Alma Mater Studiorum - Università di Bologna

DOTTORATO DI RICERCA IN MATEMATICA
Ciclo XXXVI

Settore Concorsuale: 01/A2 - GEOMETRIA E ALGEBRA
Settore Scientifico Disciplinare: MAT/02 - ALGEBRA

THE COMBINATORICS OF HILBERT–POINCARÉ SERIES OF MATROIDS

Presentata da: Lorenzo Vecchi

Coordinatrice Dottorato:
Prof.ssa Valeria Simoncini

Supervisore:
Prof. Luca Moci

Cosupervisore:
Prof. Luca Migliorini

Esame finale anno 2024

Abstract

In this thesis we explore the combinatorial properties of several polynomials arising in matroid theory. Our main motivation comes from the problem of computing them in an efficient way and from a collection of conjectures, mainly the real-rootedness and the monotonicity of their coefficients with respect to weak maps. Most of these polynomials can be interpreted as Hilbert–Poincaré series of graded vector spaces associated to a matroid and thus some combinatorial properties can be inferred via combinatorial algebraic geometry (non-negativity, palindromicity, unimodality); one of our goals is also to provide purely combinatorial interpretations of these properties, for example by redefining these polynomials as poset invariants (via the incidence algebra of the lattice of flats); moreover, by exploiting the bases polytopes and the valuativity of these invariants with respect to matroid decompositions, we are able to produce efficient closed formulas for every paving matroid, a class that is conjectured to be predominant among all matroids. One last goal is to extend part of our results to a higher categorical level, by proving analogous results on the original graded vector spaces via abelian categorification or on equivariant versions of these polynomials.

Introduction

As it often happens, we start with an example.

Example. Let $X = \left(\mathbb{P}^1(\mathbb{C})\right)^n$. This is a compact Kähler manifold, and thus its cohomology ring $\mathrm{H}^\bullet(X, \mathbb{C})$ satisfies the following cohomological theorems.

- *(Poincaré duality)* We have an isomorphism
$$\mathrm{H}^{n-k}(X, \mathbb{C}) \cong \mathrm{H}^{n+k}(X, \mathbb{C})^*.$$

- *(Hard Lefschetz Theorem)* If ω is an ample class in $\mathrm{H}^2(X, \mathbb{Z})$, define the k-Lefschetz operator L_ω^k as the k-intersection product with ω for every $k \leq n$. This gives rise to an isomorphism
$$L_\omega^k : \mathrm{H}^{n-k}(X, x) \cong \mathrm{H}^{n+k}(X, \mathbb{C})$$
for every $0 \leq k \leq n$.

Together with the Hodge-Riemann bilinear relations, these three results are known as the *Hodge package* (or *Kähler package*). The *Hilbert-Poincaré series* of X, i.e.
$$\mathrm{Hilb}(X, \mathbb{C})(x) = \sum_{k \geq 0} \dim \mathrm{H}^{2k}(X, \mathbb{C}) x^k,$$
is the polynomial $Z(x) = (x+1)^n$ (we shift the dimensions because we are only interested in the even degrees, as the cohomology in odd degree is zero). Its coefficients, also known as the *Betti numbers*, are the binomial coefficients $\binom{n}{k}$. The combinatorial properties of this sequence of numbers are overabunding. Here are some of them

- *(Non-negativity).* $\binom{n}{k}$ is a non-negative integer for every k; moreover it is strictly positive for $0 \leq k \leq n$ and zero otherwise.
- *(Symmetry).* $\binom{n}{k} = \binom{n}{n-k}$ for every k.
- *(Unimodality).* $\binom{n}{k} \geq \binom{n}{k-1}$ for every $k \leq \frac{n}{2}$.
- *(Log-concavity).* $\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}$ for every k.
- *(Real-rootedness).* Every root of the polynomial $Z(x) = \sum_{k \geq 0} \binom{n}{k} x^k$ is real.

The previous example poses the following question. Which one is more interesting, the variety X or the polynomial $Z(x)$ naturally associated to it? Surely, the whole geometric structure that one can associate to a variety (or even the graded vector space associated to it) must carry more information. In support of this point of view, the first three combinatorial properties listed above can actually be seen as a consequence of geometry, respectively of the fact that $Z(x)$ is a Hilbert–Poincaré series, and that Poincaré duality and Hard Lefschetz hold. On the other hand, some would argue that proving properties of the binomial coefficients via Hodge theory could be an overkill. Purely combinatorial proofs of all those facts exist and this is why almost everyone has first met the numbers $(1, 2, 1)$ looking at the subsets of $\{1, 2\}$ and not at the cohomology ring of $\mathbb{P}^1 \times \mathbb{P}^1$. In this sense, combinatorics is useful to geometers, as it offers clean methods to perform computations. Lastly, one might overturn the original point of view and argue the following: the symmetry and unimodality of the sequence $\{\binom{n}{k}\}$, is very easy to prove, combinatorially or with direct computations, and it foreshadows a deeper result at the algebraic or geometric level (i.e. the Hodge package); in this sense, a geometer might pose conjectures by taking combinatorial statements and trying to upgrade them.

The lines get even blurrier when we discover that combinatorial results on sequences of integers were first proved as corollaries of geometric statements. This was the case with the long-standing conjecture due to Heron, Rota, and Welsh, now a theorem by Adiprasito, Huh and Katz.

Theorem ([AHK18]). *For every matroid* M*, the sequence of Whitney numbers of the first kind* (ω_i) *is log-concave, i.e.* $\omega_i^2 \geq \omega_{i-1}\omega_{i+1}$.

The proof given by Adiprasito, Huh and Katz heavily relies on a combinatorial version of the very geometric Hodge package. On the other hand, sometimes the Betti numbers of some variety appear to have interesting combinatorial properties, but geometry alone does not have the tools to show that they hold. How would you prove that

$$\left(\dim \mathrm{H}^k(X, \mathbb{C})\right)^2 \geq \dim \mathrm{H}^{k-1}(X, \mathbb{C}) \, \dim \mathrm{H}^{k+1}(X, \mathbb{C})$$

is true when $X = (\mathbb{P}^1(\mathbb{C}))^n$ without knowing that you are talking about the binomial coefficients? How could you even try to investigate how general this statement about log-concavity is for every variety without developing new combinatorials tools?

The goal of this thesis is to make an exposition of the properties of some old and new invariants in matroid theory; these are five polynomials that were introduced and whose properties were proved with a continuous and fruitful interplay of geometry and combinatorics. They are

- The *characteristic polynomial* $\chi_{\mathsf{M}}(x)$ and its reparametrisation, the *Poincaré polynomial* $\pi_{\mathsf{M}}(x) = (-x)^{\operatorname{rk} \mathsf{M}} \chi_{\mathsf{M}}(x^{-1})$,
- The *Z-polynomial* $Z_{\mathsf{M}}(x)$,
- The *Kazhdan–Lusztig polynomial* $P_{\mathsf{M}}(x)$,
- The *Chow polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x)$,
- The *augmented Chow polynomial* $\mathrm{H}_{\mathsf{M}}(x)$.

Each of the five polynomials above can be seen as the Hilbert–Poincaré series of the cohomology of a variety associated to a hyperplane arrangement $\mathcal{A}$, or of an abstract combinatorial version of it. These are, respectively,

- The *complement* of the arrangement $\mathcal{M}(\mathcal{A})$ and the *Orlik-Solomon algebra* $\mathrm{OS}(\mathsf{M})$,
- The *Schubert variety* $Y(\mathcal{A})$ and the *intersection cohomology module* $\mathrm{IH}(\mathsf{M})$,

- The *reciprocal plane* $Y_\emptyset(\mathcal{A})$ and the *stalk at the empty flat* of the intersection cohomology module $\mathrm{IH}(\mathsf{M})_\emptyset$,
- The *De Concini-Procesi wonderful variety* $\underline{X}(\mathcal{A})$ and the *Chow ring* $\underline{\mathrm{CH}}(\mathsf{M})$,
- The *augmented wonderful variety* $X(\mathcal{A})$ and the *augmented Chow ring* $\mathrm{CH}(\mathsf{M})$.

Regarding their relationship with geometry and combinatorics, each of these polynomials has a different story to tell: historically, the characteristic polynomial was introduced combinatorially and then later given a geometric interpretation; the *Kazhdan–Lusztig–Stanley* polynomials, $P_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ were given a combinatorial and geometric interpretation in the same articles they were defined in; the two *Chow polynomials* $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ have been firstly introduced as geometric objects that additionally have interesting combinatorial properties. Regarding the way in which properties of these polynomials were proved, we have again a mixture of both approaches. Since these objects are interesting from a combinatorial point of view, great emphasis is given to trying to reprove theorems that were proved "geometrically" only using combinatorics, or to use geometric theorems to prove new combinatorial results. What one has in the end is a rich intertwined picture, and unravelling this tangle makes the task of creating a coherent exposition more challenging than expected. It now seems that our discussion can either be well-organized or chronologically accurate. We choose to prioritise the former, meaning that some polynomials will not be presented in the same way as they were first introduced.

Our exposition is based on the following articles, all of which were written in the course of the last two years, [FV22], [FNV22], [KNPV23], [FMSV22]. Section 5.2 deals with some of the contents of a work not yet published, joint with Ben Elias, Dane Miyata and Nicholas Proudfoot. We have also included (few) additional results that do not appear in the articles mentioned above.

Outline

We now describe how this thesis is organized. In Chapter 1 we start by laying down all the necessary combinatorial background that we will need: matroids in Section 1.1, posets and their incidence algebras in Section 1.2, polytopes in Section 1.3, and properties of polynomials in Section 1.4. In Chapter 2 we provide a geometric background to motivate the rest of the discussion. We start in Section 2.1 by considering hyperplane arrangements and then building varieties associated to them. In order to obtain combinatorial invariants attached to a matroid in a "geometric way", we extract information from these varieties using only the combinatorics associated to them (i.e. in our case, the *lattice of flats* of the underlying matroid). This lets us build the three main algebraic objects we are interested in, the Orlik–Solomon algebra in Section 2.2.1, the Chow ring in Section 2.2.2 and the intersection cohomology module in Section 2.2.3. In Chapter 3 we focus on the combinatorics of their Hilbert–Poincaré series. In particular, in Section 3.1 we list some open conjectures and show how to obtain answers using the *lattice of flats* and the geometry of *matroid polytope decompositions.* Chapter 4 is dedicated to the study of paving matroids, a predominant class of matroids in which we are able to obtain more interesting results. In Chapter 5 we upgrade the properties of our polynomials to a higher categorical level: Section 5.1 deals with equivariant polynomials of paving matroids, i.e. polynomials whose coefficients are virtual representations of a group of symmetries, while Section 5.2 upgrades the notion of valuativity to functors defined on a new category of matroids with weak maps.

Contents

Chapter 1

Preliminary notions

1.1 Matroid theory

A *matroid* is, loosely speaking, the combinatorial object that abstracts the notion of linear independence. The name comes from *matrix*, as these objects were introduced by Whitney in [Whi35] as a generalization of the columns of a matrix $A \in M_{k\times n}(\mathbb{K})$. The type of matroid arising in this context will be further studied in Sections 1.1.2.2 and 1.1.4.

1.1.1 First definitions

We begin our discussion by giving an abstract definition of matroids with many different equivalent axiomatic systems. For explicit examples we refer to Section 1.1.2. The fact that all these axioms define the same class of objects is not entirely trivial; however, as these results are classical in the literature and for brevity of exposition, we omit the proofs to focus more on how to use these definitions as tools. Our main references are [Oxl11] and [Wel76].

1.1.1.1 Axiomatic systems

Definition 1.1.1.1. A *matroid* M is a pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I} = \mathcal{I}(\mathsf{M}) \subset 2^E$ is a family of subsets of E that satisfies the following three conditions:

I1 $\emptyset \in \mathcal{I}$.

I2 *(Monotonicity)* If $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$.

I3 *(Augmentation)* If $I_1, I_2 \in \mathcal{I}$ with $\#I_2 > \#I_1$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the *independent subsets of* M. Properties $I1$, $I2$ and $I3$ are also called the *independence axioms of matroids.*

When working with matroids, it is customary to identify a matroid with its ground set and speak of *subsets* or *elements* of M, rather than E. Axiom $I2$ tells us that $\mathcal{I}$ is a lower ideal in the poset $(2^E, \subseteq)$. Thus, we can describe it by listing only its maximal elements.

Definition 1.1.1.2. A maximal independent subset of a matroid M is called a *basis*.

The following theorem gives a complete characterization of the family of bases of a matroid.

Theorem 1.1.1.3. *A family of subsets $\mathcal{B} \subset 2^E$ of a finite set E is the family of bases of a matroid M on a ground set E if and only if it satisfies the following two conditions:*

B1 $\mathcal{B} \neq \emptyset$.

B2 (Bases exchange) *If $B_1, B_2 \in \mathcal{B}$ and $B_1 \neq B_2$, for every $e_1 \in B_1 \setminus B_2$, there exists an element $e_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.*

If M has a family of bases $\mathcal{B} = \mathcal{B}(\mathsf{M})$, a set I is independent if and only if there exists a basis $B \in \mathcal{B}$ such that $I \subset B$.

Remark 1.1.1.4. The bases $\mathcal{B}$ of a matroid M will be used further in Section 1.3 to define a polytope associated to a matroid M called the *matroid polytope*.

Definition 1.1.1.5. A set that is not independent is said to be *dependent*. A minimally dependent subset of a matroid M is called a *circuit*.

The following theorem gives a complete characterization of the family of circuits of a matroid.

Theorem 1.1.1.6. *A family of subsets $\mathcal{C} \subset 2^E$ of a finite set E is the family of circuits of a matroid M on a ground set E if and only if it satisfies the following three conditions:*

C1 $\emptyset \notin \mathcal{C}$.

C2 If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subset C_2$, then $C_1 = C_2$.

C3 If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subset (C_1 \cup C_2) \setminus \{e\}$.

If M has a family of circuits $\mathcal{C} = \mathcal{C}(\mathsf{M})$, a set I is independent if and only if it does not contain any circuit $C \in \mathcal{C}$.

Definition 1.1.1.7. An element $e \in E$ is called a *loop* if it is contained in no bases of M. It is called a *coloop* if it is contained in every basis of M. If a set of cardinality 2 is a circuit, i.e. $\{e_1, e_2\} \in \mathcal{C}$, we say that e_1 and e_2 are *parallel*. If a matroid does not contain any loops or parallel elements, it is said to be *simple*.

To study the properties of independence it is also useful to introduce the concept of *rank*.

Definition 1.1.1.8. The *rank function* rk of a matroid M is defined as

$$\begin{aligned} \operatorname{rk} : 2^E &\to \mathbb{Z}_{\geq 0} \\ A &\mapsto \max_{I \in \mathcal{I}} \#(A \cap I). \end{aligned}$$

The following theorem gives a complete characterization of the rank function of a matroid.

Theorem 1.1.1.9. *A function $\operatorname{rk} : 2^E \to \mathbb{Z}_{\geq 0}$ is the rank function of a matroid M on a ground set E if and only if it satisfies the following three conditions:*

R1 For $A \in 2^E$, $0 \leq \operatorname{rk} A \leq \#A$.

R2 (Monotonicity) *If $A_1 \subseteq A_2$, then $\operatorname{rk} A_1 \leq \operatorname{rk} A_2$.*

R3 (Semimodularity) *For every $A_1, A_2 \in 2^E$,*

$$\operatorname{rk} A_1 + \operatorname{rk} A_2 \geq \operatorname{rk}(A_1 \cup A_2) + \operatorname{rk}(A_1 \cap A_2).$$

If M *has a rank function* $\text{rk} = \text{rk}_{\mathsf{M}}$*, a set* I *is independent if and only if* $\text{rk}\, I = \#I$*.*

We call $\text{rk}\,\mathsf{M} := \text{rk}\, E$ the *rank of the matroid*. Notice that this number is also the cardinality of every basis $B \in \mathcal{B}(\mathsf{M})$. Similarly, we define the *corank* of the matroid as $\text{crk}\,\mathsf{M} := \#E - \text{rk}\,\mathsf{M}$ and the corank function of M as $\text{crk}\, A := \text{rk}\,\mathsf{M} - \text{rk}\, A$.

Remark 1.1.1.10. Since all the theorems we have listed so far fully characterize matroids, they could be taken as axiomatic systems to define them. In particular, we can refer to them as *bases axioms*, *circuits axioms* and *rank axioms*, respectively. In matroid theory we say that these sets of axioms are *cryptomorphic*, i.e. they are isomorphic in the sense that they define the same class of objects, but the proof of this fact is not entirely trivial.

1.1.1.2 Flats

We now define in a little more details one last family of subsets of a matroid called *flats*. The family of flats will be needed in Section 1.2.3 in order to build a poset associated to a matroid M called the *lattice of flats*. As we will see, a flat can somehow be seen as a set "closed" under dependence relations.

Definition 1.1.1.11. Let M be a matroid. A subset $F \in 2^E$ is called a *flat* if for any $e \in E \setminus F$

$$\text{rk}(F \cup \{e\}) = \text{rk}\, F + 1.$$

The family of flats of M is denoted $\mathcal{F} = \mathcal{F}(\mathsf{M})$.

Definition 1.1.1.12. If $\text{rk}(A \cup \{e\}) = \text{rk}\, A$, we say that e *depends on* A and we write $e \sim A$. We also define the *closure operator* as the function $\sigma : 2^E \to 2^E$ such that $\sigma(A)$ is the set of all elements in E that depend on A.

Proposition 1.1.1.13. *The following statements are equivalent and give a characterization of the flats of a matroid:*

F1 $F \in \mathcal{F}$.

F2 $\sigma(F) = F$.

F3 If $e \in E \setminus F$*, then* $e \nsim F$.

Remark 1.1.1.14. These statements about flats can be proved directly from the definition of σ.

- $\text{rk}\, A = \text{rk}\, \sigma(A)$.
- If $e \in A$, then $e \sim A$, therefore, $A \subseteq \sigma(A)$ and if $A_1 \subseteq A_2$, then $\sigma(A_1) \subseteq \sigma(A_2)$.

From all the previous considerations, observe that the following statements hold

Proposition 1.1.1.15. *For every matroid* M*, the ground set* E *is a flat and it is maximal by containment. Moreover, the set of all the loops of a matroid is a flat and it is minimal by containment.*

Definition 1.1.1.16. A flat of rank 1 is called an *atom* of M. A flat of corank 1 is called a *hyperplane* (or *coatom*) of M.

Theorem 1.1.1.17. *If* $F_1, F_2 \in \mathcal{F}$*, then* $F_1 \cap F_2 \in \mathcal{F}$*.*

Proof. From the previous remarks $F_1 \cap F_2 \subseteq \sigma(F_1 \cap F_2)$. Conversely, since $F_1 \cap F_2 \subseteq F_1, F_2$, we have that $\sigma(F_1 \cap F_2) \subseteq \sigma(F_1), \sigma(F_2)$ and so

$$\sigma(F_1 \cap F_2) \subseteq \sigma(F_1) \cap \sigma(F_2) = F_1 \cap F_2.$$

Then, by Theorem 1.1.1.13, $F_1 \cap F_2$ is a flat. □

Remark 1.1.1.18. Since $\mathcal{F}$ is closed under taking intersections, we can fully characterize a matroid from its family of hyperplanes. One can show that a family $\mathcal{H} \subset 2^E$ is the family of hyperplanes of a matroid if and only if

- H1 For every distinct $H_1, \mathrm{H}_2 \in \mathcal{H}$, $H_1 \not\subset H_2$.
- H2 For every $e \in E$ and $H_1, H_2 \in \mathcal{H}$ with $e \notin H_1 \cup H_2$, there exists $H_3 \in \mathcal{H}$ such that $(H_1 \cap H_2) \cup \{e\} \subset H_3$.

