1}^{2n+1} \int_\Omega \left| \nabla_\epsilon \left( \eta \left( \delta + |X_i^\epsilon u|^2 \right)^{\frac{q+\beta}{4}} \right) \right|^2 dx \le c \|\nabla_\epsilon \eta\|_\infty^2 \int_{\mathrm{supp}(\eta)} \left( \delta + |\nabla_\epsilon u|_\epsilon^2 \right)^{\frac{q+\beta}{2}} dx$$
$$+ c(q+\beta)^2 \int_\Omega \eta^2 \left( \delta + |\nabla_\epsilon u|_\epsilon^2 \right)^{\frac{\beta}{2}}$$
$$\sum_{i=1}^{n} \lambda_{i,\delta}^\epsilon (\nabla_\epsilon u) \left( |\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{n+i}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2 \right) dx,$$

where $c = c(n, L)$. Applying Theorem 5.2.5

$$\sum_{i=1}^{2n+1} \int_\Omega \left| \nabla_\epsilon \left( \eta \left( \delta + |X_i^\epsilon u|^2 \right)^{\frac{q+\beta}{4}} \right) \right|^2 dx$$
$$\le cK(\beta+q)^{12} \int_{\mathrm{supp}(\eta)} \left( \delta + |\nabla_\epsilon u|_\epsilon^2 \right)^{\frac{q+\beta}{2}} dx, \quad (5.35)$$

where $c = c(q, n, L)$ and $K = \|\nabla_\epsilon \eta\|_\infty^2 + \|\eta X_{2n+1} \eta\|_\infty$.

Moreover, Theorem 1.1.7 implies that

$$\left( \eta \left( \delta + |\nabla_\epsilon u|_\epsilon^2 \right)^{\frac{\beta+q}{4}} \right)^{\frac{2N}{N-2}} \le c \, (2n+1)^{\frac{N}{N-2}(q+\beta)} \sum_{i=1}^{2n+1} \left( \eta \left( \delta + |X_i u|^2 \right)^{\frac{\beta+q}{4}} \right)^{\frac{2N}{N-2}}, \quad (5.36)$$

where $c = c(N, n)$.

Fix two concentric balls $B_\epsilon(0, \tau r) \subset B_\epsilon(0, r)$, $0 < \tau < 1$, and consider a sequence of decreasing radii

$$r_j = \tau r + \frac{r - \tau r}{2^j} \searrow \tau r,$$

and cut-off functions $\eta_j \in C^\infty(B_\epsilon(0, r_j))$ such that $\eta_j \equiv 1$ on $B_\epsilon(0, r_{j+1})$ and

$\|\nabla_\epsilon \eta\|_\infty \leq \frac{c}{r_j - r_{j+1}}$. Using (5.36), the Sobolev's inequality and (5.35), we get

$$\left(\fint_{B_\epsilon(0,r_j)} \left(\eta\left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+q}{4}}\right)^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}$$
$$\leq c'(2n+1)^{q+\beta} \sum_{i=1}^{2n+1} \left(\fint_{B_\epsilon(0,r_j)} \left(\eta\left(\delta + |X_i^\epsilon u|^2\right)^{\frac{\beta+q}{4}}\right)^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}$$
$$\leq c'(2n+1)^{q+\beta} \sum_{i=1}^{2n+1} \fint_{B_\epsilon(0,r_j)} \left|\nabla_\epsilon \left(\eta\left(\delta + |X_i^\epsilon u|^2\right)^{\frac{q+\beta}{4}}\right)\right|^2 dx$$
$$\leq \bar{c}K(\beta+q)^{12}(2n+1)^{q+\beta} \fint_{B_\epsilon(0,r_j)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx,$$

where $\bar{c} = \bar{c}(q,n)$. If we denote by $k = \frac{N}{N-2}$, by using the properties of the cut-off functions $\eta_i$ and (5.34), we get

$$\left(\fint_{B_\epsilon(0,r_{j+1})} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+q}{2}k} dx\right)^{\frac{1}{k}}$$
$$\leq \frac{\bar{c}(\beta+q)^{12}(2n+1)^{q+\beta}}{(1-\tau)^2} \fint_{B_\epsilon(0,r_j)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta+q}{2}} dx. \quad (5.37)$$

Let us choose a sequence of increasing exponents

$$\beta_j = (q+2)k^j - q \geq 2;$$

by using these exponents in (5.37) and by raising both sides to the power $\frac{1}{(q+2)k^j} = \frac{1}{\beta_j + q}$, we get

$$\left(\fint_{B_\epsilon(0,r_{j+1})} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{(q+2)k^{j+1}}{2}} dx\right)^{\frac{1}{(q+2)k^{j+1}}}$$
$$\leq \left(\frac{\bar{c}}{(1-\tau)^2}\right)^{\frac{1}{(q+2)k^j}} \left((q+2)k^j\right)^{\frac{12}{(q+2)k^j}} (2n+1)$$
$$\left(\fint_{B_\epsilon(0,r_j)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{(q+2)k^j}{2}} dx\right)^{\frac{1}{(q+2)k^j}}, \quad (5.38)$$

where we can change the constant $\bar{c}$ by including $2n+1$ in it. We will denote again by $c = c(q,n)$ this new constant. Moreover, if we denote by