The family of flats of M then contains E and all the possible intersections of elements in $\mathcal{H}$.

1.1.1.3 Maps of matroids

We are now interested in describing maps between matroids, i.e. maps between ground sets that preserve the matroid structure. These will be useful in Section 5.2 where we build a category of matroids whose morphisms are given by these maps.

Definition 1.1.1.19. Let $\mathsf{M} = (E, \mathcal{I}(\mathsf{M}))$ and $\mathsf{M}' = (E', \mathcal{I}(\mathsf{M}'))$ be matroids on ground sets E and E' with the same cardinality. A map $\varphi : E \to E'$ is called a *weak map* if

$$\varphi^{-1}(I') \in \mathcal{I}(\mathsf{M}),$$

for every $I' \in \mathcal{I}(\mathsf{M}')$. A weak map is *rank preserving* if $\varphi^{-1}(B') \in \mathcal{B}(\mathsf{M})$ for every $B' \in \mathcal{B}(\mathsf{M}')$ (which, of course, implies that $\operatorname{rk} \mathsf{M} = \operatorname{rk} \mathsf{M}'$, hence the name). In addition, we say that φ is an *isomorphism of matroids* if it extends to a bijection

$$\varphi : \mathcal{B}(\mathsf{M}) \to \mathcal{B}(\mathsf{M}').$$

If there exists an isomorphism between M and M' we say that the two matroids are *isomorphic*.

Rank preserving weak maps let us define the following poset (see [Luc75]).

Definition 1.1.1.20. Given the family of matroids over a fixed ground set and of fixed rank, the *weak order* is defined as the poset where $\mathsf{M} \leq \mathsf{M}'$ if and only if there exists a rank preserving weak map $\mathsf{M} \mapsto \mathsf{M}'$. We can also quotient this poset and work on isomorphism classes. The initial object is the uniform matroid $\mathsf{U}_{k,n}$ while the terminal object is the matroid $\mathsf{B}_k \oplus \mathsf{U}_{0,n-k}$ (see Sections 1.1.2 and 1.1.3 for the definition of these matroids).

Remark 1.1.1.21. Definition 1.1.1.19 says that a map φ is a weak map if, after a relabelling of the ground set, M' has more dependencies than M. The map φ is an isomorphism if and only if it preserves, equivalently, the families $\mathcal{I}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{F}$, $\mathcal{H}$ or the rank function rk. An example of weak map that is not rank preserving is built in Section 1.1.3.4.

Remark 1.1.1.22. We can now partition matroids into isomorphism classes. In doing so, it is useful to identify the ground set E with the interval $[n]$. When in Section 1.1.2 we talk about *the* matroid with some properties, we mean the isomorphism class of all matroids with that property. Similarly, in Section 1.1.2.5 we list all isomorphism classes of matroids on ground sets with less than five elements. It is worth noticing that the identification $E \to [n]$ involves a choice: whenever this choice is not wanted, or needed, or the ground set on which the matroid is defined is important not up to isomorphism, we will omit this identification (see also Remark 1.1.2.1).

A matroid is always isomorphic to itself via the identity map id_E. Sometimes, however, a matroid can have non-trivial automorphisms. This leads us to give the following definition.

Definition 1.1.1.23. For a matroid $\mathsf{M} = (E, \mathcal{B}(\mathsf{M}))$, we denote by $\mathrm{Aut}(\mathsf{M}) \leq \mathfrak{S}_E$ the *group of symmetries of* M, i.e. permutations of the ground set E that extend to an action on $\mathcal{B}(\mathsf{M})$.

This definition will be needed firstly in Section 5.1 where we upgrade polynomial invariants of matroids to graded representations of $\mathrm{Aut}(\mathsf{M})$. This will be then used to compute invariants by exploiting the symmetries of different matroids.

1.1.2 Examples

We now give several different examples of (isomorphism classes of) matroids that are going to be useful in the following discussions.

1.1.2.1 Uniform and Boolean matroids

Fix a set E and a non-negative number $k \leq \#E$. The *uniform matroid* of rank k over E, denoted $\mathsf{U}_{k,E}$, is the matroid with

- Family of bases $\mathcal{B}(\mathsf{U}_{k,E}) = \binom{E}{k} := \{B \subseteq E \mid \#B = k\}$,
- Family of circuits $\mathcal{C}(\mathsf{U}_{k,E}) = \binom{E}{k+1}$.

The automorphism group of the uniform matroid $\mathsf{U}_{k,E}$ is $\mathfrak{S}_E$. When $k = \#E$ we denote the matroid B_E and call it the Boolean matroid over E. This is the matroid with only one basis, E, and no circuits.

Remark 1.1.2.1. When the ground set E is not important up to isomorphisms, the isomorphism class of uniform matroids of rank k over n elements will be denoted by $\mathsf{U}_{k,n}$.

1.1.2.2 Matrices

Let $k \leq n$ and $A \in M_{k\times n}(\mathbb{K})$ be a matrix of rank k. The matroid $\mathsf{M} = \mathsf{M}(A)$ associated to A is the matroid with ground set $E = \{\text{columns of } A\}$ and family of bases $\mathcal{B} = \{\text{bases of } \mathrm{Col}(A)\}$, where $\mathrm{Col}(A)$ denotes the vector space spanned by the columns of A. From this example we understand where the terms *independent* and *rank* come from. In fact, $I \subset E$ is independent if and only if the columns are linearly independent over $\mathbb{K}$ and the rank function rk coincides exactly with the notion of rank in linear algebra. Matroids arising in this way are called $\mathbb{K}$-realizable. Realizable matroids will play a fundamental role and will be studied further in Section 1.1.4 with the equivalent notion of *hyperplane arrangements*.

1.1.2.3 Graphic matroids

Consider a multigraph $G = (V, E)$, where E denotes the set of edges. The matroid $\mathsf{M} = \mathsf{M}(G)$ associated to G is the matroid on the ground set E with

- Family of bases $\mathcal{B} = \{\text{spanning forests of } G\}$,

- Family of circuits $\mathcal{C} = \{\text{cycles of } G\}$.

A matroid M is said to be *graphic* if there exists a graph G such that M is isomorphic to $\mathsf{M}(G)$. From this example we understand where the term *circuit* comes from, as the circuits of a graphic matroids correspond to the cycles of the associated graph.

Example 1.1.2.2. Boolean matroids B_n are graphic and can be realized by any tree with n edges. Uniform matroids of corank 1, $\mathsf{U}_{n-1,n}$ are graphic and can be realized by a cycle graph with n edges.

Example 1.1.2.3. It can be shown that the matroid $\mathsf{U}_{2,4}$ is not graphic. See Section 1.1.4 for an explanation.

1.1.2.4 Paving matroids

A matroid M is said to be *paving* if every circuit has cardinality at least $\operatorname{rk}\mathsf{M}$. A matroid M is *sparse paving* if and only if every subset of E of cardinality $\operatorname{rk}\mathsf{M}$ is either a basis or a circuit. This also implies that every sparse paving matroid is paving. Equivalently, a matroid is paving if and only if every flat up to rank $\operatorname{rk}\mathsf{M} - 2$ is independent and it is sparse paving if and only if in addition to that every hyperplane has cardinality either $\operatorname{rk}\mathsf{M}$ or $\operatorname{rk}\mathsf{M} - 1$.

Remark 1.1.2.4. The hyperplanes of a sparse paving matroid of cardinality $\operatorname{rk}\mathsf{M}$ are also circuits. Therefore, they are also known as *circuit-hyperplanes*.

See Proposition 1.1.3.15, Proposition 1.1.3.17 and Proposition 4.1.1.7 for different characterizations. This class of examples is very famous in matroid theory thanks to the following conjecture by Mayhew, Newman, Welsh, and Whittle.

Conjecture 1.1.2.5 ([MNWW11])**.** *The family of sparse paving matroids is predominant among the class of all matroids.*

In other words, it is expected that asymptotically almost all matroids are sparse paving. Paving and sparse paving matroids are the main object of study in Chapter 4 The following statement is trivial.

Proposition 1.1.2.6. *Every uniform matroid is sparse paving.*

Here are some more interesting examples of paving matroids.

Proposition 1.1.2.7. *The graphic matroid on the complete graph K_4 is sparse paving.*

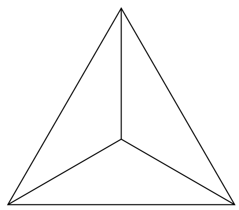

Figure 1.1: The complete graph K_4

Proof. The matroid $\mathsf{M}(K_4)$ has rank 3 and 6 elements. The 3-subsets of $E(K_4)$ are either cycles or spanning trees. □

Definition 1.1.2.8. Consider the point-line configuration in $\mathbb{P}^2$ depicted in Figure 1.2. The *Fano matroid* F_7 is the matroid of rank 3 over the seven vertices where a 3-subset of points is a basis if and only if the points are not colinear.

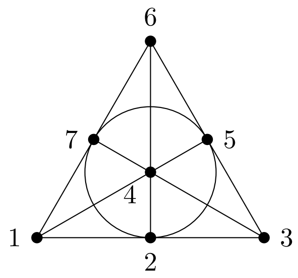

Figure 1.2: A visual representation of the Fano matroid F_7.

Definition 1.1.2.9. Consider the picture in Figure 1.3. The *Vámos matroid* V_8 is the matroid of rank 4 over 8 elements whose bases are all the 4-subsets of vertices that do not form one of the five shaded rectangles. The automorphism group of V_8 is generated by the following four elements:

$$r_1 = (12), \quad s_1 = (17)(28), \quad r_2 = (34), \quad \text{and} \quad s_2 = (35)(46).$$

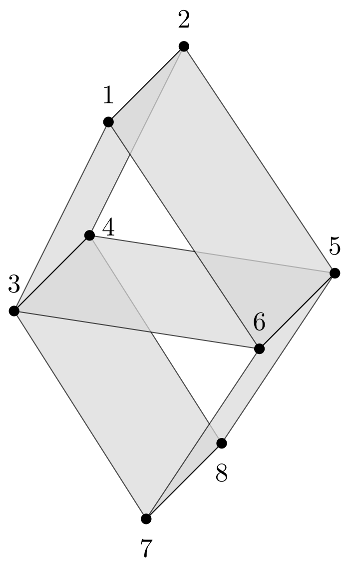

Figure 1.3: A visual representation of the circuit-hyperplanes of the Vámos matroid V_8.

Proposition 1.1.2.10. *The matroids* F_7 *and* V_8 *are sparse paving.*

Remark 1.1.2.11. The matroids F_7 and V_8 are interesting for questions regarding realizability discussed in Section 1.1.4.

Definition 1.1.2.12. A *Steiner system* of type (d, k, n) consists of a set E of cardinality n along with a family $\mathcal{H}$ of k-element subsets (called *blocks*) with the property that every d-subset of E is contained in exactly one block.

A Steiner system $(E, \mathcal{H})$ of type (d, k, n) determines a paving matroid of rank $d + 1$ on the ground set E characterized by the property that $\mathcal{H}$ is the set of hyperplanes [Wel76, Chapter 12.3]. Given a Steiner system $(E, \mathcal{H})$ of type (d, k, n) and an element $e \in E$, one can construct a new Steiner system $(E/e, \mathcal{H}/e)$ of type $(d - 1, k - 1, n - 1)$ by putting $E/e := E \setminus \{e\}$ and $\mathcal{H}/e := \{H \setminus \{e\} \mid H \in \mathcal{H}, e \in H\}$. The notation is consistent with the operations defined in Section 1.1.3.

There is a unique Steiner system of type $(5, 6, 12)$ up to isomorphism, which is typically denoted $S(5, 6, 12)$. The automorphism group of $S(5, 6, 12)$ is the Mathieu group M_{12}. This group acts 4-transitively on the ground set, and the stabilizer of a point is the Mathieu group M_{11}. Thus we may perform the aforementioned operation to obtain a Steiner system $S(4, 5, 11)$ with an action of M_{11}.

There is also a unique Steiner system of type $(5, 8, 24)$ up to isomorphism, which is denoted $S(5, 8, 24)$, and is known as the *Witt geometry.* The automorphism group of $S(5, 8, 24)$ is the Mathieu group M_{24}, which acts 5-transitively on the ground set. The stabilizer of a single point is the Mathieu group M_{23}, which acts on the corresponding Steiner system $S(4, 7, 23)$. The stabilizer of a pair of points is the Mathieu group M_{22}, which acts on the corresponding Steiner system $S(3, 6, 22)$.

1.1.2.5 Matroids on at most five elements

It is enough to list isomorphism classes of connected matroids, as all the other ones can be obtained by performing direct sums. Moreover, since a matroid is connected if and only if its dual is (see Proposition 1.1.5.4), we are going to list only the matroids up to rank $\lceil \frac{n}{2} \rceil$. We list them first by cardinality of the ground set, then by rank and by number of bases. We could have listed the possible families of bases, but it is easier to describe them using operations coming from Section 1.1.3.

$n = 1\ :\ \mathsf{U}_{0,1}$.

$n = 2\ :\ \mathsf{U}_{1,2}$.

$n = 3\ :\ \mathsf{U}_{1,3}$,

$n = 4\ :\ \mathsf{U}_{1,4},\ t(\mathsf{U}_{1,2} \oplus B_2),\ \mathsf{U}_{2,4}$,

$n = 5\ :\ \mathsf{U}_{1,5},\ t(\mathsf{U}_{1,3} \oplus \mathsf{B}_2),\ t(\mathsf{U}_{1,2} \oplus \mathsf{U}_{1,2} \oplus \mathsf{B}_1),\ t(\mathsf{U}_{1,2} \oplus \mathsf{U}_{2,3}),\ \mathsf{U}_{2,5}$.

1.1.3 Classical operations

A natural question is how to build new matroids from known ones. In this section we review some well-known operations. We give the definitions in terms of the rank function for simplicity, but of course each one could be described using all the other axiom systems.

1.1.3.1 Direct sum

Definition 1.1.3.1. Let $\mathsf{M}_1 = (E_1, \mathrm{rk}_1)$ and $\mathsf{M}_2 = (E_2, \mathrm{rk}_2)$ be two matroids. Their *direct sum* is a matroid of rank $\mathrm{rk}\,\mathsf{M}_1 + \mathrm{rk}\,\mathsf{M}_2$

$$\mathsf{M}_1 \oplus \mathsf{M}_2 = (E_1 \sqcup E_2, \mathrm{rk}),$$

where $\operatorname{rk} A := \operatorname{rk}_1(A \cap E_1) + \operatorname{rk}_2(A \cap E_2)$ for every $A \subset E_1 \sqcup E_2$.

Example 1.1.3.2. The Boolean matroid B_n can be written as a direct sum of n copies of B_1.

If a matroid M cannot be written as a direct sum of two non-empty matroids $\mathsf{M} = \mathsf{M}_1 \oplus \mathsf{M}_2$, we say that the matroid is *connected.* Otherwise we say that M is not connected and there is a unique (up to order) way of writing $\mathsf{M} = \bigoplus_i \mathsf{M}_i$ where all the matroids M_i are connected. These are called the *connected components* of M.

Example 1.1.3.3. Each loop and coloop of a matroid M forms a connected component of M. Every uniform matroid $\mathsf{U}_{k,n}$ is connected, except for when $k = 0$, in which case it is a direct sum of n loops, or when $k = n$, in which case it is a direct sum of n coloops, i.e. a Boolean matroid.

1.1.3.2 Minors

Definition 1.1.3.4. Let $\mathsf{M} = (E, \operatorname{rk})$ be a matroid and $S \subset E$. The *restriction* of M to S is a matroid of rank $\operatorname{rk} S$

$$\mathsf{M}_{|S} = (S, \operatorname{rk}_{\mathsf{M}_{|S}}),$$

where $\operatorname{rk}_{\mathsf{M}_{|S}} A := \operatorname{rk} A$ for every $A \subset S$. The restriction to the subset $E \setminus \{e\}$ is also known as the *deletion* of e from M and is denoted $\mathsf{M} \setminus e$.

Example 1.1.3.5. The restriction of a uniform matroid $\mathsf{U}_{k,n}$ to a subset S of cardinality $s \leq k$ is isomorphic to a Boolean matroid B_s. The deletion of any element of the ground set is isomorphic to

$$\mathsf{U}_{k,n} \setminus e \cong \mathsf{U}_{k,n-1}.$$

Definition 1.1.3.6. Let $\mathsf{M} = (E, \operatorname{rk})$ be a matroid and $S \subset E$. The *contraction* of M by S is a matroid of rank $\operatorname{rk} \mathsf{M} - \operatorname{rk} S$

$$\mathsf{M}/S = (E \setminus S, \operatorname{rk}_{\mathsf{M}/S}),$$

where $\operatorname{rk}_{\mathsf{M}/S} A = \operatorname{rk}(A \sqcup S) - \operatorname{rk}(S)$ for every $A \subseteq E \setminus S$.

Example 1.1.3.7. The contraction of a uniform matroid $\mathsf{U}_{k,n}$ by a subset S of cardinality $s \leq k$ is isomorphic to a uniform matroid $\mathsf{U}_{k-s,n-s}$.

Definition 1.1.3.8. We say that N is a *minor* of M if N can be obtained from M by a sequence of restrictions and contractions.

Several theorems in matroid theory are stated in terms of minors. An example appears later in this Section as Theorem 1.1.4.5.

Example 1.1.3.9. Each of the connected components of a matroid M is a minor of M. If $\mathsf{M} = \mathsf{M}_1 \oplus \mathsf{M}_2$, then equivalently $\mathsf{M}_1 = \mathsf{M}_{|E_1}$ or $\mathsf{M}_1 = \mathsf{M}/E_2$.

1.1.3.3 Dualization

Definition 1.1.3.10. Let $\mathsf{M} = (E, \operatorname{rk})$ be a matroid. Its *dual* is a matroid of rank $\#E - \operatorname{rk} \mathsf{M}$

$$\mathsf{M}^* = (E, \operatorname{rk}^*),$$

where $\operatorname{rk}^* A = \operatorname{rk}(E \setminus A) + \#A - \operatorname{rk} \mathsf{M}$. This operation is an involution, i.e. $(\mathsf{M}^*)^* = \mathsf{M}$.

Example 1.1.3.11. The dual of a uniform matroid $\mathsf{U}_{k,n}$ is isomorphic to a uniform matroid $\mathsf{U}_{n-k,n}$.

Example 1.1.3.12. If e is a coloop in M it becomes a loop in M^* and viceversa.

Example 1.1.3.13. The operations of deletion and contraction by an element are dual to each other, i.e.

$$\mathsf{M}/e = (\mathsf{M}^* \setminus e)^*.$$

Theorem 1.1.3.14. *There is a bijection between $\mathcal{C}(\mathsf{M})$ and $\mathcal{H}(\mathsf{M}^*)$ given by*

$$\begin{aligned} \mathcal{C}(\mathsf{M}) &\to \mathcal{H}(\mathsf{M}^*) \\ C &\mapsto E \setminus C \\ E \setminus C &\leftarrow\!\shortmid H. \end{aligned}$$

Proof. If $C \in \mathcal{C}(\mathsf{M})$, then $\operatorname{rk}_{\mathsf{M}}(C) = \#C - 1$ and, for every $e \in C$, $\operatorname{rk}_{\mathsf{M}}(C \setminus \{e\}) = \#C - 1$. By dualizing, this means that

$$\operatorname{rk}_{\mathsf{M}^*}(E \setminus C) = \operatorname{rk}_{\mathsf{M}}(C) + \#(E \setminus C) - \operatorname{rk}_{\mathsf{M}} E = \#E - \operatorname{rk}\mathsf{M} - 1 = \operatorname{rk}\mathsf{M}^* - 1$$

and, for every $e \notin E \setminus C$

$$\operatorname{rk}_{\mathsf{M}^*}((E \setminus C) \cup \{e\}) = \operatorname{rk}_{\mathsf{M}}(C \setminus \{e\}) + \#(E \setminus C \cup \{e\}) - \operatorname{rk}_{\mathsf{M}} E = \#E - \operatorname{rk}\mathsf{M} = \operatorname{rk}\mathsf{M}^*.$$

□

The notion of duality lets us give an equivalent definition for sparse paving matroids.

Proposition 1.1.3.15. *A matroid is sparse paving if and only if it is paving and copaving (i.e. its dual is also paving).*

1.1.3.4 Truncation

Definition 1.1.3.16. Let $\mathsf{M} = (E, \operatorname{rk})$ be a matroid. Its *truncation* is a matroid of rank $\operatorname{rk}\mathsf{M} - 1$

$$t(\mathsf{M}) = (E, \operatorname{rk}'),$$

where $\operatorname{rk}' A := \min\{\operatorname{rk} A, \operatorname{rk}\mathsf{M} - 1\}$. The notation $t^j(\mathsf{M})$ denotes the composition of the truncation j times $t^j(\mathsf{M}) = t(t^{j-1}(\mathsf{M}))$.

The operation of truncation lets us give yet another equivalent definition for paving matroids.

Proposition 1.1.3.17. *A matroid is paving if and only if its truncation is isomorphic to a uniform matroid.*

Remark 1.1.3.18. If M is a matroid on the ground set E the identity on E induces a weak map

$$\mathsf{M} \to t(\mathsf{M}),$$

which is, of course, not rank preserving. It is a weak map since if $I \in \mathcal{I}(t(\mathsf{M}))$ then $\operatorname{rk}' I = \#I$, which implies that $\min\{\operatorname{rk} I, \operatorname{rk}\mathsf{M} - 1\} = \#I$. If I is not independent in M, then $\#I = \operatorname{rk}\mathsf{M} - 1$ and $\operatorname{rk} I = \operatorname{rk}\mathsf{M}$ which is a contradiction.

1.1.4 Realizability

We now consider again matroids coming from matrices defined in Section 1.1.2.2. By taking the orthogonal complement of each of these vectors we might consider the list of the corresponding codimension-1 subspaces. These are known as *hyperplane arrangements*, one of the most famous classes of matroids. A standard reference to study hyperplane arrangements is [OT92]. This class will be exploited in Chapter 2 in order to work on matroids using geometry. Fix a field $\mathbb{K}$ and a $\mathbb{K}$-vector space V. A *hyperplane arrangement* over $\mathbb{K}$ is a finite list of codimension one affine subspaces of V,

$$\mathcal{A} = \{H_1 \dots , H_n\}.$$

If the arrangement is *central*, i.e. the intersection of all the hyperplanes in $\mathcal{A}$ is non-empty, this defines a matroid by setting

$$\begin{aligned} &E = \mathcal{A}, \\ &I \in \mathcal{I} \text{ if and only if } \operatorname{codim} \bigcap_{i \in I} H_i = \# I. \end{aligned}$$

We call this matroid $\mathsf{M}(\mathcal{A})$. It should be clear that the rank function rk is given by the codimension and the flats $\mathcal{F}$ are in bijection with the subspaces obtained by intersecting the hyperplanes in $\mathcal{A}$. If the intersection of all the hyperplanes $V_E := \bigcap_{i \in E} H_i$ is of positive dimension, we can always define an arrangement over V/V_E with the same underlying matroid. We call this arrangement *essential*. The dimension of the ambient space is now equal to the rank of the matroid $\operatorname{rk}\mathsf{M}$.

Definition 1.1.4.1. Given a matroid M, if there exists an arrangement $\mathcal{A}$ over $\mathbb{K}$ such that $\mathsf{M} = \mathsf{M}(\mathcal{A})$, then M is said to be *realizable* over $\mathbb{K}$ or $\mathbb{K}$-realizable. A matroid is *realizable* if it is realizable over some field $\mathbb{K}$ and it is *regular* if it is realizable over every field.

For our purposes it is important to understand if a matroid is realizable (hence there is some geometry we can associate to our combinatorial object) or not.

Example 1.1.4.2. Every graphic matroid is regular. We show that for a connected graph. First we observe that the cardinality of any spanning tree is equal to the number of vertices minus one, hence we can label the vertices of G as $v_1, \dots , v_{\operatorname{rk}\mathsf{M}+1}$. Consider a vector space V over $\mathbb{K}$ with a basis $\{\varphi_1, \dots , \varphi_{\operatorname{rk}\mathsf{M}+1}\}$ of V^*. Fix a spanning tree $B \in \mathcal{B}(\mathsf{M}(G))$ and assign every edge $e = (v_i, v_j)$ in B to the vector $\Phi(e) := \varphi_i - \varphi_j \in V^*$. Then, every edge $e \notin B$ forms a cycle with some elements $e_{i_1}, \dots , e_{i_r}$ of B. Define $\Phi(e) := -\sum_{j=1}^{r} \Phi(e_{i_j})$. Now by defining $H_e = \ker \Phi(e)$ for every edge e we obtain the desired arrangement $\mathcal{A}$. Notice that H_E is 1-dimensional and we can quotient by it to obtain an essential arrangement.

Example 1.1.4.3. The uniform matroid $\mathsf{U}_{2,4}$ is not realizable over $\mathbb{F}_2$, as there are only three distinct lines in $(\mathbb{F}_2)^2$. In particular it is not graphic (See Example 1.1.2.3). The matroid F_7 defined in Definition 1.1.2.8 and its dual F_7^* are realizable over $\mathbb{K}$ if and only if $\operatorname{char}\mathbb{K} = 2$.

Remark 1.1.4.4. For trivial reasons, the matroid $\mathsf{U}_{2,4} \oplus \mathsf{F}_7$ is not realizable over any field. The matroid V_8 defined in 1.1.2.9 is the smallest matroid that is not realizable over *any* field ([BF71]).

Given a matroid M, the realizability problem asks to find every field $\mathbb{K}$ over which the matroid is realizable. This is in general a difficult task. The regularity problem is instead fully solved with a list of excluded minors.

Theorem 1.1.4.5 ([Tut58]). *A matroid is regular if and only if it does not contain any of the forbidden minors* $\mathsf{U}_{2,4}$, F_7 *or* F_7^*.

Of course any matroid containing these as minors is not regular. The converse implication is harder to prove and can be found in [Sey79]. Instead of requiring a matroid to be regular, we can ask for it to be realizable over some field.

Theorem 1.1.4.6 ([Nel18, Theorem 1.1]). *Almost all matroids are not realizable.*

This means that asymptotically, as we let the cardinality of the ground set grow to infinity, the proportion of matroids that are realizable over some field approaches zero. We said that in Chapter 2 we want to build varieties associated to hyperplane arrangements and recover polynomial invariants for *matroids* from them. The previous theorem by Nelson warns us that even if we are able to do that, we then need to find a way to extend those definitions to all the other matroids (which are almost all matroids) where we cannot work with geometric tools.

1.1.5 First examples of polynomial invariants

Given a class of mathematical objects, a classic thing to do is to describe its elements via *invariants*. A matroid invariant is a function f defined on matroids such that if M and M' are isomorphic, then $f(\mathsf{M}) = f(\mathsf{M}')$. An example of invariant is, for example, the rank $\operatorname{rk}\mathsf{M}$ or the cardinality of the ground set. Of course one is interested in computing invariants that carry with them more information. This subsection is devoted to describe some invariants that are very well known in the literature.

1.1.5.1 Tutte polynomial

The *Tutte polynomial* of a matroid is an important invariant that encodes many fundamental features of the matroid. Concretely, the Tutte polynomial of $\mathsf{M} = (E, \operatorname{rk})$ is the bivariate polynomial defined by

$$T_{\mathsf{M}}(x, y) = \sum_{A \subseteq E} (x-1)^{\operatorname{rk}\mathsf{M} - \operatorname{rk} A} (y-1)^{\# A - \operatorname{rk} A}. \tag{1.1}$$

Much of the relevance of this polynomial comes from the fact that it is the most general "deletion-contraction" invariant, that is, every deletion-contraction invariant is a specialization of T_{M}. The following theorem could be taken as an equivalent recursive definition.

Theorem 1.1.5.1 ([Whi86]). *Let* M *be a matroid. If* M *is the empty matroid, set* $T_{\mathsf{M}}(x,y) = 1$. *Otherwise for every* $e \in E$

$$T_{\mathsf{M}}(x,y) = \begin{cases} x\, T_{\mathsf{M}\setminus e}(x,y) & \text{if } e \text{ is a coloop of } \mathsf{M}, \\ y\, T_{\mathsf{M}\setminus e}(x,y) & \text{if } e \text{ is a loop of } \mathsf{M}, \\ T_{\mathsf{M}\setminus e}(x,y) + T_{\mathsf{M}/e}(x,y) & \text{otherwise.} \end{cases}$$

As a direct consequence of this recursion, one has that if M is not empty, then $T_{\mathsf{M}}(0,0) = 0$. From the previous result, one can show by induction that the Tutte polynomial always has non-negative coefficients. Moreover, the following results are easily shown by induction.

Proposition 1.1.5.2. *The Tutte polynomial of the uniform matroid* $\mathsf{U}_{k,n}$ *is*

$$T_{\mathsf{U}_{k,n}}(x,y) = \sum_{i=0}^{k-1} \binom{n}{i} (x-1)^{k-i} + \binom{n}{k} + \sum_{i=k+1}^{n} \binom{n}{i} (y-1)^{i-k}.$$

In particular,

$$T_{\mathsf{B}_n}(x, y) = x^n.$$

Closed formulas for most of the classical operations are also well known.

Theorem 1.1.5.3. *The following equalities for the Tutte polynomial hold:*

- $T_{\mathsf{M}_1 \oplus \mathsf{M}_2}(x, y) = T_{\mathsf{M}_1}(x, y)\, T_{\mathsf{M}_2}(x, y)$.
- $T_{\mathsf{M}^*}(x, y) = T_{\mathsf{M}}(y, x)$.
- $(x - 1)\, T_{t(\mathsf{M})}(x, y) = T_{\mathsf{M}}(x, y) + (xy - x - y)T_{\mathsf{M}}(1, y)$.

1.1.5.2 The β-invariant

In [Cra67, Proposition 3], Crapo proved that if $\#E \geq 2$, then M is connected if and only if the coefficient of the monomial x^1y^0 in the Tutte polynomial of M is strictly positive. The coefficient of this monomial is known in the literature as the *β-invariant* and is denoted by $\beta(\mathsf{M})$. The β-invariant can also be computed with the following recursive formula.

$$\beta(\mathsf{M}) = \begin{cases} 0 & \text{if } \mathsf{M} = \mathsf{U}_{0,1}, \\ 1 & \text{if } \mathsf{M} = \mathsf{U}_{1,1}, \\ \beta(\mathsf{M} \setminus e) + \beta(\mathsf{M}/e) & \text{if } e \text{ is neither a loop nor a coloop.} \end{cases}$$

By inspecting the formulas in Theorem 1.1.5.3 one has

Proposition 1.1.5.4. *The following equalities hold:*

- *If M_1 and M_2 are not empty, then $\beta(\mathsf{M}_1 \oplus \mathsf{M}_2) = 0$.*
- *If $\#E \geq 2$, then $\beta(\mathsf{M}) = \beta(\mathsf{M}^*)$.*

1.1.5.3 Characteristic polynomial

The last polynomial we introduce in this section is the *characteristic polynomial* $\chi_{\mathsf{M}}(x)$. We define it as

$$\chi_{\mathsf{M}}(x) = (-1)^{\operatorname{rk} \mathsf{M}} T_{\mathsf{M}}(1 - x, 0).$$

This gives us directly the formula

$$\begin{aligned} \chi_{\mathsf{M}}(x) &= (-1)^{\operatorname{rk} \mathsf{M}} \sum_{A \subseteq E} (-x)^{\operatorname{rk} \mathsf{M} - \operatorname{rk} A} (-1)^{\#A - \operatorname{rk} A} \\ &= \sum_{A \subseteq E} (-1)^{\#A} x^{\operatorname{rk} \mathsf{M} - \operatorname{rk} A}. \end{aligned}$$

Equivalently, the characteristic polynomial is defined by the following recursion.

Theorem 1.1.5.5. *For every matroid M, the following deletion-contraction formula holds.*

$$\chi_{\mathsf{M}}(x) = \begin{cases} 1 & \textit{if } \mathsf{M} = \mathsf{U}_{0,0} \\ (x - 1)\chi_{\mathsf{M} \setminus e}(x) & \textit{if } e \textit{ is a coloop} \\ \chi_{\mathsf{M} \setminus e}(x) - \chi_{\mathsf{M}/e}(x) & \textit{otherwise.} \end{cases}$$

As a direct consequence of its definition we also notice that

$$\chi_{\mathsf{M}}(1) = (-1)^{\operatorname{rk}\mathsf{M}} T_{\mathsf{M}}(0,0) = 0.$$

This motivates us to define the *reduced characteristic polynomial* as

$$\overline{\chi}_{\mathsf{M}}(x) := \frac{\chi_{\mathsf{M}}(x)}{x-1}.$$

Proposition 1.1.5.6. *The following holds:*

$$\beta(\mathsf{M}) = (-1)^{\operatorname{rk}\mathsf{M}-1}\overline{\chi}_{\mathsf{M}}(1) = (-1)^{\operatorname{rk}\mathsf{M}-1}\left(\frac{d}{dx}\chi_{\mathsf{M}}(x)\right)_{|x=1}.$$

By inspecting the formulas in Theorem 1.1.5.3, one has the following.

Proposition 1.1.5.7. *The following hold:*

- $\chi_{\mathsf{M}_1\oplus\mathsf{M}_2}(x) = \chi_{\mathsf{M}_1}(x)\,\chi_{\mathsf{M}_2}(x)$.
- $x\chi_{t(\mathsf{M})}(x) = \chi_{\mathsf{M}}(x) + (-1)^{\operatorname{rk}\mathsf{M}}\,(x-1)\,\chi_{\mathsf{M}}(0)$.

Lastly, we record here the values of $\chi_{\mathsf{M}}(x)$ and $\overline{\chi}_{\mathsf{M}}(x)$ for uniform matroids.

Proposition 1.1.5.8. *The characteristic polynomial of the uniform matroid $\mathsf{U}_{k,n}$ is*

$$\chi_{\mathsf{U}_{k,n}} = \sum_{i=0}^{k-1}(-1)^i\binom{n}{i}x^{k-i} + (-1)^k\binom{n-1}{k-1}.$$

In particular,

$$\chi_{\mathsf{B}_n}(x) = (x-1)^n.$$

Lemma 1.1.5.9. *The reduced characteristic polynomial of the uniform matroid $\mathsf{U}_{k,n}$ is*

$$\overline{\chi}_{\mathsf{U}_{k,n}}(x) = \sum_{j=0}^{k-1}(-1)^j\binom{n-1}{j}x^{k-1-j}.$$

1.2 Poset theory

The family of flats give a cryptomorphic set of axioms for matroids, i.e. one is able to fully reconstruct the matroid by only knowing $\mathcal{F}(\mathsf{M})$. When they are ordered by inclusion in a poset, they form a lattice known as the *lattice of flats* of the matroid, $\mathcal{L}(\mathsf{M})$. Isomorphic lattices of flats define matroids that are isomorphic up to simplification, i.e. up to removal of loops and parallel elements. In our geometric approach to matroids this is good enough, as in the realizable case loops correspond to identically zero 1-forms (whose kernel is the whole ambient space and not a codimension 1 subspace) and parallel elements correspond to parallel 1-forms (which then define the same hyperplane). Then, to define a matroid invariant we can first work on $\mathcal{L}(\mathsf{M})$ and then extend the definition to non-simple matroids by properly defining the "corner cases". In this section we want to revise the theory of incidence algebras of posets, applied in particular to geometric lattices defined in Section 1.2.3. One of the goals is to redefine the characteristic polynomial defined in Section 1.1.5.3 only in terms of this lattice of flats. Once we are able to obtain this, in Section 1.2.4 we are able to give the definition of several polynomial invariants that will be of both geometric and combinatorial interest.

1.2.1 Incidence algebra

Most of the ideas that we revise in this Section can be found in [Pro18]. For a poset $(P, \leq)$ we use the following notations. The *interval* between u and v is the subposet $[u, v] = \{w \in P \mid u \leq w \leq v\}$. We will only work with *locally finite* posets, i.e. where every interval $[u, v]$ is finite. The *maximal* (resp. *minimal*) elements of P are $\max P = \{u \in P \mid \text{if } u \leq v \text{ (resp. } u \geq v) \text{ then } v = u\}$. Even if it is not needed for the whole section, we will always assume that P has a unique maximal and minimal element, denoted $\widehat{1}$ and $\widehat{0}$, respectively.

Definition 1.2.1.1. The *incidence algebra* of P is defined as

$$I(P) = \prod_{[u,v]} \mathbb{Z}[x].$$

An element $f \in I(P)$ is a function f that maps every interval $[u, v]$ to a polynomial $f([u, v]) := f_{[u,v]}(x)$. We can extend a function $f \in I(P)$ to a function $f : P \times P \to \mathbb{Z}[x]$ by setting $f_{[u,v]}(x) = 0$ whenever $u \not\leq v$.

Since P is locally finite, $I(P)$ admits the following *convolution* product. If f and g are in $I(P)$, $f * g$ is the element of $I(P)$ whose components are given by

$$(f * g)_{[u,v]}(x) = \sum_{u \leq w \leq v} f_{[u,w]}(x) g_{[w,v]}(x).$$

With the convolution product, $I(P)$ becomes an associative algebra. Its unity is given by the function

$$\delta_{[u,v]}(x) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

Theorem 1.2.1.2. *The function δ is the multiplicative unit of $I(P)$.*

Proof. Let $f \in I(P)$. We show that $f * \delta = \delta * f = f$.

$$\begin{aligned}(f * \delta)_{[u,v]}(x) &= \sum_{u \leq w \leq v} f_{[u,w]}(x)\delta_{[w,v]}(x) \\ &= f_{[u,v]}(x)\delta_{[v,v]}(x) + \sum_{u \leq w \lneq v} f_{[u,w]}(x)\delta_{[w,v]}(x) \\ &= f_{[u,v]}(x).\end{aligned}$$

For $(\delta * f)_{[u,v]}(x)$ the proof is analogous. □

One of the easiest functions inside $I(P)$ is the function ζ.

Definition 1.2.1.3. Let P be a locally finite poset. We define $\zeta \in I(P)$ as

$$\zeta_{[u,v]}(x) = 1$$

for every $u \leq v$.

We are now interested in computing inverses with respect to the convolution product. First of all, we observe that left and right inverses coincide.

Proposition 1.2.1.4. *Let f be a function that admits a left and right inverse, i.e. there exist two functions g_1 and g_2 such that $g_1 * f = f * g_2 = \delta$. Then, $g_1 = g_2$.*

Proof. We just need to compute $g_1 * f * g_2$ and exploit the associativity of the product.

$$g_1 = g_1 * \delta = g_1 * (f * g_2) = (g_1 * f) * g_2 = \delta * g_2 = g_2.$$

□

The following result lets us characterize the elements of $I(P)$ that admit a multiplicative inverse.