$\alpha_j = (q+2)k^j$ and we iterate (5.38), we get

$$\left( \fint_{B_\epsilon(0,r_{m+1})} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\alpha_{m+1}}{2}} dx \right)^{\frac{1}{\alpha_{m+1}}} \leq \left( \frac{c}{(1-\tau)^2} \right)^{\sum_{j=0}^{m} \frac{1}{\alpha_j}} \prod_{j=0}^{m} \alpha_j^{\frac{12}{\alpha_j}} \left( \fint_{B_\epsilon(0,r)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+2}{2}} dx \right)^{\frac{1}{q+2}}$$
$$\leq \left( \frac{c}{(1-\tau)^2} \right)^{\sum_{j=0}^{\infty} \frac{1}{\alpha_j}} \prod_{j=0}^{\infty} \alpha_j^{\frac{12}{\alpha_j}} \left( \fint_{B_\epsilon(0,r)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+2}{2}} dx \right)^{\frac{1}{q+2}}.$$

Now
$$\sum_{j=0}^{\infty} \frac{1}{\alpha_j} = \frac{1}{q+2} \sum_{j=0}^{\infty} \frac{1}{k^j} = \frac{k}{(k-2)(q+2)} = \frac{N}{2(q+2)},$$

and
$$\log(\prod_{j=0}^{\infty} \alpha_j^{\frac{12}{\alpha_j}}) = \sum_{j=0}^{\infty} \frac{12}{\alpha_j} \log \alpha_j = 12 \frac{\log(q+2)}{q+2} \frac{k}{k-1} + \frac{12}{q+2} \log k \sum_{j=0}^{\infty} \frac{j}{k^j},$$

that is a constant depending on $q$ and $N$. We still keep the notation $c = c(N, n, q)$. Hence

$$\left( \fint_{B_\epsilon(0,r_{m+1})} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\alpha_{m+1}}{2}} dx \right)^{\frac{1}{\alpha_{m+1}}} \leq \left( \frac{c}{(1-\tau)^{\frac{N}{q+2}}} \right) \left( \fint_{B_\epsilon(0,r)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+2}{2}} dx \right)^{\frac{1}{q+2}}.$$

Now since $\alpha_m \to \infty$ when $m \to \infty$, and the averages on the left hand side of the previous inequality tend to the essential supremum of the integrand we get

$$\sup_{B_\epsilon(0,\tau r)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{1}{2}} \leq \left( \frac{c}{(1-\tau)^{\frac{N}{q+2}}} \right) \left( \fint_{B_\epsilon(0,r)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+2}{2}} dx \right)^{\frac{1}{q+2}},$$

where $c = c(N, n, q)$ and this holds for all $B_\epsilon(0, r) \subset \Omega$ and for all $0 < \delta < 1$.

Another iteration argument, see for instance [80, Theorem 5.1] or [62, Lemma 3.38] implies that

$$\sup_{B_\epsilon(0,\tau r)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{1}{2}} \leq \left( \frac{c'}{(1-\tau)^{\frac{N}{s}}} \right) \left( \fint_{B_\epsilon(0,r)} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{s}{2}} dx \right)^{\frac{1}{s}}, \quad (5.39)$$

for any $s > 0$, where $c' = c'(n, L, s, q) > 0$. The thesis follows by choosing $s = q$. $\square$

Let us prove the following interpolation inequality that will be useful later.

**Lemma 5.2.7.** *There exists a constant $c = c(q, n, L) > 0$, independent of $\epsilon$ and $\delta$, such that for every $u \in C_c^\infty(\Omega)$, for every $\alpha, \beta \geq 0$ and for every $\eta \in C_c^\infty(\Omega)$, one has*

$$\begin{aligned}
&\int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} dx \\
&\qquad \leq c(\alpha+2)^2 \|u\|_\infty^2 \int \eta^\alpha \left(\eta^2 + |\nabla_\epsilon \eta|_\epsilon^2\right) \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx \\
&\qquad\quad + c(\beta+q+1)^2 \|u\|_\infty^2 \int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \\
&\qquad\quad \sum_{i=1}^n \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) \left(|\nabla_\epsilon X_i^\epsilon u|^2 + |\nabla_\epsilon X_{n+i}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx.
\end{aligned}$$

*Proof.* Let $i \in \{1, \ldots, 2n+1\}$, we write

$$\left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} = \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right),$$

hence, calling $I = \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} dx$,

$$I = \delta \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx + \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} X_i^\epsilon u X_i^\epsilon u dx$$

Now we integrate by parts the second term and we obtain

$$\begin{aligned}
I &= \delta \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx - \int_\Omega X_i^\epsilon \left(\eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} X_i^\epsilon u\right) u dx \\
&= \delta \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx - (\alpha+2) \int_\Omega \eta^{\alpha+1} X_i^\epsilon \eta \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} u X_i^\epsilon u dx \\
&\qquad - (q+\beta) \int_\Omega \eta^{\alpha+2} u \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta-2}{2}} (X_i^\epsilon u)^2 X_i^\epsilon X_i^\epsilon u dx \\
&\qquad - \int_\Omega \eta^{\alpha+2} u \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} X_i^\epsilon X_i^\epsilon u dx \\
&\leq \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx + (\alpha+2) \|u\|_\infty \int_\Omega \eta^{\alpha+1} |\nabla_\epsilon \eta|_\epsilon \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+1}{2}} dx \\
&\qquad + (q+\beta+1) \|u\|_\infty \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} |X_i^\epsilon X_i^\epsilon u| dx =: I_1^i + I_2^i + I_3^i
\end{aligned}$$