Theorem 1.2.1.5. *Let $f \in I(P)$. There exists a multiplicative inverse $g = f^{-1}$ if and only if $f_{[u,u]} = \pm 1$ for every $u \in P$.*

Proof. Suppose there exists $g \in I(P)$ such that $f * g = g * f = \delta$. Then,

$$\begin{aligned}1 = \delta_{[u,u]}(x) &= (f * g)_{[u,u]}(x) \\ &= f_{[u,u]}(x) * g_{[u,u]}(x),\end{aligned}$$

meaning that $f_{[u,u]}(x)$ is invertible in $\mathbb{Z}[x]$, i.e. $f_{[u,u]}(x) = \pm 1$. Conversely, we build g by induction by setting $g_{[u,u]}(x) = f_{[u,u]}(x)$ for every u and

$$g_{[u,v]}(x) = -\frac{1}{f_{[v,v]}(x)} \sum_{u \leq w \lneq v} g_{[u,w]}(x) f_{[w,v]}(x).$$

□

Since $\zeta_{[u,u]}(x) = 1$, the function ζ admits an inverse. This is known as the *Möbius function* μ.

Theorem 1.2.1.6. *For every poset P, the function $\mu := \zeta^{-1}$ is the function*

$$\mu_{[u,v]}(x) = \begin{cases} 1 & \text{if } u = v, \\ -\sum_{u \leq w \lneq v} \mu_{[u,w]} & \text{if } u \lneq v. \end{cases}$$

Proof. This follows directly by expanding the convolution $(\mu * \zeta)_{[u,v]}(x)$ as in the proof of Theorem 1.2.1.5. □

1.2.2 Graded lattices

The following definition can be found in [Bre99, Section 2].

Definition 1.2.2.1. A *weak rank function* r is a function $r \in I(P)$ that satisfies the following properties

- $r_{[u,v]} \in \mathbb{Z} \subset \mathbb{Z}[x]$ for every $u \le v$,
- If $u < v$ then $r_{[u,v]} > 0$,
- If $u \le w \le v$ then $r_{[u,w]} + r_{[w,v]} = r_{[u,v]}$.

We say that a poset P is *weakly ranked* if it is a locally finite poset equipped with a weak rank function. If P has a unique minimal element $\widehat{0}$ we can reconstruct the whole weak rank function r from the values $r_u := r_{[\widehat{0},u]}$, which we call *rank of* u. If in addition to that P also has a unique maximal element $\widehat{1}$, we call $r_P := r_{\widehat{1}}$ the *rank of the poset* P.

The incidence algebra of weakly ranked posets has some additional structure that we want to exploit. We denote by $\mathcal{I}(P)$ the subring of functions such that $\deg f_{[u,v]}(x) \le r_{[u,v]}$. This subring is also equipped with an involution

$$\widetilde{f}_{[u,v]}(x) := x^{r_{[u,v]}} f_{[u,v]}(x^{-1}).$$

Definition 1.2.2.2. A function $f \in \mathcal{I}(P)$ is *symmetric* if $\widetilde{f} = f$.

Definition 1.2.2.3. An element $\kappa \in \mathcal{I}(P)$ is a P*-kernel* if $\kappa_{[u,u]} = 1$ and $\kappa^{-1} = \widetilde{\kappa}$.

Definition 1.2.2.4. If P is a weakly ranked poset, its characteristic function $\chi = \chi_P$ can be defined as a function on the incidence algebra

$$\chi = \mu * \widetilde{\zeta},$$

or more explicitly,

$$\chi_{[u,v]}(x) = \sum_{u \le w \le v} \mu_{[u,w]} x^{r_{[w,v]}}.$$

If P has $\widehat{0}$ and $\widehat{1}$, we define *the characteristic polynomial* $\chi_P(x) := \chi_{[\widehat{0},\widehat{1}]}(x)$. From the definition, this is a monic polynomial of degree r_P.

Remark 1.2.2.5. The characteristic function is a P-kernel, as

$$\chi * \widetilde{\chi} = (\mu * \widetilde{\zeta}) * (\widetilde{\mu} * \zeta) = \delta.$$

Theorem 1.2.2.6 ([Pro18, Theorem 2.2])**.** *If κ is a P-kernel, there exist unique functions f and g such that*

- $f_{[u,u]}(x) = g_{[u,u]}(x) = 1$,
- *if* $u < v$, $\deg f_{[u,v]}, \deg g_{[u,v]} < \frac{r_{[u,v]}}{2}$,
- $\widetilde{f} = \kappa * f$ *and* $\widetilde{g} = g * \kappa$.

We call these functions the right (resp. left) Kazhdan–Lusztig–Stanley function *associated to* κ.

Remark 1.2.2.7. The right and left Kazhdan–Lusztig–Stanley functions of κ are both invertible. Their inverse functions are the (unique) functions that satisfy

$$\widetilde{f^{-1}} = f^{-1} * \kappa,$$
$$\widetilde{g^{-1}} = \kappa * g^{-1}.$$

Example 1.2.2.8. If $\kappa = \chi$, the left Kazhdan–Lusztig–Stanley function is trivial, i.e. $g = \zeta$, as

$$\zeta * \chi = \zeta * \mu * \widetilde{\zeta} = \delta * \widetilde{\zeta} = \widetilde{\zeta}.$$

The right Kazhdan–Lusztig–Stanley function of χ is more interesting. For matroids, this is defined in Section 1.2.4.

Definition 1.2.2.9. For every P-kernel κ, we define the associated Z-function to be

$$Z = g * \kappa * f = \widetilde{g} * f = g * \widetilde{f}.$$

From the definition, we quickly see that Z is a symmetric function. As the P-kernels can be seen as generalizations of the characteristic polynomials, we want to introduce a generalized version of the reduced characteristic polynomial.

Theorem 1.2.2.10. *If κ is a P-kernel, then $\kappa_{[u,v]}(1) = 0$ for every $u \lneq v$.*

Proof. Since $\kappa * \widetilde{\kappa} = \delta$, we write

$$\kappa_P(x) + x^{\mathrm{rk}_P} \kappa_P(x^{-1}) = - \sum_{\widehat{0} \lneq u \lneq \widehat{1}} x^{\mathrm{rk}_u} \kappa_{[\widehat{0},u]}(x^{-1}) \kappa_{[u,\widehat{1}]}(x).$$

By induction, since every interval in the sum is non-trivial and $r_u < r_P$ for every $u \neq \widehat{1}$, the whole sum evaluates to 0. Therefore,

$$2\,\kappa_P(1) = 0,$$

from which the result follows. □

This motivates us to define the following

Definition 1.2.2.11. For every poset P with $\widehat{0}$ and $\widehat{1}$ and for every P-kernel κ, the *reduced P-kernel* $\overline{\kappa}$ is the function

$$\overline{\kappa}_{[u,v]}(x) = \begin{cases} -1 & \text{if } u = v \\ \frac{\kappa_{[u,v]}(x)}{x-1} & \text{otherwise.} \end{cases}$$

If $u \neq v$, $\overline{\kappa}_{[u,v]}(x)$ is a polynomial of degree $r_v - r_u - 1$. The value for the corner case $u = v$ is set to be -1 to make computations in Theorem 1.2.4.22 easier.

Example 1.2.2.12. By definition of $\chi_P(x)$,

$$\chi_P(1) = \sum_{u \in P} \mu_{[\widehat{0},u]} = 0,$$

where the sum is zero from Theorem 1.2.1.6. The *reduced characteristic polynomial* $\overline{\chi}_P(x)$ is defined as

$$\overline{\chi}_P(x) := \frac{\chi_P(x)}{x-1}.$$

We also record here a result that will be useful later.

Lemma 1.2.2.13. *The following equalities hold for $\overline{\kappa}$:*

$$(g * \overline{\kappa})_P(x) = \frac{\widetilde{g}_P(x) - x g_P(x)}{x-1},$$

$$(\overline{\kappa} * f)_P(x) = \frac{\widetilde{f}_P(x) - x f_P(x)}{x-1}.$$

Proof. Let us prove the statement for g. Since $(g * \kappa)_P(x) = \widetilde{g}_P(x)$, by adding and subtracting $(x-1)\,(g_P(x)\,\overline{\kappa}_{[\widehat{1},\widehat{1}]})$, we obtain

$$\widetilde{g}_P(x) = g_P(x) + (x-1)\,(g * \overline{\kappa})_P(x) + g_P(x)\,(x-1),$$

from which the statement follows. The proof for f is identical. □

Lemma 1.2.2.14. *If $\kappa = \chi$, then the identity for g reduces to*

$$\sum_{u \in P} \overline{\chi}_{[u,\widehat{1}]} = \frac{x\,(x^{r_P - 1} - 1)}{x-1}.$$

While $\overline{\kappa}$ is clearly not a P-kernel, we observe, thanks to Theorem 1.2.1.5 that it admits an inverse.

Definition 1.2.2.15. The function $\underline{\iota}$ such that

$$\overline{\kappa} * \underline{\iota} = \underline{\iota} * \overline{\kappa} = \delta,$$

is called the *reduced inverse of the P-kernel.*

By expanding the convolution product, one notices that the degree of $\underline{\iota}_P(x)$ is $r_P - 1$, which directly implies that the function is not symmetric.

Example 1.2.2.16. We call the reduced inverse of χ the *Chow function of P* and, if P has a minimal and maximal element, we call

$$\underline{\mathrm{H}}_P(x) = -\underline{\iota}_{[\widehat{0},\widehat{1}]}(x)$$

the *Chow polynomial of P*.

We call this function the Chow function in analogy to what we define later in Section 1.2.4. The polynomial invariant $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ associated to a matroid M is proved to be the Hilbert–Poincaré series of the Chow ring of the matroid.

In analogy with Definition 1.2.2.9, we also give the following definition.

Definition 1.2.2.17. If $\underline{\iota}$ is the reduced inverse of a P-kernel κ, its augmented version is

$$\iota = \widetilde{g} * \underline{\iota},$$

where g is the left Kazhdan–Lusztig–Stanley function of κ.

Example 1.2.2.18. If $\kappa = \chi$, then ι will be called *augmented Chow function* of P. If P has $\widehat{0}$ and $\widehat{1}$, we call

$$\mathrm{H}_P(x) := -\iota_{[\widehat{0},\widehat{1}]}(x).$$

the *augmented Chow polynomial of P*.

1.2.3 Geometric lattices

In this section we show that the family of flats $\mathcal{F}(\mathsf{M})$ of a matroid M has the structure of a lattice called *lattice of flats*.

Definition 1.2.3.1. Let M be a matroid. The poset $\mathcal{L}(\mathsf{M}) = (\mathcal{F}, \subseteq)$ is called the *lattice of flats of* M.

Remark 1.2.3.2. The following properties of $\mathcal{L}(\mathsf{M})$ hold:

- $\mathcal{L}(\mathsf{M})$ is finite with a minimum, $\widehat{0} = \sigma(\emptyset) = \{\text{loops of } \mathsf{M}\}$, and a maximum, $\widehat{1} = E$;
- An element of $\mathcal{L}(\mathsf{M})$ covers $\widehat{0}$ if and only if it is a flat of rank 1;
- An element of $\mathcal{L}(\mathsf{M})$ is covered by $\widehat{1}$ if and only if it is a flat of corank 1;
- A flat F_1 covers another flat F_2 in $\mathcal{L}(\mathsf{M})$ if and only if $F_2 \subseteq F_1$ and $\operatorname{rk} F_1 = \operatorname{rk} F_2 + 1$. This implies that the poset $\mathcal{L}(\mathsf{M})$ is ranked, with rank function equal to the length function of chains.

Theorem 1.2.3.3. *The lattice of flats $\mathcal{L}(\mathsf{M})$ is indeed a lattice. Moreover, the rank is semimodular, that is,*

$$\operatorname{rk}(F_1 \vee F_2) + \operatorname{rk}(F_1 \wedge F_2) \leq \operatorname{rk} F_1 + \operatorname{rk} F_2.$$

Proof. By Theorem 1.1.1.17, the meet of two elements exists and it is well defined,

$$F_1 \wedge F_2 := F_1 \cap F_2.$$

The join is defined as $F_1 \vee F_2 := \sigma(F_1 \cup F_2)$. To prove the semimodularity we use the fact that the rank function is semimodular on every subset and Remark 1.1.1.14 to obtain

$$\begin{aligned} \operatorname{rk}(F_1 \vee F_2) + \operatorname{rk}(F_1 \wedge F_2) &= \operatorname{rk}(\sigma(F_1 \cup F_2)) + \operatorname{rk}(F_1 \cap F_2) \\ &= \operatorname{rk}(F_1 \cup F_2) + \operatorname{rk}(F_1 \cap F_2) \\ &\leq \operatorname{rk} F_1 + \operatorname{rk} F_2. \end{aligned}$$

□

Definition 1.2.3.4. A finite ranked lattice $\mathcal{L}$ is called *geometric* if it is semimodular and *atomistic*, i.e. any element $F \in \mathcal{L}$ can be written as the join of atoms of the lattice,

$$F \in \mathcal{L} \Leftrightarrow F = \bigvee_{\substack{e \text{ atom} \\ e \leq F}} e.$$

Theorem 1.2.3.5. *A finite lattice $\mathcal{L}$ is isomorphic to the lattice of flats $\mathcal{L}(\mathsf{M})$ of a matroid M if and only if it is geometric.*

Proof. We have already proved that $\mathcal{L}(\mathsf{M})$ is semimodular. Let then F be a rank r flat. Then, there exists an independent set $\{e_1, \dots, e_r\} \in \mathcal{I}(\mathsf{M})$ contained in F. Each of its elements is independent as a singleton and $\operatorname{rk}\{e_i, e_j\} = 2$ for $i \neq j$, therefore $e_i \notin \sigma(e_j)$. Hence, the atoms $\sigma(e_1), \dots \sigma(e_r)$ are distinct and

$$F = \bigvee_{i=1}^{r} \sigma(e_i).$$

This proves that $\mathcal{L}(\mathsf{M})$ is geometric. Conversely, let $\mathcal{L}$ be a geometric lattice with rank function h. Define $\operatorname{rk} A := h\left(\bigvee_{\substack{e \text{ atom} \\ e \leq A}} e\right)$. The properties of the rank function then follow directly. □

It is noteworthy to see that M is completely described by $\mathcal{L}(\mathsf{M})$, if we decide to overlook loops and parallel elements.

Theorem 1.2.3.6. *The correspondence between a geometric lattice $\mathcal{L}$ and the matroid $\mathsf{M}(\mathcal{L})$ defined on the family of atoms of $\mathcal{L}$ is a bijection between the family of finite geometric lattices and the family of simple matroids.*

Proof. Let e_1, e_2 be two distinct atoms of a geometric lattice $\mathcal{L}$. If rk is the rank function of $\mathsf{M}(\mathcal{L})$, clearly $\operatorname{rk}\{e_1\} = \operatorname{rk}\{e_2\} = 1$, and $\operatorname{rk}\{e_1, e_2\} = 2$, therefore we can conclude that $\mathsf{M}(\mathcal{L})$ is simple and

$$\mathcal{L}(\mathsf{M}(\mathcal{L})) = \mathcal{L}$$

using Remark 1.1.1.14. Conversely, if $\mathcal{L} = \mathcal{L}(\mathsf{M})$ is the geometric lattice of a simple matroid M, clearly

$$\mathsf{M}(\mathcal{L}(\mathsf{M})) \cong \mathsf{M}.$$

□

From now on, unless otherwise stated, when working on geometric lattices we will always assume that the underlying matroid is simple. In particular, then the minimal element is $\sigma(\emptyset) = \emptyset$ and the set of atoms coincides with the ground set E.

1.2.3.1 Intervals and products

Most of the operations on matroids have a very clear interpretation on $\mathcal{L}(\mathsf{M})$.

Proposition 1.2.3.7. *Let M_1 and M_2 be two matroids on ground sets E_1 and E_2, respectively. Then,*

$$\mathcal{L}(\mathsf{M}_1 \oplus \mathsf{M}_2) = \mathcal{L}(\mathsf{M}_1) \times \mathcal{L}(\mathsf{M}_2),$$

where $\mathcal{L}_1 \times \mathcal{L}_2$ is a poset where $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$.

Proof. We only need to show that the flats of a direct sum are disjoint unions of a flat F_1 of M_1 and a flat F_2 of M_2. Recall that $\operatorname{rk} A = \operatorname{rk}_{\mathsf{M}_1}(A \cap E_1) + \operatorname{rk}_{\mathsf{M}_2}(A \cap E_2)$. For every flat F of $\mathsf{M}_1 \oplus \mathsf{M}_2$ denote by $F_1 = F \cap E_1$ and $F_2 = F \cap E_2$. Now, for every $e \notin F$, by using the semimodularity of the rank function of $\mathsf{M}_1 \oplus \mathsf{M}_2$, the fact that $F_1 \cap F_2 = \emptyset$ and the fact that $\operatorname{rk}(F \cup \{e\}) = \operatorname{rk} F + 1$ since F is a flat, we get that

$$\operatorname{rk}(F_1 \cup \{e\}) + \operatorname{rk} F_2 \geq \operatorname{rk}(F_1 \cup F_2) + 1.$$

Since F_1 and F_2 are disjoint, we know that $\operatorname{rk}(F_1 \cup F_2) = \operatorname{rk} F_1 + \operatorname{rk} F_2$. The claim then holds. □

What makes geometric lattices work so well with the incidence algebra is that they are a downward-closed class of posets. Formally, the following result holds.

Theorem 1.2.3.8. *Every interval in a geometric lattice is a geometric lattice.*

Proof. Fix F in $\mathcal{L}$ and consider the initial interval $[\emptyset, F]$. This subposet is still clearly atomistic and is equipped with a semimodular rank function, given by the restriction of $\operatorname{rk}_{\mathcal{L}}$ to F. Consider now the final interval $[F, E]$. This subposet is still a lattice and it is ranked with rank function $\operatorname{rk}_{[F,E]}(G) := \operatorname{rk}_{\mathcal{L}}(G) - \operatorname{rk}_{\mathcal{L}}(F)$. This rank function is still semimodular for trivial reasons. We only need to check whether it is still atomistic. However, the atoms of $[F, E]$, i.e. the flats of

$\mathcal{L}$ that cover F are in bijection with the set $\{\sigma(F \cup \{e\}) \mid e \in E \setminus F\}$. Since $G \in [F, E]$ implies that $G = F \cup \bigcup_{e \in G \setminus F} e$, we can write

$$G = \bigcup_{e \in G \setminus F} \sigma(F \cup e)$$

and the claim follows. Lastly, for a generic interval $[F, G] \subset \mathcal{L}$ we just need to observe that $[\emptyset, G]$ is initial in $\mathcal{L}$ and $[F, G]$ is final in $[\emptyset, G]$. □

Corollary 1.2.3.9. *The interval $[F, G] \subset \mathcal{L}(\mathsf{M})$ is isomorphic to the lattice of flats of the matroid $\left(\mathsf{M}_{|G}\right)/F$, where we first restricted to the flat G and then contracted by the flat F.*

Proof. This follows directly by the construction of the two rank functions in the previous proof. □

Theorem 1.2.3.10. *The family of hyperplanes of the truncation of M is*

$$\mathcal{H}(t(\mathsf{M})) = \{F \in \mathcal{F} \mid \operatorname{crk} F = 2\},$$

i.e. the lattice of flats of $t(\mathsf{M})$ is obtained by deleting all the hyperplanes from $\mathcal{L}(\mathsf{M})$.