The integral $I_1$ can be trivially estimated as

$$I_1^i \le \int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx,$$

while for $I_2^i$ and $I_3^i$ we use Young's inequality. In particular

$$\begin{aligned} I_2^i \le (\alpha+2)\|u\|_\infty \int_\Omega \eta^{\alpha+1} |\nabla_\epsilon \eta|_\epsilon \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{4}} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{4}} dx \\ \le \tau \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} dx \\ + \frac{1}{\tau}(\alpha+2)^2 \|u\|_\infty^2 \int \eta^\alpha |\nabla_\epsilon \eta|_\epsilon^2 \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx. \end{aligned}$$

As for $I_3$, if $i \in \{1, \dots, n\}$

$$\begin{aligned} I_3^i \le \tau \int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} dx \\ + \frac{(\beta+q+1)^2 \|u\|_\infty^2}{\tau} \int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \lambda_{i,\delta}^\epsilon(\nabla_\epsilon u) |\nabla_\epsilon X_i^\epsilon u|^2 dx \\ \le \tau \int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} dx + \frac{(\beta+q+1)^2 \|u\|_\infty^2}{\tau} \\ \int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \\ \sum_{j=1}^n \lambda_{j,\delta}^\epsilon(\nabla_\epsilon u) \left(|\nabla_\epsilon X_j^\epsilon u|^2 + |\nabla_\epsilon X_{j+n}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx; \end{aligned}$$

In the end we get that for any $i \in \{1, \dots, 2n+1\}$

$$\begin{aligned} \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} dx \\ \le (\alpha+2)^2 \|u\|_\infty^2 \int \eta^\alpha \left(\eta^2 + |\nabla_\epsilon \eta|_\epsilon^2\right) \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta}{2}} dx \\ + (\beta+q+1)^2 \|u\|_\infty^2 \int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{\beta}{2}} \\ \sum_{j=1}^n \lambda_{j,\delta}^\epsilon(\nabla_\epsilon u) \left(|\nabla_\epsilon X_j^\epsilon u|^2 + |\nabla_\epsilon X_{n+j}^\epsilon u|^2 + |\nabla_\epsilon X_{2n+1}^\epsilon u|^2\right) dx. \end{aligned}$$

Using the fact that

$$\int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} dx \le c(q,n) \sum_{i=1}^{2n+1} \int_\Omega \eta^{\alpha+2} \left(\delta + |X_i^\epsilon u|_\epsilon^2\right)^{\frac{q+\beta+2}{2}} dx,$$

we get the thesis. □

**Corollary 5.2.8.** *There exists a constant $c = c(q, n, L) > 0$, independent of $\epsilon$ and $\delta$, such that for every weak solution $u \in W^{1,q}_\epsilon(\Omega)$ to (5.22), for every $\beta \geq 2$ and for every $\eta \in C^\infty_c(\Omega)$ one has*

$$\int_\Omega \eta^{q+\beta+2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta+2}{2}} dx \leq c\|u\|^2_\infty (2\beta + q + 2)^{12} \overline{K} \int_\Omega \eta^{q+\beta} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta}{2}} dx,$$

*where $\overline{K} = (\|\nabla_\epsilon \eta\|^2_\infty + \|\eta X_{2n+1}\eta\|_\infty + 1)$.*

*Proof.* The thesis follows by Lemma 5.2.7 and Theorem 5.2.5 with $\alpha = q+\beta$. Indeed, Lemma 5.2.7 implies that

$$\begin{aligned}\int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta+2}{2}} dx &\leq c(\alpha+2)^2 \|u\|^2_\infty \int \eta^\alpha \left(\eta^2 + |\nabla_\epsilon \eta|^2_\epsilon\right) \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta}{2}} dx \\ &\quad + c(\alpha+1)^2 \|u\|^2_\infty \int_\Omega \eta^{\alpha+2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{\beta}{2}} \\ &\quad \sum_{i=1}^n \lambda^\epsilon_{i,\delta}(\nabla_\epsilon u) \left(|\nabla_\epsilon X^\epsilon_i u|^2 + |\nabla_\epsilon X^\epsilon_{n+i} u|^2 + |\nabla_\epsilon X^\epsilon_{2n+1} u|^2\right) dx\end{aligned}$$

(we continue by estimating the second term in the right hand-side by using Theorem 5.2.5)

$$\begin{aligned}&\leq c(\alpha+2)^2 \|u\|^2_\infty \overline{K} \int \eta^\alpha \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta}{2}} dx \\ &\quad + c(\alpha+1)^2 \|u\|^2_\infty (\beta+\alpha+2)^{10} K \int \eta^\alpha \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta}{2}} dx \\ &\leq \overline{c} \|u\|^2_\infty (\beta+\alpha+2)^{12} \overline{K} \int \eta^\alpha \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta}{2}} dx,\end{aligned}$$

where $\overline{c} = \overline{c}(n, q, L)$. □

**Corollary 5.2.9.** *There exists a constant $c = c(q, n, L) > 0$, independent of $\epsilon$ and $\delta$, such that for every weak solution $u \in W^{1,q}_\epsilon(\Omega)$ to (5.22), for every $\beta \geq 2$ and for every $\eta \in C^\infty_c(\Omega)$ one has*

$$\int_\Omega \eta^{q+\beta+2} \left(\delta + |\nabla_\epsilon u|^2_\epsilon\right)^{\frac{q+\beta+2}{2}} dx \leq c\|u\|^{q+\beta+2}_\infty (2\beta + q + 2)^{6(q+\beta+2)} \overline{K}^{\frac{q+\beta+2}{2}}$$