Proof. The matroid $t(\mathsf{M})$ has rank equal to $\operatorname{rk} \mathsf{M} - 1$. We just observe that every independent set up to rank $\operatorname{rk} \mathsf{M} - 2$ remains independent, therefore the rank function is also unchanged and flats of rank $\operatorname{rk} \mathsf{M} - 2$ are still flats in $t(\mathsf{M})$. Now, since the new bases have cardinality $\operatorname{rk} \mathsf{M} - 1$, the only flat with that rank has to be E, meaning that every hyperplane in M cannot be a flat in $t(\mathsf{M})$. □

For dualization the situation becomes slightly trickier. The lattice of flats can be derived from Theorem 1.1.3.14.

1.2.4 Polynomial invariants on the lattice of flats

We are finally ready to join the two previous sections and produce polynomial invariants for matroids using the incidence algebra of $\mathcal{L}(\mathsf{M})$. We know that the poset $\mathcal{L}(\mathsf{M})$ is finite, hence locally finite, and that it is weakly ranked with the rank function being the matroid rank function rk as shown in Theorem 1.2.3.3. We have already observed in Proposition 1.1.1.15 that the poset $\mathcal{L}(\mathsf{M})$ has a minimal and a maximal element. Moreover, since by Theorem 1.2.3.8 every interval in $\mathcal{L}(\mathsf{M})$ is itself a matroid, this means that a function f in $I(\mathcal{L}(\mathsf{M}))$ can actually be thought as a function

$$f : \{\text{simple matroids}\} \to \mathbb{Z}[x]$$
$$\mathsf{M} \mapsto f_{\mathsf{M}}(x).$$

By defining separately the values of f on matroids with loops we are able to obtain a function $f : \mathrm{Mat} \to \mathbb{Z}[x]$, which is invariant up to isomorphism. Let us start by showing how to obtain the characteristic polynomial.

Theorem 1.2.4.1. *There is a unique way to assign to each loopless matroid M a polynomial $\chi_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ called the* characteristic polynomial of M *defined as*

$$\chi_{\mathsf{M}}(x) = \sum_{F \in \mathcal{L}(\mathsf{M})} \mu_{[\emptyset, F]} x^{\operatorname{rk} \mathsf{M} - \operatorname{rk} F}.$$

By setting $\chi_{\mathsf{M}}(x) = 0$ if M is a matroid with loops, we completely define χ as a polynomial invariant $\chi : \mathrm{Mat} \to \mathbb{Z}[x]$. This polynomial coincides with the polynomial defined in Section 1.1.5.3 thanks to the following result.

Theorem 1.2.4.2. *Let* M *be a simple matroid, then*

$$\sum_{\substack{A\subseteq E\\ \operatorname{rk} A=j}} (-1)^{\#A} = \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \operatorname{rk} F=j}} \mu(\emptyset, F).$$

Proof. By induction on j. If $j \leq 1$ the statement trivially holds since M is simple. Suppose now it holds for every $j < \operatorname{rk}\mathsf{M}$ and let us prove it for $j = \operatorname{rk}\mathsf{M}$. Then,

$$\begin{aligned}\mu(\emptyset, E) &= -\sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G<E}} \mu(\emptyset, G) = -\sum_{j=0}^{\operatorname{rk}\mathsf{M}-1}\sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ \operatorname{rk} G=j}} \mu(\emptyset, G)\\ &= -\sum_{j=0}^{\operatorname{rk}\mathsf{M}-1}\sum_{\substack{A\subseteq E\\ \operatorname{rk} A=j}} (-1)^{\#A} = -\sum_{\substack{A\subseteq E\\ \operatorname{rk} A<\operatorname{rk}\mathsf{M}}} (-1)^{\#A}\\ &= \sum_{\substack{A\subseteq E\\ \operatorname{rk} A=\operatorname{rk}\mathsf{M}}} (-1)^{\#A} - \sum_{A\subseteq E} (-1)^{\#A},\end{aligned}$$

where the latter sum is zero from the known combinatorial fact that

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j = 0.$$

Hence, we have just proved that

$$\mu(\emptyset, F) = \sum_{\substack{A\subseteq F\\ \operatorname{rk} A=\operatorname{rk} F}} (-1)^{\#A}.$$

Adding on all flats of rank j we get

$$\sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \operatorname{rk} F=j}} \mu(\emptyset, F) = \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \operatorname{rk} F=j}} \sum_{\substack{A\subseteq F\\ \operatorname{rk} A=\operatorname{rk} F}} (-1)^{\#A} = \sum_{\operatorname{rk} A=j} (-1)^{\#A},$$

where the last equality follows since if $\operatorname{rk} A = j$ there is a unique flat of rank j that contains A, namely $\sigma(A)$. □

The following fact will be useful later. A proof can be found in [Rot64].

Lemma 1.2.4.3. *The coefficients of the characteristic polynomial* $\chi_{\mathsf{M}}(x)$ *alternate in sign. More specifically,*

$$(-1)^{\operatorname{rk}\mathsf{M}-i}\,[x^i]\chi_{\mathsf{M}}(x) > 0.$$

Lemma 1.2.4.4. *The Möbius function* μ *and the* ζ *function are both multiplicative under direct sums, i.e.*

$$\mu_{\mathcal{L}(\mathsf{M}_1\oplus\mathsf{M}_2)} = \mu_{\mathcal{L}(\mathsf{M}_1)}\,\mu_{\mathcal{L}(\mathsf{M}_2)}$$

and similarly for ζ.

Proof. This follows from Theorem 1.2.3.7. It is sufficient to prove it only for initial intervals of the form $[\emptyset, F_1 \sqcup F_2]$ where $F_1 \in \mathcal{L}(\mathsf{M}_1)$ and $F_2 \in \mathcal{L}(\mathsf{M}_2)$. If $F_1 \sqcup F_2 = \emptyset$, the result is trivial. Otherwise,

$$\begin{aligned}
\mu_{[\emptyset, F_1 \sqcup F_2]} &= - \sum_{G_1 \sqcup G_2 \lneq F_1 \sqcup F_2} \mu_{[\emptyset, G_1 \sqcup G_2]} \\
&= - \sum_{G_1 \lneq F_1} \mu_{[\emptyset, G_1 \sqcup F_2]} - \sum_{G_2 \lneq F_2} \mu_{[\emptyset, F_1 \sqcup G_2]} - \sum_{G_1 \lneq F_1} \sum_{G_2 \lneq F_2} \mu_{[\emptyset, G_1 \sqcup G_2]} \\
&= -\mu_{[\emptyset, F_2]} \sum_{G_1 \lneq F_1} \mu_{[\emptyset, G_1]} - \mu_{[\emptyset, F_1]} \sum_{G_2 \lneq F_2} \mu_{[\emptyset, G_2]} - \sum_{G_1 \lneq F_1} \mu_{[\emptyset, G_1]} \sum_{G_2 \lneq F_2} \mu_{[\emptyset, G_2]} \\
&= \mu_{[\emptyset, F_1]} \, \mu_{[\emptyset, F_2]} + \mu_{[\emptyset, F_1]} \, \mu_{[\emptyset, F_2]} - \mu_{[\emptyset, F_1]} \, \mu_{[\emptyset, F_2]},
\end{aligned}$$

where the third equality follows by the inductive hypothesis. The proof for ζ is trivial as $\emptyset \leq F_1 \sqcup F_2$ if and only if $\emptyset \leq F_1$ and $\emptyset \leq F_2$ in $\mathcal{L}(\mathsf{M}_1)$ and $\mathcal{L}(\mathsf{M}_2)$ respectively. □

Theorem 1.2.4.5. *The characteristic polynomial is multiplicative under direct sums, i.e.*

$$\chi_{\mathsf{M}_1 \oplus \mathsf{M}_2}(x) = \chi_{\mathsf{M}_1}(x) \, \chi_{\mathsf{M}_2}(x).$$

Proof. If either M_1 or M_2 has loops, then $\mathsf{M}_1 \oplus \mathsf{M}_2$ also has loops and the claim trivially holds. Otherwise, the result follows from the multiplicativity of μ and ζ. □

Theorem 1.2.4.6. *For every matroid* M *the following equality holds*

$$\sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F} \chi_{\mathsf{M}_{|F}}(x^{-1}) \chi_{\mathsf{M}/F}(x) = 0.$$

Proof. If M has loops, every restriction $\mathsf{M}_{|F}$ has loops and thus the claim trivially holds. If M is loopless this is equivalent to saying that χ_{M} is a $\mathcal{L}(\mathsf{M})$-kernel. □

Since χ is a $\mathcal{L}(\mathsf{M})$-kernel, this motivates us to introduce the Kazhdan–Lusztig–Stanley functions for matroids. As we have observed in Example 1.2.2.8, we are interested only in the right function. This was first studied by Elias, Proudfoot and Wakefield in [EPW16] without using the language of incidence algebras.

Theorem 1.2.4.7. *There is a unique way to assign to each loopless matroid* M *a polynomial* $P_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) *If* $\operatorname{rk} \mathsf{M} = 0$*, then* $P_{\mathsf{M}}(x) = 1$.

(ii) *If* $\operatorname{rk} \mathsf{M} > 0$*, then* $\deg P_{\mathsf{M}}(x) < \frac{1}{2} \operatorname{rk} \mathsf{M}$.

(iii) *For every matroid* M*, the following recursion holds:*

$$x^{\operatorname{rk} \mathsf{M}} P_{\mathsf{M}}(x^{-1}) = \sum_{F \in \mathcal{L}(\mathsf{M})} \chi_{\mathsf{M}_{|F}}(x) \, P_{\mathsf{M}/F}(x).$$

Remark 1.2.4.8. Written as is, the third condition does not give us a recursive definition for $P_{\mathsf{M}}(x)$ in terms of matroids with smaller rank. In fact, for $F = \emptyset$ one finds the term $P_{\mathsf{M}}(x)$ inside the sum. To overcome this issue, one needs to write the formula as

$$x^{\operatorname{rk} \mathsf{M}} P_{\mathsf{M}}(x^{-1}) - P_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \emptyset}} \chi_{\mathsf{M}_{|F}}(x) \, P_{\mathsf{M}/F}(x)$$

and then observe that the condition on the degree lets us compute the polynomial by truncating the sum on the right-hand-side at degree $\left\lfloor \frac{1}{2} \operatorname{rk} \mathsf{M} \right\rfloor$.

Remark 1.2.4.9. Notice that if M is loopless the recursion implies that P is the right Kazhdan–Lusztig–Stanley function of χ in $I(\mathcal{L}(\mathsf{M}))$. If M is a matroid with loops, every restriction $\mathsf{M}_{|F}$ contains loops and therefore the right hand side of the defining equation is equal to zero. This implies that we can then set $P_{\mathsf{M}}(x) = 0$ for every matroid with loops. This defines uniquely the *Kazhdan–Lusztig polynomial of* M for every matroid M.

Theorem 1.2.4.10. *The Kazhdan–Lusztig polynomial* $P_{\mathsf{M}}(x)$ *is multiplicative under direct sum, i.e.*

$$P_{\mathsf{M}_1 \oplus \mathsf{M}_2}(x) = P_{\mathsf{M}_1}(x)\, P_{\mathsf{M}_2}(x).$$

Proof. If either M_1 or M_2 has loops, the claim is trivial. Otherwise, this follows by induction from the multiplicativity of χ. □

As discussed in Remark 1.2.2.7, we can also compute inverses of the Kazhdan–Lusztig–Stanley functions. Again, the left function of χ is trivial ($g^{-1} = \mu$), while the right one gives us a more interesting polynomial. This motivates us to give the following definition, first introduced in [GX21, Theorem 1.2].

Theorem 1.2.4.11. *There is a unique way to assign to each loopless matroid* M *a polynomial* $Q_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) *If* $\operatorname{rk} \mathsf{M} = 0$*, then* $Q_{\mathsf{M}}(x) = 1$.

(ii) *If* $\operatorname{rk} \mathsf{M} > 0$*, then* $\deg Q_{\mathsf{M}}(x) < \frac{1}{2} \operatorname{rk} \mathsf{M}$.

(iii) *For every matroid* M*, the following recursion holds:*

$$(-x)^{\operatorname{rk} \mathsf{M}} Q_{\mathsf{M}}(x^{-1}) = \sum_{F \in \mathcal{L}(\mathsf{M})} (-1)^{\operatorname{rk} \mathsf{M}_{|F}} Q_{\mathsf{M}_{|F}}(x) x^{\operatorname{rk} \mathsf{M} - \operatorname{rk} F} \chi_{\mathsf{M}/F}(x^{-1}).$$

Remark 1.2.4.12. Notice that if M is loopless the function with components $(-1)^{\operatorname{rk} \mathsf{M}} Q_{\mathsf{M}}(x)$ is the inverse of P in $I(\mathcal{L}(\mathsf{M}))$. This means that

$$\sum_{F \in \mathcal{L}(\mathsf{M})} P_{\mathsf{M}_{|F}}(x)\, (-1)^{\operatorname{rk} \mathsf{M} - \operatorname{rk} F} Q_{\mathsf{M}/F}(x) = 0$$

and

$$\sum_{F \in \mathcal{L}(\mathsf{M})} (-1)^{\operatorname{rk} \mathsf{M}_{|F}} Q_{\mathsf{M}_{|F}}(x)\, P_{\mathsf{M}/F}(x) = 0.$$

By using the first relation, one can see that if M is a matroid with loops then $Q_{\mathsf{M}}(x) = 0$. This defines uniquely the *inverse Kazhdan–Lusztig polynomial of* M for every matroid M.

Theorem 1.2.4.13. *The inverse Kazhdan–Lusztig polynomial* $Q_{\mathsf{M}}(x)$ *is multiplicative under direct sum, i.e.*

$$Q_{\mathsf{M}_1 \oplus \mathsf{M}_2}(x) = Q_{\mathsf{M}_1}(x)\, Q_{\mathsf{M}_2}(x).$$

Proof. The proof is identical to the one for $P_{\mathsf{M}}(x)$. □

To conclude the discussion on the Kazhdan–Lusztig–Stanley functions, we last introduce the Z-polynomial, first studied in [PXY18].

Definition 1.2.4.14. For every matroid M, the *Z-polynomial of* M is defined as

$$Z_{\mathsf{M}}(x) = \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F} P_{\mathsf{M}/F}(x).$$

Remark 1.2.4.15. Notice that if M is loopless, the recursion defines the Z-function of χ in $I(\mathcal{L}(\mathsf{M}))$. Moreover, one can directly check that the Z-polynomial of a matroid with loops is equal to the one corresponding to its simplification.

From the general discussion in Definition 1.2.2.9, the Z-polynomial is a symmetric polynomial of degree $\operatorname{rk} \mathsf{M}$. This actually gives us a way of defining both $P_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ simultaneously, as shown in [BV20].

Theorem 1.2.4.16. *There is a unique way to assign to each matroid* M *a polynomial* $P_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following properties hold:*

(i) *If* $\operatorname{rk} \mathsf{M} = 0$*, then* $P_{\mathsf{M}}(x) = 0$ *unless* E *is empty, in which case* $P_{\mathsf{M}}(x) = 1$.

(ii) *If* $\operatorname{rk} \mathsf{M} > 0$*, then* $\deg P_{\mathsf{M}}(x) < \frac{1}{2} \operatorname{rk} \mathsf{M}$.

(iii) *For every matroid* M*, the polynomial*

$$Z_{\mathsf{M}}(x) := \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F} P_{\mathsf{M}/F}(x)$$

is palindromic, i.e. $x^{\operatorname{rk} \mathsf{M}} Z_{\mathsf{M}}(x^{-1}) = Z_{\mathsf{M}}(x)$.

This defines uniquely the Z-polynomial of M *for every matroid* M.

Corollary 1.2.4.17. *The Z-polynomial $Z_{\mathsf{M}}(x)$ is multiplicative under direct sum, i.e.*

$$Z_{\mathsf{M}_1 \oplus \mathsf{M}_2}(x) = Z_{\mathsf{M}_1}(x)\, Z_{\mathsf{M}_2}(x).$$

In [FMSV22], we prove the following theorem that shows a first parallelism with the Z-polynomial.

Theorem 1.2.4.18. *There is a unique way to assign to each matroid* M *a polynomial* $\underline{\widehat{\mathrm{H}}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) *If* $\operatorname{rk} \mathsf{M} = 0$*, then* $\underline{\widehat{\mathrm{H}}}_{\mathsf{M}}(x) = 0$*, unless* E *is empty, in which case* $\underline{\widehat{\mathrm{H}}}_{\mathsf{M}}(x) = 1$.

(ii) *If* $\operatorname{rk} \mathsf{M} > 0$*, then* $\deg \underline{\widehat{\mathrm{H}}}_{\mathsf{M}}(x) < \operatorname{rk} \mathsf{M}$ *and* $x^{\operatorname{rk} \mathsf{M} - 1}\, \underline{\widehat{\mathrm{H}}}_{\mathsf{M}}(x^{-1}) = \underline{\widehat{\mathrm{H}}}_{\mathsf{M}}(x)$.

(iii) *For every matroid* M*, the polynomial*

$$\widehat{\mathrm{H}}_{\mathsf{M}}(x) := \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F}\, \underline{\widehat{\mathrm{H}}}_{\mathsf{M}/F}(x)$$

is palindromic.

This defines uniquely the Chow polynomial *and the* augmented Chow polynomial of M *for every matroid* M*, as*

$$\underline{\widehat{\mathrm{H}}}_{\mathsf{M}}(x) = \underline{\mathrm{H}}_{\mathsf{M}}(x) \qquad \textit{and} \qquad \widehat{\mathrm{H}}_{\mathsf{M}}(x) = \mathrm{H}_{\mathsf{M}}(x).$$

As it turns out, although the second condition only requires that $\deg \widehat{\underline{\mathrm{H}}}_{\mathsf{M}}(x) < \operatorname{rk} \mathsf{M}$, the last part of the statement will end up guaranteeing that in fact $\deg \widehat{\underline{\mathrm{H}}}_{\mathsf{M}}(x) = \deg \underline{\mathrm{H}}_{\mathsf{M}}(x) = \operatorname{rk} \mathsf{M} - 1$, and therefore the condition $x^{\operatorname{rk} \mathsf{M}-1}\widehat{\underline{\mathrm{H}}}_{\mathsf{M}}(x^{-1}) = \widehat{\underline{\mathrm{H}}}_{\mathsf{M}}(x)$ states that $\widehat{\underline{\mathrm{H}}}(x)$ is palindromic.

For the proof, we rely on an elementary symmetric decomposition that is of particular interest in Ehrhart theory due to the work of Stapledon (see also [BS21] and [AT21] for related work on real-rootedness of these decompositions). Precisely, we need a slight modification of [Sta09, Lemma 2.3].

Lemma 1.2.4.19. *Let $p(x)$ be a polynomial of degree d. There exist unique polynomials $a(x)$ of degree d and $b(x)$ of degree at most $d-1$ with the properties that $a(x) = x^d a(x^{-1})$ and $b(x) = x^{d-1}b(x^{-1})$, and that satisfy*

$$p(x) = a(x) + b(x).$$

Proof. Let us denote by p_i, a_i, and b_i the coefficients of x^i in each of $p(x)$, $a(x)$, and $b(x)$. The condition that $\deg a(x) = \deg p(x)$ and $\deg b(x) < \deg p(x)$ implies that $a_d = p_d$ and the condition that $x^d a(x^{-1}) = a(x)$ yields that $a_0 = p_d$ as well. This together with $a(x) + b(x)$ determines $b_0 = p_0 - p_d$, and this in turn determines $b_{d-1} = b_0 = p_0 - p_d$. Continuing this way, we determine all the coefficients of $a(x)$ and $b(x)$ inductively. Indeed, for each i the coefficients are determined by the equations

$$\begin{aligned} a_i &= p_d + \cdots + p_{d-i} - p_0 - \cdots - p_{i-1}, \\ b_i &= p_0 + \cdots + p_i - p_d - \cdots - p_{d-i}. \end{aligned}$$

□

Proof of Theorem 1.2.4.18. In Section 1.2.2 we have already produced two polynomials, $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ that satisfy all these properties. Let us prove the statement by induction on the size of the ground set of M. We need to establish the uniqueness of $\widehat{\underline{\mathrm{H}}}_{\mathsf{M}}(x)$, as $\widehat{\mathrm{H}}_{\mathsf{M}}(x)$ is determined by the former. If M has cardinality $n = 0$, then $\operatorname{rk} \mathsf{M} = 0$ and the polynomial $\widehat{\underline{\mathrm{H}}}_{\mathsf{M}}(x)$ is uniquely defined and equal to 1 by the first property. Now, assume the uniqueness has already been established for matroids with cardinality at most $n-1$, and consider a matroid M of cardinality n. The polynomial

$$\mathrm{S}_{\mathsf{M}}(x) := \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \emptyset}} x^{\operatorname{rk} F}\, \widehat{\underline{\mathrm{H}}}_{\mathsf{M}/F}(x),$$

is uniquely determined because all the matroids M/F for flats $F \neq \emptyset$ have ground sets with cardinality at most $n-1$. Observe that since $F = E$ (the ground set of M) is a nonempty flat of M, in $\mathrm{S}_{\mathsf{M}}(x)$ we have a summand of degree $\operatorname{rk} \mathsf{M}$, whereas for $F \subsetneq E$, condition (ii) guarantees that $\deg \left(x^{\operatorname{rk} F}\, \widehat{\underline{\mathrm{H}}}_{\mathsf{M}/F}(x)\right) \leq \operatorname{rk} F + \operatorname{rk}(\mathsf{M}/F) - 1 = \operatorname{rk} \mathsf{M} - 1$. In particular $\deg \mathrm{S}_{\mathsf{M}}(x) = \operatorname{rk} \mathsf{M}$.

Now, using Lemma 1.2.4.19, we can find unique polynomials $a(x)$ and $b(x)$ such that $\deg a(x) = \operatorname{rk} \mathsf{M}$, $\deg b(x) \leq \operatorname{rk} \mathsf{M} - 1$, the polynomial $a(x)$ is palindromic, the polynomial $b(x)$ satisfies $b(x) = x^{\operatorname{rk} \mathsf{M}-1}b(x^{-1})$, and the following property holds:

$$\mathrm{S}_{\mathsf{M}}(x) = a(x) + b(x).$$

In particular, by defining $\widehat{\underline{\mathrm{H}}}_{\mathsf{M}}(x) := -b(x)$, which satisfies the requirements of (ii), we obtain that

$$\sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F}\, \widehat{\underline{\mathrm{H}}}_{\mathsf{M}/F}(x) = \widehat{\underline{\mathrm{H}}}_{\mathsf{M}}(x) + \mathrm{S}_{\mathsf{M}}(x) = -b(x) + \mathrm{S}_{\mathsf{M}}(x) = a(x),$$

which is palindromic, as required. Notice that the uniqueness of the decomposition of Lemma 1.2.4.19 yields the uniqueness for $\widehat{\mathrm{H}}_{\mathsf{M}}(x)$ as we claimed. □

Remark 1.2.4.20. Notice that if M is a matroid with loops $\mathrm{H}_{\mathsf{M}}(x)$ is equal to the one corresponding to its simplification.

Remark 1.2.4.21. Since $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ are palindromic and their degrees are $\operatorname{rk}\mathsf{M}-1$ and $\operatorname{rk}\mathsf{M}$ respectively, it follows from the above recursion that

$$\begin{aligned}\mathrm{H}_{\mathsf{M}}(x) = x^{\operatorname{rk}\mathsf{M}}\mathrm{H}_{\mathsf{M}}(x^{-1}) &= x^{\operatorname{rk}\mathsf{M}} \sum_{F\in\mathcal{L}(\mathsf{M})} x^{-\operatorname{rk}F}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x^{-1}) \\ &= 1 + x \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq E}} \underline{\mathrm{H}}_{\mathsf{M}/F}(x).\end{aligned}$$

Clearly, this new recursion also implies the previous one.

Lastly, the following discussion lets us define $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ independently from $\mathrm{H}_{\mathsf{M}}(x)$ as the inverse of $\overline{\chi}_{\mathsf{M}}(x)$. Its uniqueness is guaranteed by Theorem 1.2.4.18. Since χ is a $\mathcal{L}(\mathsf{M})$-kernel in a poset with unique minimal and maximal elements, this lets us assign to each non-empty loopless matroid M the polynomial

$$\overline{\chi}_{\mathsf{M}}(x) := \frac{\chi_{\mathsf{M}}(x)}{x-1}.$$

By setting $\overline{\chi}_{\mathsf{M}}(x) = 0$ for every matroid with loops and $\overline{\chi}_{\mathsf{M}}(x) = -1$ for the empty matroid, this defines uniquely the *reduced characteristic polynomial of* M for every matroid M. As we observed, $\overline{\chi}$ is invertible in $I(\mathcal{L}(\mathsf{M}))$; this lets us give the following definitions.

Theorem 1.2.4.22. *There is a unique way to assign to each matroid* M *a polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) *If* $\operatorname{rk}\mathsf{M} = 0$*, then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 0$*, unless E is empty, in which case* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.

(ii) *For every matroid* M*, the following recursion holds:*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq\emptyset}} \overline{\chi}_{\mathsf{M}|_F}(x)\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x).$$

This defines uniquely the Chow polynomial of M *for every matroid* M.

1.3 Polytope theory

Now that we have defined our polynomials in terms of $\mathcal{L}(\mathsf{M})$ it might be worth it to see if we have other ways of computing them using different axiomatic systems, for example the bases axioms. To work with them, we start by arranging the bases to form a polytope $\mathcal{P}(\mathsf{M})$ and then we define the notion of *valuativity* with respect to it.

1.3.1 Matroid polytopes

We are now ready to define a polytope associated to a matroid M. For every finite set E we denote by $\mathbb{R}^E$ the finite dimensional real vector space of the formal linear combinations of elements $\{e_i \mid i \in E\}$. For every subset $S \subset E$ we use the notation

$$e_S := \sum_{i\in S} e_i.$$

Definition 1.3.1.1. Let $\mathsf{M} = (E, \mathcal{B})$ a matroid. We define the *matroid polytope* $\mathcal{P}(\mathsf{M})$ as

$$\mathcal{P}(\mathsf{M}) := \operatorname{conv}\{e_B \mid B \in \mathcal{B}\}.$$

Example 1.3.1.2. The matroid polytope of the Boolean matroid $\mathcal{P}(\mathsf{B}_n)$ consists of one point.

It should be clear that every x_B is a $\{0,1\}$-vector. A polytope whose vertices are all $\{0,1\}$-vectors is called a $\{0,1\}$*-polytope*. Something less evident at first sight is that the points x_B are exactly the vertices of $\mathcal{P}(\mathsf{M})$.

Proposition 1.3.1.3 ([Fer22a, Proposition 1.2.4])**.** *For every matroid* M*,*

$$\operatorname{Vert}\mathcal{P}(\mathsf{M}) = \{e_B \mid B \in \mathcal{B}\}.$$

It turns out that the matroid polytope $\mathcal{P}(\mathsf{M})$ encodes all the information regarding $\mathcal{B}(\mathsf{M})$ and therefore all the information regarding the matroid itself.

Theorem 1.3.1.4 ([GGMS87])**.** *A polytope $\mathcal{P} \subset \mathbb{R}^E$ is the matroid polytope of a matroid M on a ground set E if and only if it satisfies the following two conditions:*

P1 $\mathcal{P}$ is a $\{0,1\}$-polytope,

P2 Every edge of $\mathcal{P}$ is equal to $e_i - e_j$ for some $i \neq j$.

Remark 1.3.1.5. Notice that if the vertices give us the bases $\mathcal{B}(\mathsf{M})$, the edges are exactly the exchanges that we can perform with the exchange axiom. As a direct consequence of this cryptomorphic definition, we also notice the following. Denote by $(x_i)_{i\in E}$ the coordinates of a point $x \in \mathbb{R}^E$. As every basis $B \in \mathcal{B}$ has the same cardinality equal to $\operatorname{rk}\mathsf{M}$, then

$$\sum_{i\in E}(x_B)_i = \sum_{i\in B} 1 = \operatorname{rk}\mathsf{M}$$

and the same holds for every point in their convex hull. Therefore, a matroid polytope $\mathcal{P}(\mathsf{M})$ actually lives in the affine hyperplane $\{\sum_{i\in E} x_i = \operatorname{rk}\mathsf{M}\} \subset \mathbb{R}^E$.

Proposition 1.3.1.6. *Let M be a matroid on N elements and let $c(\mathsf{M})$ denote the number of its connected components. Then,*

$$d(\mathsf{M}) := \dim\mathcal{P}(\mathsf{M}) = n - c(\mathsf{M}).$$

In particular, a matroid M is connected if and only if $\dim\mathcal{P}(\mathsf{M}) = n - 1$.

Example 1.3.1.7. Consider the uniform matroid $\mathsf{U}_{k,n}$. Its matroid polytope is also known as the *hypersimplex* $\Delta_{k,n}$. This is a $(n-1)$-dimensional polytope unless $k=0$ or $k=n$, in which case it is a point.

Some of the classical operations from Section 1.1.3 are relatively easy to describe at the level of matroid polytopes.

Proposition 1.3.1.8. *If* $\mathsf{M}_1 = (E_1, \mathcal{B}_1)$ *and* $\mathsf{M}_2 = (E_2, \mathcal{B}_2)$ *are two matroids, then*

$$\mathcal{P}(\mathsf{M}_1 \oplus \mathsf{M}_2) = \mathcal{P}(\mathsf{M}_1) \times \mathcal{P}(\mathsf{M}_2) \subset \mathbb{R}^{E_1 \sqcup E_2}.$$

Proposition 1.3.1.9. *Let* M *be a matroid. Then*

$$\mathcal{P}(\mathsf{M}^*) = (1, \dots, 1) - \mathcal{P}(\mathsf{M}),$$

i.e. the matroid polytope of the dual can be obtained performing the involution $(x_i) \mapsto (1 - x_i)$.

Proposition 1.3.1.10. *Let* M *be a matroid. Fix a subset* $S \subset E$ *and consider the linear functional*

$$\varphi_S : x \mapsto \sum_{i \in S} x_i.$$

The matroid polytope $\mathcal{P}(\mathsf{M}_{|S} \oplus \mathsf{M}/S) = \mathcal{P}(\mathsf{M}_{|S}) \times \mathcal{P}(\mathsf{M}/S)$ *is the (unique) highest dimensional face of* $\mathcal{P}(\mathsf{M})$ *on which* φ_S *is maximized.*

Remark 1.3.1.11. Lastly, we observe that, thanks to the notion of matroid polytope, we can finally have a more geometric description of the (rank preserving) weak maps described in 1.1.1.19. A bijection $\varphi : E \to E'$ is a rank preserving weak map between M and M' if and only if

$$\varphi^{-1}(\mathcal{P}(\mathsf{M}')) \subseteq \mathcal{P}(\mathsf{M}).$$

The weak order introduced in Definition 1.1.1.20 can now be interpreted as the poset of matroid polytopes with respect to inclusion (up to a rigid transformation given by a relabeling of the elements of the ground set). In Section 5.2 we provide a categorification of this poset. With this interpretation, it is clear that, after fixing the rank k and the ground set E, id_E induces a weak map from $\mathsf{U}_{k,E}$ to any matroid, as every matroid polytope is contained in the respective hypersimplex. The identity also induces a weak map from a matroid M to $\mathsf{M}_{|S} \oplus \mathsf{M}/S$ for every subset $S \subset E$, as $\mathcal{P}(\mathsf{M}_{|S} \oplus \mathsf{M}/S)$ is a face of $\mathcal{P}(\mathsf{M})$.

1.3.2 Decompositions and the valuative group

We now proceed to describe how to decompose matroid polytopes in smaller matroid polytopes. Among many, standard references for this are [AFR10] and[DF10]. Let us fix a ground set E. For every matroid M, let $\mathbb{1}_{\mathsf{M}}$ be the function on $\mathbb{R}^E$ that takes the value 1 on $\mathcal{P}(\mathsf{M})$ and 0 otherwise. Let $\mathrm{Mat}(E)$ be the free abelian group with basis given by matroids on E. Let $\mathfrak{i}(E) \subset \mathrm{Mat}(E)$ be the kernel of the homomorphism from $\mathrm{Mat}(E)$ to the group of $\mathbb{Z}$-valued functions on $\mathbb{R}^E$ taking M to $\mathbb{1}_{\mathsf{M}}$, and let $\mathrm{Val}(E)$ be the quotient of $\mathrm{Mat}(E)$ by $\mathfrak{i}(E)$. The group $\mathrm{Val}(E)$ is called the *valuative group* on E.

The subgroup $\mathfrak{i}(E) \subset \mathrm{Mat}(E)$ admits a concrete presentation, which we now describe. Let M be a matroid on E. A *decomposition* of M is a collection $\mathcal{N}$ of matroids on E with the following properties:

- If $\mathsf{N} \in \mathcal{N}$, then every nonempty face of $\mathcal{P}(\mathsf{N})$ is equal to $\mathcal{P}(\mathsf{N}')$ for some $\mathsf{N}' \in \mathcal{N}$.

- If $\mathsf{N}, \mathsf{N}' \in \mathcal{N}$, then $\mathcal{P}(\mathsf{N}) \cap \mathcal{P}(\mathsf{N}')$ is a (possibly empty) face of both $\mathcal{P}(\mathsf{N})$ and $\mathcal{P}(\mathsf{N}')$.
- We have $\mathcal{P}(\mathsf{M}) = \bigcup_{\mathsf{N} \in \mathcal{N}} \mathcal{P}(\mathsf{N})$.

More informally, $\mathcal{N}$ is a collection of matroids on E whose polytopes are the closed cells of a cellular decomposition of $\mathcal{P}(\mathsf{M})$.

Example 1.3.2.1. Let $E = \{1, 2, 3, 4\}$. Let M be the uniform matroid of rank 2 on E, let N be the matroid whose bases are all subsets of cardinality 2 except for $\{3, 4\}$, let N' be the matroid whose bases are all subsets of cardinality 2 except for $\{1, 2\}$, and let N'' be the matroid whose bases are all subsets of cardinality 2 except for $\{1, 2\}$ and $\{3, 4\}$. In Figure 1.4, the matroid polytope of M is the octahedron, the matroid polytopes of N and N' are the two pyramids, and the matroid polytope of N'' is the square. The family that consists of N, N' and N'' and all their faces forms a decomposition of the matroid M. It is worth it to point out that as $\mathcal{P}(\mathsf{N}) \cong \mathcal{P}(\mathsf{N}')$ (i.e. there exists a rigid transformation of $\mathbb{R}^4$ that sends $\mathcal{P}(\mathsf{N})$ to $\mathcal{P}(\mathsf{N}')$ given by a permutation of the ground set E), then $\mathsf{N} \cong \mathsf{N}'$. Moreover, $\mathcal{P}(\mathsf{N}'')$ is the face of $\mathcal{P}(\mathsf{N})$ on which $\varphi_{\{3,4\}}$ is maximized, and the face of $\mathcal{P}(\mathsf{N}')$ on which $\varphi_{\{1,2\}}$ is maximized. In fact,

$$\mathsf{N}'' \cong \mathsf{N}_{|\{3,4\}} \oplus \mathsf{N}/\{3, 4\}$$

$$\mathsf{N}'' \cong \mathsf{N}'_{|\{1,2\}} \oplus \mathsf{N}'/\{1, 2\}.$$

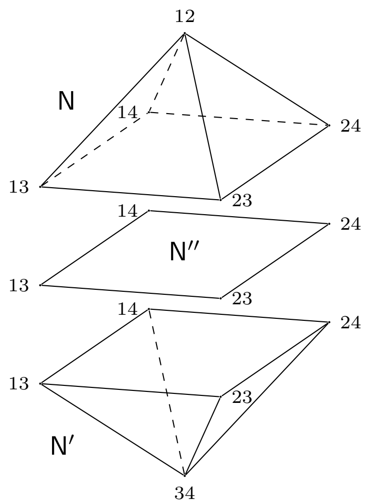

Figure 1.4: A decomposition of the matroid $\mathsf{M} = \mathsf{U}_{2,4}$. The label ij refers to the point that takes the value 1 in the i-th and j-th coordinates, such as $12 = (1, 1, 0, 0)$.

Elements of $\mathcal{N}$ are called *faces*. We say that a face $\mathsf{N} \in \mathcal{N}$ is *internal* if $\mathcal{P}(\mathsf{N})$ is not contained in the boundary of $\mathcal{P}(\mathsf{M})$. We write $\mathcal{N}_k$ to denote the set of internal faces $\mathsf{N} \in \mathcal{N}$ with $d(\mathsf{N}) = k$. In Example 1.3.2.1, the internal faces are N, N' and N''. The following theorem follows from [AFR10, Theorem 3.5] and [DF10, Corollary 3.9].

Theorem 1.3.2.2. *If $\mathcal{N}$ is a decomposition of* M*, then*

$$(-1)^{d(\mathsf{M})}\mathsf{M} - \sum_k (-1)^k \sum_{\mathsf{N}\in\mathcal{N}_k} \mathsf{N} \in \mathfrak{i}(E).$$

Furthermore, $\mathfrak{i}(E)$ *is spanned by elements of this form.*

Example 1.3.2.3. Any matroid M has a *trivial decomposition* consisting of M itself along with all of the matroids N such that $\mathcal{P}(\mathsf{N})$ is a face of $\mathcal{P}(\mathsf{M})$. In this example, M is the only internal face.

Example 1.3.2.4. The matroids N, N', and N'' in Example 1.3.2.1 are the three internal faces of $\mathcal{N}$. There are also many faces that are not internal, corresponding to the eight facets, twelve edges, and six vertices of $\mathcal{P}(\mathsf{M})$. The generator of $\mathfrak{i}(E)$ corresponding to this decomposition is depicted in Figure 1.5.

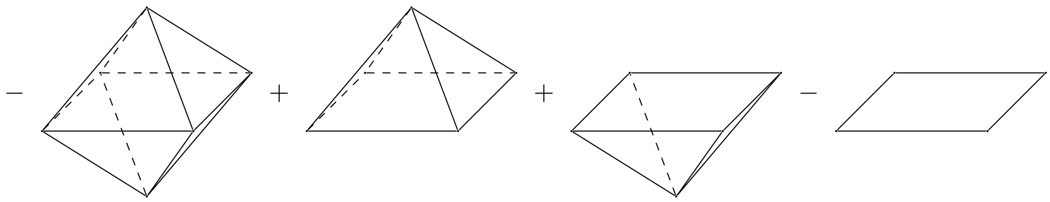

Figure 1.5: The generator of $\mathfrak{i}(E)$ from the decomposition of $\mathsf{M} = \mathsf{U}_{2,4}$.

We are finally ready to state the main definition of this section.

Definition 1.3.2.5. Let A be an abelian group. A homomorphism $f : \mathrm{Mat}(E) \to A$ is called a *valuative matroid invariant* if f vanishes on $\mathfrak{i}(E)$, or equivalently if f factors through $\mathrm{Val}(E)$.

More concretely, f is valuative if for every M and every decomposition $\mathcal{N}$,

$$(-1)^d f(\mathsf{M}) = \sum_k (-1)^k \sum_{\mathsf{N}\in\mathcal{N}_k} f(\mathsf{N}).$$

1.3.3 Examples of valuative invariants

Unexpectedly, it turns out that a lot of invariants associated to matroids are valuative.