*where $\overline{K} = (\|\nabla_\epsilon \eta\|^2_\infty + \|\eta X_{2n+1}\eta\|_\infty + 1)$.*

### 5.2.2 Lipschitz regularity for solutions

We first state the weak comparison principle for weak solutions to (5.18). This will be useful in the proof of the $L^\infty$ estimate for the gradient of solutions.

**Lemma 5.2.10** (Weak comparison principle)**.** *Let $u, v$ be two weak solutions to* (5.18) *in a sub-Riemannian ball $B$. If $u \leq v$ on the boundary $\partial B$, then it holds that $u \leq v$ in $B$.*

*Proof.* Since $u, v$ are weak solutions to (5.18) in $B$, it holds that

$$\int_B \langle D\mathcal{G}^*(\nabla_H u), \nabla_H \psi\rangle dx = 0, \quad \forall \psi \in HW_0^{1,p}(B),$$

and

$$\int_B \langle D\mathcal{G}^*(\nabla_H v), \nabla_H \psi\rangle dx = 0, \quad \forall \psi \in HW_0^{1,p}(B).$$

By subtracting the previous equalities and by choosing $\psi = (u-v)_+ = \max\{u-v, 0\} \in HW_0^{1,p}(B)$

$$\begin{aligned}
0 &= \int_{B\cap\{u>v\}} \langle D\mathcal{G}^*(\nabla_H u) - D\mathcal{G}^*(\nabla_H v), \nabla_H u - \nabla_H v\rangle dx \\
&= \int_{B\cap\{u>v\}} \langle \int_0^1 \frac{d}{dt}\left(D\mathcal{G}^*\left(t\nabla_H u + (1-t)\nabla_H v\right)\right) dt, \nabla_H u - \nabla_H v\rangle dx \\
&= \int_{B\cap\{u>v\}} \langle \int_0^1 D^2\mathcal{G}^*\left(t\nabla_H u + (1-t)\nabla_H v\right)\left(\nabla_H u - \nabla_H v\right) dt, \nabla_H u - \nabla_H v\rangle dx \\
&= \int_{B\cap\{u>v\}} \langle D^2\mathcal{G}^*\left(\bar{t}\nabla_H u + (1-\bar{t})\nabla_H v\right)\left(\nabla_H u - \nabla_H v\right), \nabla_H u - \nabla_H v\rangle dx \\
&\geq \int_{B\cap\{u>v\}} \sum_{i=1}^n \lambda_i(\bar{t}\nabla_H u + (1-\bar{t})\nabla_H v)\left((X_i u - X_i v)^2 + (X_{n+i} u - X_{n+i} v)^2\right) dx
\end{aligned}$$

Hence, either $\mathcal{L}^{2n+1}\left(B \cap \{u > v\}\right) = 0$ or $X_i u = X_i v$ a.e. in $B \cap \{u > v\}$, for every $i \in \{1, \ldots, 2n\}$. If the latter equality holds true, then $u - v$ is constant a.e. in $B$: since $(u-v)_+ \in HW_0^{1,p}(B)$, then $u - v = 0$ a.e. in $B$. Otherwise, $\mathcal{L}^{2n+1}\left(B \cap \{u > v\}\right) = 0$, which precisely means that $u \leq v$ a.e. in $B$. $\square$

**Theorem 5.2.11** ($L^\infty$ estimate for the gradient of solutions)**.** *Let $2 \le q < \infty$ and $u \in HW^{1,q}(\Omega)$ be a weak solution of* (5.18)*, then $\nabla_H u \in L^\infty_{loc}(\Omega)$. Moreover, for any ball $B(0,2r) \subset \Omega$ it holds that*

$$\|\nabla_H u\|_{L^\infty(B(0,r))} \le c \left( \fint_{B(0,2r)} |\nabla_H u|_H^q dx \right)^{\frac{1}{q}},$$

*where $c = c(q,n,L) > 0$.*

*Proof.* Let us first note that from [42, Theorem 3.4] we known that $u \in L^\infty_{loc}(\Omega)$.

Let $B(0,r_0) \subset \Omega$, such that $B(0,2r_0) \subset \Omega$ and let $u^\epsilon$ be the unique weak solution to

$$\begin{cases} \sum_{i=1}^{2n+1} X_i^\epsilon(A_{i,\delta}^\epsilon(\nabla_\epsilon u^\epsilon)) = 0 & \text{in } B(0,r_0), \\ u^\epsilon = u & \text{in } \partial B(0,r_0). \end{cases} \tag{5.40}$$