1.3.3.1 $\mathcal{G}$-invariant

We begin this discussion by introducing the Derksen invariant $\mathcal{G}$.

Definition 1.3.3.1. Let $\mathsf{M} = (E, \mathrm{rk})$ be a matroid on a ground set E of cardinality n. Consider the set of all possible chains of subsets of E

$$\underline{S} := S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_n$$

with $\#S_j = j$. For every such $\underline{S}$ consider the $\{0,1\}$-vector $r(\underline{S}) = [r_i] \in \{0,1\}^n$ given by

$$r_i = \mathrm{rk}(S_i) - \mathrm{rk}(S_{i-1}),$$

for every $1 \leq i \leq n$. Then, we define the *$\mathcal{G}$-invariant* of M to be

$$\mathcal{G}(\mathsf{M}) := \sum_{\underline{S}} r(\underline{S}).$$

Example 1.3.3.2. The value of the $\mathcal{G}$-invariant for the uniform matroid $\mathsf{U}_{k,n}$ is

$$\mathcal{G}(\mathsf{U}_{k,n}) = n![1\dots 10\dots 0],$$

with exactly k ones and $n-k$ zeros.

The $\mathcal{G}$-invariant is proved to be valuative in [Der09]. Moreover, the following stronger result holds.

Theorem 1.3.3.3 ([DF10, Theorem 1.4])**.** *The $\mathcal{G}$-invariant is universal for all valuative matroid invariants, i.e. the coefficients of $\mathcal{G}$ span the vector space of all valuative matroid invariants with values in $\mathbb{Q}$.*

This means that any invariant that is obtained by specializing $\mathcal{G}$ is also valuative.

1.3.3.2 Tutte polynomial

Theorem 1.3.3.4 ([Der09, Theorem 1.1])**.** *Let M be a matroid of rank k over n elements. The Tutte polynomial of M can be obtained from $\mathcal{G}(\mathsf{M})$ by the specialization*

$$r \mapsto \sum_{m=0}^{n} \frac{(x-1)^{k-\mathrm{wt}[r_1\dots r_m]}(y-1)^{m-\mathrm{wt}[r_1\dots r_m]}}{m!(n-m)!},$$

where $\mathrm{wt}[r_1\dots r_m]$ is the sum of the first m entries of r. In particular, $T_{\mathsf{M}}(x,y)$ is valuative.

Corollary 1.3.3.5. *The characteristic polynomial $\chi_{\mathsf{M}}(x)$ can be obtained from $\mathcal{G}(\mathsf{M})$ by the specialization*

$$r \mapsto \sum_{m=0}^{n} \frac{(-1)^m x^{k-\mathrm{wt}[r_1\dots r_m]}}{m!(n-m)!},$$

hence $\chi_{\mathsf{M}}(x)$ is valuative and so is the reduced characteristic polynomial $\overline{\chi}_{\mathsf{M}}(x)$.

The valuativity of the characteristic polynomial was first proved by Speyer in [Spe08, Lemma 3.4].

1.3.3.3 Hopf monoids

One issue with valuative invariants and matroid polytope decompositions is that there is no combinatorial algorithm to produce all such decompositions for a given matroid. Proving that a function satisfies an inclusion-exclusion formula with respect to every decomposition therefore seems to be an impossible task. However, we can rely on some results due to Ardila and Sanchez [AS23], who were able to find an algorithm to combine valuative invariants to produce new valuative invariants using the theory of Hopf monoids as developed by Aguiar and Mahajan in [AM10].

Definition 1.3.3.6. A *Hopf monoid* H consists of the following data:

- A vector space $\mathrm{H}[E]$ for every finite set E and an isomorphism $\mathrm{H}[f] : \mathrm{H}[E] \to \mathrm{H}[E']$ for every bijection $f : E \to E'$.
- A *product* $m_{S,T} : \mathrm{H}[S] \otimes \mathrm{H}[T] \to \mathrm{H}[E]$ for each decomposition $E = S \sqcup T$ such that the following axioms hold:

  (Naturality) Let E and E' be two sets and $f : E \to E'$ be a bijection. Let $E = S \sqcup T$ be a decomposition and let $f_{|S}$ and $f_{|T}$ be the restrictions of f to S and T, respectively. This gives us a decomposition of $E' = f(S) \sqcup f(T)$ and a pair of bijections $f_{|S} : S \to f(S)$ and $f_{|T} : T \to f(T)$. Then, we have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{H}[S] \otimes \mathrm{H}[T] & \xrightarrow{m_{S,T}} & \mathrm{H}[E] \\
\Big\downarrow{\scriptstyle \mathrm{H}[f_{|S}] \otimes \mathrm{H}[f_{|T}]} & & \Big\downarrow{\scriptstyle \mathrm{H}[f]} \\
\mathrm{H}[f(S)] \otimes \mathrm{H}[f(T)] & \xrightarrow{m_{f(S),f(T)}} & \mathrm{H}[E']
\end{array}$$

(Unitality) We have $\mathrm{H}[\emptyset] \cong \mathbb{K}$. Denote the unit of that vector space by 1. For every $x \in \mathrm{H}[E]$ and for the two trivial decompositions $E = E \sqcup \emptyset = \emptyset \sqcup E$, we have

$$m_{E,\emptyset}(x, 1) = m_{\emptyset,E}(1, x) = x.$$

(Associativity) Let $E = R \sqcup S \sqcup T$ be a decomposition. Then we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}[R] \otimes \mathrm{H}[S] \otimes \mathrm{H}[T] & \xrightarrow{\mathrm{id} \otimes m_{S,T}} & \mathrm{H}[R] \otimes \mathrm{H}[S \sqcup T] \\
\Big\downarrow{\scriptstyle m_{R,S} \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle m_{R,S \sqcup T}} \\
\mathrm{H}[R \sqcup S] \otimes \mathrm{H}[T] & \xrightarrow{m_{R \sqcup S,T}} & \mathrm{H}[E]
\end{array}$$

This allows us to define a multiplication map $m_{S_1,S_2,\ldots,S_k}$ for any set decomposition $E = S_1 \sqcup \cdots \sqcup S_k$.

- A *coproduct* $\Delta_{S,T} : \mathrm{H}[E] \to \mathrm{H}[S] \otimes \mathrm{H}[T]$ for each decomposition $E = S \sqcup T$ such that the following axioms hold:

  (Naturality) Let E and E' be two sets and $f : E \to E'$ be a bijection. Let $E = S \sqcup T$ be a decomposition and let $f_{|S}$ and $f_{|T}$ be the restrictions of f to S and T, respectively. This gives us a decomposition of $E' = f(S) \sqcup f(T)$ and a pair of bijections $f_{|S} : S \to f(S)$ and $f_{|T} : T \to f(T)$. Then, we have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{H}[S] \otimes \mathrm{H}[T] & \xleftarrow[\Delta_{S,T}]{} & \mathrm{H}[E] \\
\Big\downarrow{\scriptstyle \mathrm{H}[f_{|S}] \otimes \mathrm{H}[f_{|T}]} & & \Big\downarrow{\scriptstyle \mathrm{H}[f]} \\
\mathrm{H}[f(S)] \otimes \mathrm{H}[f(T)] & \xleftarrow[\Delta_{f(S),f(T)}]{} & \mathrm{H}[E']
\end{array}$$

  (Counitality) We have $\mathrm{H}[\emptyset] \cong \mathbb{K}$. Denote the counit of that vector space by 1. For every $x \in \mathrm{H}[E]$ and for the two trivial decompositions $E = E \sqcup \emptyset = \emptyset \sqcup E$, we have

$$\Delta_{E,\emptyset}(x) = x \otimes 1,$$
$$\Delta_{\emptyset,E}(x) = 1 \otimes x.$$

  (Coassociativity) Let $E = R \sqcup S \sqcup T$ be a decomposition. Then we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}[R] \otimes \mathrm{H}[S] \otimes \mathrm{H}[T] & \xleftarrow[\mathrm{id} \otimes \Delta_{S,T}]{} & \mathrm{H}[R] \otimes \mathrm{H}[S \sqcup T] \\
{\scriptstyle \Delta_{R,S} \otimes \mathrm{id}}\Big\uparrow & & {\scriptstyle \Delta_{R,S \sqcup T}}\Big\uparrow \\
\mathrm{H}[R \sqcup S] \otimes \mathrm{H}[T] & \xleftarrow[\Delta_{R \sqcup S,T}]{} & \mathrm{H}[E]
\end{array}$$

- Moreover, m and Δ have to satisfy an additional axiom:

  (Compatibility) Let $E = S_1 \sqcup S_2 = T_1 \sqcup T_2$ be two decompositions of E. Let $A = S_1 \cap T_1$, $B = S_1 \cap T_2$, $C = S_2 \cap T_1$ and $D = S_2 \cap T_2$ be their pairwise intersections. Then, we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}[S_1] \otimes \mathrm{H}[S_2] & \xrightarrow{m_{S_1,S_2}} & \mathrm{H}[E] & \xrightarrow{\Delta_{T_1,T_2}} & \mathrm{H}[T_1] \otimes \mathrm{H}[T_2] \\ \downarrow \Delta_{A,B} \otimes \Delta_{C,D} & & & & \uparrow m_{A,C} \otimes m_{B,D} \\ \mathrm{H}[A] \otimes \mathrm{H}[B] \otimes \mathrm{H}[C] \otimes \mathrm{H}[D] & & \xrightarrow{\mathrm{id} \otimes \beta \otimes \mathrm{id}} & & \mathrm{H}[A] \otimes \mathrm{H}[C] \otimes \mathrm{H}[B] \otimes \mathrm{H}[D] \end{array}$$

  where β is the braiding map $\beta(x \otimes y) = (y \otimes x)$.

We can now give the class of matroids a Hopf monoid structure.

Definition 1.3.3.7. We construct the Hopf monoid Mat as follows.

- $\mathrm{Mat}[E]$ is the $\mathbb{K}$-vector space generated by every matroid on the ground set E.
- The product of $\mathsf{M}_S \in \mathrm{Mat}[S]$ and $\mathsf{M}_T \in \mathrm{Mat}[T]$ is

$$m_{S,T}(\mathsf{M}_S \otimes \mathsf{M}_T) := \mathsf{M}_S \oplus \mathsf{M}_T \in \mathrm{Mat}[E].$$

- The coproduct of M is

$$\Delta_{S,T}(\mathsf{M}) := \mathsf{M}_{|S} \otimes \mathsf{M}/S.$$

Remark 1.3.3.8. We would like to remark how $m_{S,T}$ and $\Delta_{S,T}$ interact with each other. On one hand, we have $\Delta_{S,T} \circ m_{S,T} = id$; on the other, we have that

$$(m_{S,T} \circ \Delta_{S,T})(\mathsf{M}) = \mathsf{M}_{|S} \oplus \mathsf{M}/S.$$

In terms of matroid polytopes, this is a function that sends a matroid polytope $\mathcal{P}(\mathsf{M})$ to its face $\mathcal{P}(\mathsf{M}_{|S}) \times \mathcal{P}(\mathsf{M}/S)$ as discussed in Proposition 1.3.1.10.

We are finally ready to state the main theorem that we need to link Hopf monoids to the theory of valuative invariants.

Theorem 1.3.3.9 ([AS23, Theorem C])**.** *Let $E = S_1 \sqcup \ldots \sqcup S_k$ be a decomposition and consider valuative matroid invariants $f_i : \mathrm{Mat}[S_i] \to R$ for every $1 \le i \le k$, where R is a ring with multiplication m. Then, the function $f_1 \star \ldots \star f_k : \mathrm{Mat}[E] \to R$ defined as*

$$f_1 \star \ldots \star f_k := m \circ (f_1 \otimes \ldots \otimes f_k) \circ \Delta_{S_1,\ldots,S_k}$$

is a valuative matroid invariant.

Corollary 1.3.3.10. *Using Theorem 1.3.3.9, we observe that if f and g are valuative matroid invariants, then so is*

$$(f * g)(\mathsf{M}) := \sum_{S \subseteq E} f(\mathsf{M}_{|S})\, g(\mathsf{M}/S).$$

Moreover, if $g(\mathsf{M}) = 0$ whenever M has loops, then the previous sum reduces to

$$(f * g)(\mathsf{M}) := \sum_{F \in \mathcal{L}(\mathsf{M})} f(\mathsf{M}_{|F}) g(\mathsf{M}/F).$$

The previous Corollary implies that the convolution product of two valuative functions in the incidence algebra of $\mathcal{L}(\mathsf{M})$ is again valuative. This will be used in Section 3.1.4 to prove that the polynomials that arose in Section 1.2.4 are valuative.

1.4 Polynomial invariants

In Section 1.1.5 and Section 1.2.4 we have associated some polynomials to every matroid. We are now interested in understanding more about the properties of these polynomials. In this Section we recap all the known results that we need in the theory of polynomials, mostly following [Brä15].

1.4.1 Properties of a polynomial

When looking at a polynomial the first thing that comes to mind is to study its roots. In this section we give some way to relate the roots of a polynomial to its coefficients. To do so we will work with the following toy example.

Example 1.4.1.1. Denote by $b_n(x)$ the polynomial defined as

$$b_n(x) := \sum_{i=0}^{n} \binom{n}{i} x^i.$$

This polynomial is clearly *real-rooted*, i.e. all its roots are real. In fact $b_n(x) = (x+1)^n$ and therefore it has a root $\lambda = -1$ with multiplicity n.

We state here a result known as the Gauss–Lucas Theorem that we will need later.

Theorem 1.4.1.2. *Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree at least one. All zeros of $\frac{d}{dx}f(x)$ lie in the convex hull of the zeros of $f(x)$. In particular, if $f(x)$ is real-rooted then so is $\frac{d}{dx}f(x)$.*

A much less strict condition to require on a polynomial is the one of unimodality.

Definition 1.4.1.3. A sequence of integers $(a_0, \dots, a_n)$ is *unimodal* if there exists $0 \leq i \leq n$ such that

$$a_{j-1} \leq a_j \text{ for every } 1 \leq j \leq i \text{ and } a_j \geq a_{j+1} \text{ for every } i \leq j \leq n-1.$$

A polynomial is unimodal if its coefficients form a unimodal sequence.

Example 1.4.1.4. The polynomial $b_n(x)$ is unimodal, as

$$\binom{n}{i-1} < \binom{n}{i}$$

for every $i \leq \frac{n}{2}$. The unimodality then follows from the symmetric property

$$\binom{n}{i} = \binom{n}{n-i}.$$

As we will see in Sections 1.4.1.1 and 1.4.1.2, the notion of real-rootedness and unimodality are related.

1.4.1.1 Log-concavity

Definition 1.4.1.5. A sequence of integers $(a_0, \dots, a_n)$ is *log-concave* if for every $1 \leq i \leq n-1$

$$a_{i-1}\, a_{i+1} \leq a_i^2.$$

The sequence is *ultra log-concave* if $(a_i/\binom{n}{i})$ is log-concave. A polynomial is (ultra) log-concave if its coefficients form a (ultra) log-concave sequence.

Example 1.4.1.6. The polynomial $b_n(x)$ is log-concave, in fact with a direct check one can show that

$$\binom{n}{i}^2 \geq \binom{n}{i-1}\binom{n}{i+1} \Leftrightarrow \frac{i+1}{i}\frac{n-i+1}{n-i} \geq 1,$$

which is clearly true. Moreover, it is also trivially ultra log-concave, as a constant sequence is log-concave.

We also have the following chain of implications.

Proposition 1.4.1.7 ([Gal05])**.** *Let $f(x)$ be a polynomial with non-negative coefficients. We have the following strict implications*

$f(x)$ is real-rooted $\Rightarrow$ $f(x)$ is ultra log-concave $\Rightarrow$ $f(x)$ is log-concave $\Rightarrow$ $f(x)$ is unimodal.

Proof. Let $f(x) = \sum_{i=0}^n a_i x^i$ and write $a_i = \binom{n}{i} b_i$. Then by Theorem 1.4.1.2 the polynomial

$$\frac{1}{n}\frac{d}{dx}f(x) = \sum_{i=0}^{n} \frac{i}{n}\binom{n}{i} b_i x^{i-1} = \sum_{i=0}^{n-1}\binom{n-1}{i} b_{i+1}\, x^i$$

is real-rooted. Moreover,

$$x^n f(x^{-1}) = \sum_{i=0}^{n}\binom{n}{i} b_{n-i} x^i$$

is also real-rooted. By applying repeatedly these two operations, we obtain the real-rooted polynomial

$$b_{j-1} + 2b_j x + b_{j+1}x^2.$$

Since this is real-rooted, its discriminant is non-negative, which then implies that $b_j^2 \geq b_{j-1}b_{j+1}$ and concludes the proof of the first implication. For the second implication, since (a_i) is ultra log-concave we know that

$$\frac{a_i^2}{\binom{n}{i}^2} \geq \frac{a_{i-1}}{\binom{n}{i-1}}\frac{a_{i+1}}{\binom{n}{i+1}}$$

for every $1 \leq i \leq n-1$. Then,

$$a_i^2 \geq \frac{\binom{n}{i}^2}{\binom{n}{i-1}\binom{n}{i+1}} a_{i-1}a_{i+1}.$$

The claim then follows from the computations made in Example 1.4.1.6. Lastly, if we had an index j such that

$$a_{j-1} > a_j < a_{j+1},$$

this would imply that $a_j^2 < a_{j-1}a_{j+1}$ which contradicts the hypothesis of log-concavity. □

1.4.1.2 Palindromicity and γ-positivity

Definition 1.4.1.8. A sequence of integers $(a_0, \ldots, a_n)$ is *symmetric* if $a_i = a_{n-i}$ for every $0 \leq i \leq \frac{n}{2}$. A polynomial $f(x) = \sum_i a_i x^i$ is *symmetric* with center of symmetry $\frac{d}{2}$ if $a_i = a_{d-i}$ for every $i \in \mathbb{Z}$. This can be rephrased by saying that $f(x)$ satisfies $x^d f(x^{-1}) = f(x)$. If the center of symmetry is $\frac{\deg f(x)}{2}$, then $f(x)$ is called *palindromic*.

Example 1.4.1.9. The polynomial $b_n(x)$ is palindromic as

$$\binom{n}{i} = \binom{n}{n-i}$$

for every $0 \leq i \leq \frac{n}{2}$ and $\deg b_n(x) = n$.

Palindromic polynomials are ubiquitous objects in combinatorics. A powerful tool to prove the unimodality of a non-negative palindromic polynomial is provided by the notion of *γ-positivity* or *γ-non-negativity.*

The first step is to state a basic result that allows one to encode a palindromic polynomial inside a new polynomial with half of the number of terms.