Then, $u^\epsilon$ is $C^\infty$ on the sub-Riemannian ball $B(0,r_0)$. From the weak maximum principle it follows that $(u^\epsilon)_{\epsilon>0}$ is uniformly bounded in $\epsilon$ in $L^\infty(B(0,r_0))$. By Corollary 5.2.9 the sequence $(\nabla_\epsilon u^\epsilon)_{\epsilon>0}$ is bounded in $L^q_{loc}(\Omega)$, uniformly in $\epsilon$ and $\delta$. Moreover, Theorem 5.2.6 implies that for any ball $B_\epsilon(0,2r) \subset B(0,r_0)$ it holds that

$$\|\nabla_\epsilon u^\epsilon\|_{L^\infty(B_\epsilon(0,r))} \le c \left( \fint_{B_\epsilon(0,2r)} \left(\delta + |\nabla_\epsilon u^\epsilon|_\epsilon^2\right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}}, \tag{5.41}$$

where $c = c(q,n,L) > 0$. Hence, from the previous remark there exists $M_1 > 0$, independent of $\epsilon$ and $\delta$, such that

$$\|\nabla_\epsilon u^\epsilon\|_{L^\infty(B_\epsilon(0,r))} \le M_1$$

Ascoli-Arzela's Theorem implies that $u^\epsilon \to u_0$ uniformly on compact subsets of $B(0,r_0)$ to a function $u_0$. Then, up to subsequence, $\nabla_\epsilon u^\epsilon \overset{*}{\rightharpoonup} \nabla_H u_0 \in L^\infty(B(0,\tau r))$, for any $B(0,\tau r) \subseteq B_\epsilon(0,r)$, for every $\epsilon$. Moreover Theorem 5.2.5 and Corollary 5.2.9 imply that there exists $M_2 > 0$, independent of $\epsilon$ and $\delta$, such that

$$\|\nabla_\epsilon \left(|X_i^\epsilon u^\epsilon|^{q+\beta}\right)\|_{L^2(B_\epsilon(0,r))} \le M_2, \quad \forall \beta \ge 2.$$

It follows that $|X_i^\epsilon u^\epsilon|^{q+\beta}$ converges in $L^r(B(0,\tau r))$, up to subsequences, for any $r < 2^*$. Hence, $X_i^\epsilon u^\epsilon$ converges in $L^s(B(0,\tau r))$ for any $s \in (1,\infty)$. It follows that $X_i^\epsilon u^\epsilon$ admits a subsequence converging pointwise a.e., hence $\nabla_\epsilon u \to \nabla_H u_0$ a.e. and we can pass to the limit in

$$\|\nabla_\epsilon u^\epsilon\|_{L^\infty(B(0,\tau r))} \leq c \left( \fint_{B(0,2\tau r)} \left(\delta + |\nabla_\epsilon u^\epsilon|_\epsilon^2\right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}},$$

both in $\epsilon$ and $\delta$, and get

$$\|\nabla_H u_0\|_{L^\infty(B(0,\tau r))} \leq c \left( \fint_{B(0,2\tau r)} |\nabla_H u_0|^q dx \right)^{\frac{1}{q}},$$

and using the fact that

$$\left(A_{1,\delta}^\epsilon(\xi^\epsilon), \dots, A_{2n+1,\delta}^\epsilon(\xi^\epsilon)\right) \underset{\epsilon,\delta\to 0^+}{\longrightarrow} \left(D_1\mathcal{G}^*(\xi_1,\dots,\xi_{2n}), \dots, D_{2n}\mathcal{G}^*(\xi_1,\dots,\xi_{2n}), 0\right),$$

we can take the limit in the weak formulation of (5.40) and we get that $u_0$ is a weak solution to (5.18). The comparison principle implies that $u_0 = u$ in $B(0,r)$. □

**Corollary 5.2.12** (Lipschitzianity)**.** *Let $2 \leq q < \infty$ and $u \in HW^{1,q}(\Omega)$ be a weak solution of (5.18), then $u$ is locally Lipschitz continuous in $\Omega$. Moreover, for any ball $B(0,2r) \subset \Omega$ it holds that*

$$|u(x) - u(y)| \leq c \left( \fint_{B(0,2r)} |\nabla_H u|_H^q dz \right)^{\frac{1}{q}} d_{SR}(x,y), \quad \forall x,y \in B(0,r),$$

*where $c = c(q,n,L) > 0$.*

# Bibliography


[1] A. Agrachev, D. Barilari, and U. Boscain. *A comprehensive introduction to sub-Riemannian geometry*. Vol. 181. Cambridge University Press, 2019.

[2] L. Ambrosio. “Lecture notes on optimal transport problems”. In: *Mathematical aspects of evolving interfaces*. Springer, 2003, pp. 1–52.

[3] L. Ambrosio, E. Brué, D. Semola, et al. *Lectures on optimal transport*. Springer, 2021.

[4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434.

[5] L. Ambrosio and S. Rigot. “Optimal mass transportation in the Heisenberg group”. In: *J. Funct. Anal.* 208.2 (2004), pp. 261–301.

[6] L. Ambrosio and D. Trevisan. “Lecture notes on the DiPerna-Lions theory in abstract measure spaces”. In: *Ann. Fac. Sci. Toulouse Math. (6)* 26.4 (2017), pp. 729–766.

[7] Lorenzo B. and Guillaume C. “On certain anisotropic elliptic equations arising in congested optimal transport: Local gradient bounds”. In: *Advances in Calculus of Variations* 7.3 (2014), pp. 379–407.

[8] Z. Badreddine and L. Rifford. "Measure contraction properties for two-step analytic sub-Riemannian structures and Lipschitz Carnot groups". In: *Annales de l'Institut Fourier*. Vol. 70. 6. 2020, pp. 2303–2330.

[9] J.-B. Baillon and G. Carlier. "From discrete to continuous Wardrop equilibria". In: *Networks and Heterogenous Media* 7.2 (2012), pp. x–xx.

[10] A. Baldi and F. Montefalcone. "The distributional divergence of horizontal vector fields vanishing at infinity on Carnot groups". In: *arXiv preprint arXiv:2210.15490* (2022).

[11] Z. M. Balogh. "Size of characteristic sets and functions with prescribed gradient". In: *Journal für die reine und angewandte Mathematik* 2003.564 (2003), pp. 63–83. DOI: `doi:10.1515/crll.2003.094`.

[12] D. Barilari and L. Rizzi. "Sharp measure contraction property for generalized H-type Carnot groups". In: *Communications in Contemporary Mathematics* 20.06 (2018), p. 1750081.

[13] M. Beckmann. "A continuous model of transportation". In: *Econometrica* 20 (1952), pp. 643–660.

[14] M. Beckmann, C. B. McGuire, and C. B. Winsten. *Studies in the Economics of Transportation*. Tech. rep. 1956.

[15] P. Bernard. "Young measures, superposition and transport". In: *Indiana Univ. Math. J.* 57.1 (2008), pp. 247–275.

[16] S. Bianchini and F. Cavalletti. "The Monge problem for distance cost in geodesic spaces". In: *Comm. Math. Phys.* 318.3 (2013), pp. 615–673.

[17] P. Billingsley. *Convergence of probability measures*. Second. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. x+277.

[18] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007, Vol. I: xviii+500 pp., Vol. II: xiv+575.

[19] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007, pp. xxvi+800.

[20] G. Bouchitté and G. Buttazzo. "Characterization of optimal shapes and masses through Monge-Kantorovich equation". In: *J. Eur. Math. Soc. (JEMS)* 3.2 (2001), pp. 139–168.

[21] G. Bouchitté, G. Buttazzo, and P. Seppecher. "Shape optimization solutions via Monge-Kantorovich equation". In: *C. R. Acad. Sci. Paris Sér. I Math.* 324.10 (1997), pp. 1185–1191.

[22] P. Bousquet and L. Brasco. "C1 regularity of orthotropic p-harmonic functions in the plane". In: *Anal. PDE* 11.4 (2018), pp. 813–854.

[23] P. Bousquet and L. Brasco. "Global Lipschitz continuity for minima of degenerate problems". In: *Mathematische Annalen* 366.3-4 (2016), pp. 1403–1450.

[24] P. Bousquet and L. Brasco. "Lipschitz regularity for orthotropic functionals with nonstandard growth conditions". In: *Revista matemática iberoamericana* 36.7 (2020), pp. 1989–2032.

[25] P. Bousquet, L. Brasco, and V. Julin. "Lipschitz regularity for local minimizers of some widely degenerate problems". In: *arXiv preprint arXiv:1409.2162* (2014).

[26] P. Bousquet, L. Brasco, and C. Leone. “Singular orthotropic functionals with nonstandard growth conditions”. In: *Revista Matemática Iberoamericana* (2023).

[27] P. Bousquet, L. Brasco, C. Leone, and A. Verde. “On the Lipschitz character of orthotropic p-harmonic functions”. In: *Calculus of Variations and Partial Differential Equations* 57 (2018), pp. 1–33.

[28] L. Brasco. “Global $L^\infty$ gradient estimates for solutions to a certain degenerate elliptic equation”. In: *Nonlinear Analysis: Theory, Methods & Applications* 74.2 (2011), pp. 516–531.

[29] L. Brasco and G. Carlier. “Congested traffic equilibria and degenerate anisotropic PDEs”. In: *Dynamic Games and Applications* 3 (2013), pp. 508–522.

[30] L. Brasco, G. Carlier, and F. Santambrogio. “Congested traffic dynamics, weak flows and very degenerate elliptic equations”. In: *J. Math. Pures Appl. (9)* 93.6 (2010), pp. 652–671.

[31] L. Brasco, C. Leone, G. Pisante, and A. Verde. “Sobolev and Lipschitz regularity for local minimizers of widely degenerate anisotropic functionals”. In: *Nonlinear Analysis: Theory, Methods & Applications* 153 (2017), pp. 169–199.

[32] L. Brasco and M. Petrache. “A continuous model of transportation revisited”. In: *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 411.Teoriya Predstavlenii, Dinamicheskie Sistemy, Kombinatornye Metody. XXII (2013), pp. 5–37, 240.

[33] Lorenzo Brasco. “Geodesics and PDE methods in transport models”. PhD thesis. Université Paris Dauphine-Paris IX, 2010.

[34] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry.* Vol. 33. American Mathematical Society, 2022.

[35] O. Calin and D.-C. Chang. *Sub-Riemannian geometry: general theory and examples*. Vol. 126. Cambridge University Press, 2009.

[36] L. Capogna. “Regularity of quasi-linear equations in the Heisenberg group”. In: *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences* 50.9 (1997), pp. 867–889.

[37] L. Capogna and G. Citti. “Regularity for subelliptic PDE through uniform estimates in multi-scale geometries”. In: *Bulletin of Mathematical Sciences* 6 (2016), pp. 173–230.

[38] L. Capogna, G. Citti, and N. Garofalo. “Regularity for a class of quasilinear degenerate parabolic equations in the Heisenberg group”. In: *arXiv preprint arXiv:2001.08303* (2020).

[39] L. Capogna, G. Citti, E. Le Donne, and A. Ottazzi. “Conformality and Q-harmonicity in sub-Riemannian manifolds”. In: *Journal de Mathématiques Pures et Appliquées* 122 (2019), pp. 67–124.

[40] L. Capogna, G. Citti, and X. Zhong. “Lipschitz regularity for solutions of the parabolic $p$-Laplacian in the Heisenberg group”. In: *arXiv preprint arXiv:2106.05998* (2021).

[41] L. Capogna, G. Citti, and X. Zhong. “Regularity theory of quasilinear elliptic and parabolic equations in the Heisenberg group”. In: *Vietnam Journal of Mathematics* (2023), pp. 1–21.

[42] L. Capogna, D. Danielli, and N. Garofalo. “An embedding theorem and the Harnack inequality for nonlinear subelliptic equations”. In: *Communications in Partial Differential Equations* 18.9-10 (1993), pp. 1765–1794.

[43] G. Carlier, C. Jimenez, and F. Santambrogio. “Optimal transportation with traffic congestion and Wardrop equilibria”. In: *SIAM J. Control Optim.* 47.3 (2008), pp. 1330–1350.

[44] G. Carlier and F. Santambrogio. "A continuous theory of traffic congestion and Wardrop equilibria". In: *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 390.Teoriya Predstavlenii, Dinamicheskie Sistemy, Kombinatornye Metody. XX (2011), pp. 69–91, 307–308.

[45] G. Carlier and F. Santambrogio. "A variational model for urban planning with traffic congestion". In: *ESAIM Control Optim. Calc. Var.* 11.4 (2005), pp. 595–613.

[46] F. Cavalletti. "An overview of $L^1$ optimal transportation on metric measure spaces". In: *arXiv preprint arXiv:1809.04859* (2018).

[47] F. Cavalletti and M. Huesmann. "Existence and uniqueness of optimal transport maps". In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 32.6 (2015), pp. 1367–1377.

[48] F. Cavalletti and A. Mondino. "Optimal maps in essentially non-branching spaces". In: *Commun. Contemp. Math.* 19.6 (2017), pp. 1750007, 27.

[49] M. Colombo and A. Figalli. "Regularity results for very degenerate elliptic equations". In: *Flows of Non-smooth Vector Fields and Degenerate Elliptic Equations: with Applications to the Vlasov-Poisson and Semigeostrophic Systems* (2017), pp. 119–157.

[50] G. E Comi and V. Magnani. "The Gauss–Green theorem in stratified groups". In: *Advances in Mathematics* 360 (2020), p. 106916.

[51] B. Dacorogna and J. Moser. "On a partial differential equation involving the Jacobian determinant". In: *Annales de l'Institut Henri Poincaré C, Analyse non linéaire.* Vol. 7. 1. Elsevier. 1990, pp. 1–26.

[52] L. De Pascale and S. Rigot. "Monge's transport problem in the Heisenberg group". In: *Adv. Calc. Var.* 4.2 (2011), pp. 195–227.

[53] A. Domokos, J. J. Manfredi, and D. Ricciotti. “Riemannian approximation in Carnot groups”. In: *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 152.5 (2022), pp. 1139–1154.

[54] I. Ekeland. *Convexity methods in Hamiltonian mechanics*. Vol. 19. Springer Science & Business Media, 2012.

[55] L. C. Evans and W. Gangbo. “Differential equations methods for the Monge-Kantorovich mass transfer problem”. In: *Mem. Amer. Math. Soc.* 137.653 (1999), pp. viii+66.

[56] M. Feldman and R. J. McCann. “Monge’s transport problem on a Riemannian manifold”. In: *Trans. Amer. Math. Soc.* 354.4 (2002), pp. 1667–1697.

[57] M. Feldman and R. J. McCann. “Uniqueness and transport density in Monge’s mass transportation problem”. In: *Calc. Var. Partial Differential Equations* 15.1 (2002), pp. 81–113.

[58] A. Figalli and L. Rifford. “Mass transportation on sub-Riemannian manifolds”. In: *Geom. Funct. Anal.* 20.1 (2010), pp. 124–159.

[59] G. B. Folland and Elias M. Stein. *Hardy spaces on homogeneous groups*. Vol. 28. Mathematical Notes. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982, pp. xii+285.

[60] B. Franchi, R. Serapioni, and F. Serra Cassano. “Rectifiability and perimeter in the Heisenberg group”. In: *Math. Ann.* 321.3 (2001), pp. 479–531.

[61] N. Garofalo and D. Nhieu. “Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces”. In: *Communications on Pure and Applied Mathematics* 49.10 (1996), pp. 1081–1144.

[62] J. Heinonen, T. Kipelainen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Courier Dover Publications, 2018.

[63] Lars Hörmander. "Hypoelliptic second order differential equations". In: *Acta Math.* 119 (1967), pp. 147–171. ISSN: 0001-5962,1871-2509. DOI: 10.1007/BF02392081. URL: https://doi.org/10.1007/BF02392081.

[64] D. Jerison. "The Poincaré inequality for vector fields satisfying Hörmander's condition". In: (1986).

[65] N. Juillet. "Diffusion by optimal transport in Heisenberg groups". In: *Calculus of Variations and Partial Differential Equations* 50.3-4 (2014), pp. 693–721.

[66] N. Juillet. "Geometric inequalities and generalized Ricci bounds in the Heisenberg group". In: *Int. Math. Res. Not. IMRN* 13 (2009), pp. 2347–2373.

[67] P. Lindqvist and D. Ricciotti. "Regularity for an anisotropic equation in the plane". In: *Nonlinear analysis* 177 (2018), pp. 628–636.

[68] V. Magnani and D. Trevisan. "On Lipschitz vector fields and the Cauchy problem in homogeneous groups". In: *Communications in Contemporary Mathematics* 20.05 (2018), p. 1750057.

[69] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*. 91. American Mathematical Soc., 2002.

[70] R. Monti. "Distances, boundaries and surface measures in Carnot-Carathéodory spaces". In: *UTM PhDTS, Department of Mathematics, University of Trento* (2001).

[71] R. Monti. "Some properties of Carnot-Carathéodory balls in the Heisenberg group". In: *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 11.3 (2000), 155–167 (2001).

[72] R. Monti, M. Rickly, et al. "Geodetically convex sets in the Heisenberg group". In: *J. Convex Anal* 12.1 (2005), pp. 187–196.

[73] J. Moser. "On the volume elements on a manifold". In: *Transactions of the American Mathematical Society* 120.2 (1965), pp. 286–294.

[74] S. Mukherjee and Y. Sire. "Regularity of inhomogeneous quasilinear equations on the Heisenberg group". In: *arXiv preprint arXiv:1904.03778* (2019).

[75] S. Mukherjee and X. Zhong. "$C^{1,\alpha}$-regularity for variational problems in the Heisenberg group". In: *Anal. PDE* 14 (2021), pp. 567–594.

[76] A. Nagel, E. M. Stein, and S. Wainger. "Balls and metrics defined by vector fields. I. Basic properties". In: *Acta Math.* 155.1-2 (1985), pp. 103–147.

[77] S. Ohta. "On the measure contraction property of metric measure spaces". In: *Commentarii Mathematici Helvetici* 82.4 (2007), pp. 805–828.

[78] P. Pansu. "Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un". In: *Ann. of Math. (2)* 129.1 (1989), pp. 1–60.

[79] D. Ricciotti. "On the $C^{1,\alpha}$-regularity of p-harmonic functions in the Heisenberg group". In: *Proceedings of the American Mathematical Society* 146.7 (2018), pp. 2937–2952.

[80] D. Ricciotti. *p-Laplace equation in the Heisenberg group: regularity of solutions*. Springer, 2015.

[81] D. Ricciotti. "Regularity of the derivatives of p-orthotropic functions in the plane for $1 < p < 2$". In: *Annales Academiae scientiarum Fennicae. Mathematica.* Vol. 44. 2. 2019, pp. 1093–1099.

[82] L. Rifford. "Ricci curvatures in Carnot groups". In: *Mathematical Control and Related Fields* 3.4 (2013), pp. 467–487.

[83] S. Rigot. "Mass transportation in groups of type H". In: *Communications in Contemporary Mathematics* 7.04 (2005), pp. 509–537.

[84] L. Rizzi. "Measure contraction properties of Carnot groups". In: *Calculus of Variations and Partial Differential Equations* 55 (2016), pp. 1–20.

[85] F. Santambrogio. "A Dacorogna-Moser approach to flow decomposition and minimal flow problems". In: *ESAIM: Proceedings and Surveys* 45 (2014), pp. 265–274.

[86] F. Santambrogio. "Absolute continuity and summability of transport densities: simpler proofs and new estimates". In: *Calculus of variations and partial differential equations* 36.3 (2009), pp. 343–354.

[87] F. Santambrogio. *Optimal transport for applied mathematicians*. Vol. 87. Progress in Nonlinear Differential Equations and their Applications. Calculus of variations, PDEs, and modeling. Birkhäuser/Springer, Cham, 2015, pp. xxvii+353.

[88] F. Santambrogio. "Regularity via duality in calculus of variations and degenerate elliptic PDEs". In: *Journal of Mathematical Analysis and Applications* 457.2 (2018), pp. 1649–1674.

[89] F. Santambrogio and V. Vespri. "Continuity in two dimensions for a very degenerate elliptic equation". In: *Nonlinear Anal.* 73.12 (2010), pp. 3832–3841.

[90] E. M. Stein and T. S. Murphy. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Vol. 3. Princeton University Press, 1993.

[91] G. Strang. "$L^1$ and $L^\infty$ approximation of vector fields in the plane". In: *Nonlinear partial differential equations in applied science* 5 (2000), pp. 273–288.

[92] L. W. Tu. "Manifolds". In: *An Introduction to Manifolds*. Springer, 2011, pp. 47–83.

[93] C. Villani. *Optimal transport. Old and New*. Vol. 338. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2009, pp. xxii+973.

[94] J. G. Wardrop. "Some theoretical aspects of road traffic research." In: *Proceedings of the institution of civil engineers* 1.3 (1952), pp. 325–362.

[95] X. Zhong. "Regularity for variational problems in the Heisenberg group". In: *arXiv preprint arXiv:1711.03284* (2017).