Proposition 1.4.1.10 ([Gal05, Proposition 2.1.1])**.** *If $f(x) \in \mathbb{Z}[x]$ is a palindromic polynomial of degree n, then there exist integers $\gamma_0, \dots, \gamma_{\lfloor \frac{n}{2} \rfloor}$ such that*

$$f(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i x^i (1+x)^{n-2i}. \tag{1.2}$$

Proof. Let us prove first that the family of polynomials $u_i(x) = x^i(1+x)^{n-2i}$ for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$ is linearly independent. In fact, if we call v_i the vector having the coefficients of degrees $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$ of the polynomial u_i, we see that the first non-zero entry of v_i is exactly a 1 in the i-th position. In other words, the square matrix S of size $\left(\lfloor \frac{n}{2} \rfloor + 1\right) \times \left(\lfloor \frac{n}{2} \rfloor + 1\right)$ obtained by putting each v_i as the i-th row, is upper-triangular and has ones in the diagonal. This means that the polynomials u_i are linearly independent as we claimed, so that they generate the space of palindromic polynomials of degree d (which itself has dimension $\lfloor \frac{n}{2} \rfloor + 1$). In particular P can be written as a real linear combination of the u_i's. The fact that $\det(S) = 1$ guarantees that whenever P has integer coefficients, the γ_i's are integers too. □

Definition 1.4.1.11. Let $f(x)$ be a palindromic polynomial of degree n. If $\gamma_0, \dots, \gamma_{\lfloor \frac{n}{2} \rfloor}$ are as in equation (1.2), we define the *γ-polynomial* associated to f by

$$\gamma_f(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i x^i.$$

If $f(x)$ is a palindromic polynomial of degree n, we will say that $f(x)$ is *γ-positive* if all the coefficients of $\gamma_f(x)$ are non-negative.

We have the following important result by Gal [Gal05], which establishes links between properties of $f(x)$ and properties of $\gamma_f(x)$.

Proposition 1.4.1.12. *Let $f(x)$ be a palindromic polynomial of degree n with positive coefficients. We have the following strict implications*

$$\gamma_f(x) \textit{ is negative real-rooted} \Leftrightarrow f(x) \textit{ is real-rooted} \Rightarrow f(x) \textit{ is } \gamma\textit{-positive} \Rightarrow f(x) \textit{ is unimodal.}$$

Proof. For the first "if and only if", notice that

$$f(x) = \gamma_f\left(\frac{x}{(1+x)^2}\right)(1+x)^n. \tag{1.3}$$

If $f(x)$ is real-rooted so is $\gamma_f(x)$. Moreover, as $f(x)$ is assumed to have positive coefficients, all the roots of $f(x)$ are negative, and thus so are all the roots of $\gamma_f(x)$. On the other hand, let us assume that $\gamma_f(x)$ has only negative real roots. Assume that z is a complex number such that $f(z) = 0$. We want to prove that z is a negative real number. If $z = -1$, then there is nothing to prove. Otherwise, by the negative real-rootedness of $\gamma_f(x)$, it follows that $\frac{z}{(1+z)^2} \in \mathbb{R}^-$. By noticing that $\frac{z}{(1+z)^2} = \left(\frac{1}{\sqrt{z}+\sqrt{z^{-1}}}\right)^2$, we obtain that $\sqrt{z} + \sqrt{z^{-1}}$ is a pure imaginary number. However, for every complex number z, Re $\sqrt{z}$ and Re $\sqrt{z^{-1}}$ have the same sign. As in our case their sum has real part zero, it means that actually both of them were pure imaginary numbers. In particular $\sqrt{z}$ is a pure imaginary number, which tells us that z is a negative real number.

For the second implication, let us assume that $f(x)$ is real-rooted. As before, since the coefficients of $f(x)$ are positive, all the roots of $f(x)$ must be negative. Also, as $f(x)$ was assumed to be palindromic, we may pair the zeros of $f(x)$ into groups of the form r and $\frac{1}{r}$ and write

$$f(x) = A(x+1)^{\varepsilon} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (x+r_i)(x+\tfrac{1}{r_i}),$$

where $\varepsilon = 0, 1$ according to the parity of d and A is some constant. Observe that

$$(x+r_i)(x+\frac{1}{r_i}) = (1+x)^2 + (r_i + \frac{1}{r_i} - 2)x,$$

which is a non-negative[1] linear combination of the polynomials $x^0(1+x)^2$ and $x^1(1+x)^0$. After multiplying all such factors, this property still holds, and thus $\gamma_f(x)$ has positive coefficients.

The last implication follows directly from the fact that a positive sum of the unimodal palindromic polynomials $x^i(x+1)^{n-2i}$ (all of which can be thought as having "degree n", completing with zeros accordingly) will be again a palindromic unimodal polynomial. □

Example 1.4.1.13. The last two implications are strict. Consider, for example, the polynomial $f_1(x) = x^4 + 4x^3 + 7x^2 + 4x + 1$. It is not difficult to show that $\gamma_{f_1}(x) = x^2 + 1$. In particular, f_1 is γ-positive but not real-rooted. On the other hand, if we take $f_2(x) = x^2 + x + 1$, this is a unimodal polynomial whose γ-polynomial is $\gamma_{f_2}(x) = -x + 1$.

Observe that a nonzero symmetric polynomial $f(x) \in \mathbb{Z}[x]$ has a unique center of symmetry. In other words, there is exactly one integer d such that $x^d f(x^{-1}) = f(x)$. In particular, whenever f is symmetric there is no ambiguity in writing $\gamma_f(x)$, even when the degree of the polynomial $f(x)$ is not specified. We have the following toolbox of basic identities that exhibit the behavior of the assignment $f(x) \mapsto \gamma_f(x)$ under simple operations.

Lemma 1.4.1.14. *Let $f(x)$ and $g(x)$ be symmetric polynomials. Then, we have:*

(i) $\gamma_{fg}(x) = \gamma_f(x) \cdot \gamma_g(x)$.

(ii) $\gamma_{xf}(x) = x \cdot \gamma_f(x)$.

(iii) $\gamma_{(x+1)f}(x) = \gamma_f(x)$.

(iv) *If $f(x)$ and $g(x)$ have the same center of symmetry, then $\gamma_{f+g}(x) = \gamma_f(x) + \gamma_g(x)$.*

1.4.2 Some nice families of polynomials

We present some families of polynomials that will be of much importance when we deal with the Hilbert–Poincaré series arising in Section 2.4.

[1] As r_i is positive, we may use the inequalities between the arithmetic and geometric mean and obtain that $1 \leq \sqrt{r_i \cdot \frac{1}{r_i}} \leq \frac{r_i + \frac{1}{r_i}}{2}$, from where it follows that $r_i + \frac{1}{r_i} - 2 \geq 0$.

1.4.2.1 Eulerian polynomials

One of the most pervasive objects in enumerative combinatorics is the family of Eulerian polynomials. Given a permutation $\sigma \in \mathfrak{S}_n$, $n \geq 1$, written in one-line notation as $\sigma = \sigma_1 \cdots \sigma_n$, the number of *descents* of σ is defined as the cardinality of the set $\{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$, and is denoted by $\mathrm{des}(\sigma)$. For $n \geq 1$, we define the n-th *Eulerian polynomial* as

$$A_n(x) := \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)}.$$

We note that this differs by a factor of x from the definition in [Sta12, p. 33]. Furthermore, we define $A_0(x) = 1$. Explicitly, we have:

$$A_n(x) = \begin{cases} 1 & \text{if } n = 0, 1, \\ x + 1 & \text{if } n = 2, \\ x^2 + 4x + 1 & \text{if } n = 3, \\ x^3 + 11x^2 + 11x + 1 & \text{if } n = 4, \\ x^4 + 26x^3 + 66x^2 + 26x + 1 & \text{if } n = 5, \\ \text{etc.} \end{cases}$$

The polynomials $A_n(x)$ are palindromic and $\deg A_n(x) = n - 1$ for every positive n. It is a classical result attributed to Frobenius that these polynomials are real-rooted (for a proof, see [Brä15, Example 7.3]). The coefficients of the Eulerian polynomials admit several combinatorial interpretations, many of which can be found in [Sta12, Chapter 1].

1.4.2.2 Derangement polynomials

A permutation $\sigma \in \mathfrak{S}_n$ is said to be a *derangement* if $\sigma_i \neq i$ for all i, i.e. if σ has no fixed points. The set of all derangements on n elements is usually denoted by $\mathfrak{D}_n$. For each $n \geq 1$, the n-th *derangement polynomial*, denoted $d_n(x)$, is defined by

$$d_n(x) := \sum_{\sigma \in \mathfrak{D}_n} x^{\mathrm{exc}(\sigma)}.$$

where $\mathrm{exc}(\sigma) := \#\{i \in [n] : \sigma_i > i\}$ denotes the number of *excedances* of σ. We extend this to $n = 0$ by defining $d_0(x) := 1$. The first few values of $d_n(x)$ are:

$$d_n(x) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ x & \text{if } n = 2, \\ x^2 + x & \text{if } n = 3, \\ x^3 + 7x^2 + x & \text{if } n = 4, \\ x^4 + 21x^3 + 21x^2 + x & \text{if } n = 5, \\ \text{etc.} \end{cases}$$

We have $\deg d_n(x) = n - 1$ for every positive n. With only the exception of $d_0(x)$, the polynomials $d_n(x)$ are a multiple of x and are symmetric with center of symmetry $\frac{n}{2}$. Derangement polynomials are known to be real-rooted (see for example [GS20, Theorems 3.5 and 4.1]).

1.4.2.3 Binomial Eulerian polynomials

A related family of polynomials that will play an important role is that of the *binomial Eulerian polynomials*, which were named this way, e.g. in [SW20, Ath20]. The n-th binomial Eulerian polynomial $\widetilde{A}_n(x)$ is defined by

$$\widetilde{A}_n(x) := 1 + x\sum_{j=1}^{n}\binom{n}{j}A_j(x).$$

In particular, the first few values of these polynomials are given by:

$$\widetilde{A}_n(x) = \begin{cases} 1 & \text{if } n=0,\\ x+1 & \text{if } n=1,\\ x^2+3x+1 & \text{if } n=2,\\ x^3+7x^2+7x+1 & \text{if } n=3,\\ x^4+15x^3+33x^2+15x+1 & \text{if } n=4,\\ x^5+31x^4+131x^3+131x^2+31x+1 & \text{if } n=5,\\ \text{etc.} \end{cases}$$

We note that $\deg \widetilde{A}_n(x) = n$ for every n. It is a non-obvious fact that these polynomials are palindromic and γ-positive, see for example [PRW08, Theorem 11.6] or [Ath20, Theorem 1.1]. Furthermore, they are real-rooted, by [HZ19, Theorem 3.1] or [BJ22, Theorem 4.4].

1.4.2.4 Haglund–Zhang polynomials

To any sequence $\underline{s} = (s_1, \dots, s_n) \in \mathbb{Z}^n_{>0}$, Haglund and Zhang associate a *generalized binomial Eulerian polynomial* $\widetilde{E}^{\underline{s}}_n(x)$ in the following way. First, define the set

$$\mathcal{I}^{\underline{s}}_n = \{\underline{e} = (e_1, \dots, e_n) \in \mathbb{Z}^n : 0 \le e_i < s_i \text{ for all } 0 \le i \le n\},$$

where we set $e_0 = e_{n+1} = 0$ and $s_0 = s_{n+1} = 1$. Furthermore, we say that $i \in [0, n]$ is an ascent of $\underline{e} \in \mathcal{I}^{\underline{s}}_n$ if $\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}$, and that it is a collision if $\frac{e_i}{s_i} = \frac{e_{i+1}}{s_{i+1}}$. We write $\operatorname{asc}(\underline{e})$ and $\operatorname{col}(\underline{e})$ for the number of ascents and collisions of $\underline{e}$, respectively. Now define the polynomial

$$\widetilde{E}^{\underline{s}}_n(x) := \sum_{\underline{e}\in\mathcal{I}^{\underline{s}}_n} (1+x)^{\operatorname{col}(\underline{e})} x^{\operatorname{asc}(\underline{e})}.$$

The main result of Haglund and Zhang [HZ19, Theorem 1.1] proves that all such polynomials are real-rooted.

Theorem 1.4.2.1. *Let $\underline{s} = (s_1, \dots, s_n) \in \mathbb{Z}^n_{>0}$. Then $\widetilde{E}^{\underline{s}}_n(x)$ is real-rooted.*

We mention explicitly that one of the motivations of Haglund and Zhang to define their polynomials originates in the work of Savage and Visontai [SV15] and Gustafsson and Solus [GS20], in which they define similar real-rooted polynomials which are indexed by vectors of positive integers.

Chapter 2

Geometry

2.1 Varieties

The purpose of this section is to define some varieties associated to hyperplane arrangements, namely the *complement* of an arrangement, the *De Concini-Procesi wonderful model*, the *reciprocal plane*, the *Schubert variety* and the *augmented wonderful model.* Their cohomology (be it ordinary cohomology or some more refined version of it like the intersection cohomology) can be extracted in a purely combinatorial way by working on the intersection poset of the arrangement, i.e. the lattice of flats of the associated matroid. This motivates us to find a combinatorial version of these geometric structures even for non-realizable matroids. This will be done in the remaining of the chapter, in Section 2.2.1, 2.2.2 and 2.2.3.

We discussed in Section 1.1.4 how one can build a matroid $\mathsf{M} = \mathsf{M}(\mathcal{A})$ from a central hyperplane arrangement $\mathcal{A}$ on a vector space V (where we can require $\mathcal{A}$ to be essential if we want the rank of the matroid to be equal to the dimension of V). Here is an example we will carry throughout this section to build all the varieties.

Example 2.1.0.1. Let $\mathcal{A} = \{H_1, H_2, H_3\}$ be an arrangement over a 2-dimensional vector space given by the following equations,

$$\begin{aligned} H_1 =& \{x_1 = 0\}, \\ H_2 =& \{x_1 + x_2 = 0\}, \\ H_3 =& \{x_2 = 0\}, \end{aligned}$$

The underlying matroid has as bases $\binom{E}{2}$ and is isomorphic to $\mathsf{U}_{2,3}$. This is also a graphic matroid realized by the 3-cycle, (the complete graph over 3 vertices). In particular, then, it is a regular matroid, therefore we do not need to worry about questions of representability.

We also recall the following constructions in analogy with the usual matroid operations. For simplicity, and since we will not need the general case, we will only define them in terms of flats.

Definition 2.1.0.2. Let $\mathcal{A}$ be an arrangement over V and denote by $\mathsf{M} = \mathsf{M}(\mathcal{A})$ its associated matroid. Consider a flat $F \in \mathcal{L}(\mathsf{M})$ and the corresponding subspace $V_F = \bigcap_{i \in F} H_i$. We define the following arrangements:

- The *restriction* to F, denoted $\mathcal{A}_{|F}$ is the arrangement $\mathcal{A}_{|F} = \{H_i\}_{i\in F}$. This is the arrangement obtained by considering all the hyperplanes that contain V_F. After quotienting by H_F we obtain an arrangement in V/V_F that corresponds to the matroid restriction $\mathsf{M}_{|F}$.
- The *localization* to F, denoted $\mathcal{A}/F$ is the arrangement over V_F given by $\{H_i \cap V_F \mid i \in E \setminus F\}$. This corresponds to the operation of matroid contraction M/F.

Instead of working with the previous definition, it will be convenient for us to define an arrangement as follows.

Definition 2.1.0.3. A hyperplane arrangement $\mathcal{A}$ over a $\mathbb{K}$-vector space V is a finite list of non-zero linear forms

$$\mathcal{A} = \{\varphi_1, \dots, \varphi_n\}.$$

We recover the previous definition by setting $H_i = \ker\varphi_i$. Of course the same hyperplane H_i can be defined with any scalar multiple of φ_i, but we claim that this is not important for our purposes.

Remark 2.1.0.4. With the latter definition, we exclude identically zero forms from our list because those would not give us codimension 1 subspaces. This amounts to say that the underlying matroid is loopless.

The vector space V can now be seen as a subspace of $\mathbb{K}^n$ by means of the linear map

$$\begin{aligned} f : V &\to \mathbb{K}^n \\ x &\mapsto (\varphi_i(x)). \end{aligned}$$

Consider a subset $S \subseteq E$ such that there exist $a_i \in \mathbb{K}$, $(a_i)_{i\in S} \neq (0, \dots, 0)$ such that

$$\sum_{i\in S} a_i\varphi_i = 0.$$

We call such a subset a *dependency* of the arrangement $\mathcal{A}$. A minimal dependency C will be called a *circuit* of the arrangement; these correspond exactly to the circuits $\mathcal{C}(\mathsf{M})$ of the underlying (realizable) matroid M. The circuits give a defining set of equations for V as a subspace of $\mathbb{K}^n$. Conversely, any subspace of $\mathbb{K}^n$ gives rise to a matroid by following the same procedure in reverse. If we want to work with loopless matroids, this means that V cannot be contained in any coordinate hyperplane.

An arrangement $\mathcal{A}$ can be stratified in the following way. Consider

$$J_S = \bigcap_{i\in S} H_i \text{ and } H_S = J_S \setminus \bigcup_{S\subsetneq T} J_T.$$

Then, H_S is non-empty if and only if S is a flat of M and

$$\bigcup_{i\in E} H_i = \bigsqcup_{\substack{F\in\mathcal{L}(\mathsf{M}) \\ F\neq\emptyset}} H_F.$$

Observe also that $x \in H_F \subset V$ if and only if the set of indices $\{i \in E \mid z_i = 0\}$ is exactly F. Moreover, the non-zero entries give us exactly the arrangement $\mathcal{A}/F$.

Example 2.1.0.5. The matroid B_1 is realized by $V = \mathbb{K}^1$, which can be stratified as $\{0\}$ and $\mathbb{K}^\times$. A different way of realizing the arrangement in Example 2.1.0.1 is to consider

$$V = \{(s, s+t, t) \mid s, t \in \mathbb{K}\} \subset \mathbb{K}^3.$$

If we use (z_i) as coordinates of $\mathbb{K}^3$, V is the subspace given by the equation

$$\{z_1 - z_2 + z_3 = 0\}.$$

This is exactly the only circuit of M.
To study the stratification with the following interpretation, we can fix a point $x \in V$ and consider the vector $(\varphi_i(x)) \in \mathbb{K}^3$. These are the strata of $\mathcal{A}$ we obtain:

- $H_\emptyset = \{(s, s+t, t) \mid s, s+t, t \neq 0\}$,
- $H_1 = \{(0, t, t) \mid t \neq 0\}$,
- $H_2 = \{(s, 0, -s) \mid s \neq 0\}$,
- $H_3 = \{(s, s, 0) \mid s \neq 0\}$,
- $H_E = \{(0, 0, 0)\}$.

It is also useful to consider the *projectivization* of an arrangement, by taking the projectivization $\mathbb{P}(H_i)$ for every hyperplane $H_i \in \mathcal{A}$. If V is the subspace of $\mathbb{K}^n$ that corresponds to $\mathcal{A}$, we can look at $\mathbb{P}(V)$ to recover all the proper flats.

Example 2.1.0.6. The projectivization of the Boolean arrangement B_1 gives a single point $\{[1]\} =$ "$\mathbb{P}^0(\mathbb{K})$". In Example 2.1.0.1, the corresponding projective strata are

- $\mathbb{P}(H_\emptyset) = \{[s : s+t : t] \mid s, s+t, t \neq 0\}$,
- $\mathbb{P}(H_1) = \{[0 : 1 : 1]\}$,
- $\mathbb{P}(H_2) = \{[1 : 0 : -1]\}$,
- $\mathbb{P}(H_3) = \{[1 : 1 : 0]\}$.

2.1.1 Complement of an arrangement

The first geometric object associated to $\mathcal{A}$ that we study is the complement of the arrangement $\mathcal{M}(\mathcal{A})$,

$$\mathcal{M}(\mathcal{A}) := V \setminus \bigcup_{i \in E} H_i,$$

or, equivalently, the stratum $H_\emptyset \subset (\mathbb{K}^\times)^E$.

The projectivization $\mathbb{P}(\mathcal{M}(\mathcal{A}))$ is defined by

$$\mathbb{P}(\mathcal{M}(\mathcal{A})) = \mathbb{P}(V) \setminus \bigcup_i \mathbb{P}(H_i),$$

or equivalently as the projective stratum $\mathbb{P}(H_\emptyset)$.

Let us start with some remarks on the combinatorial and geometric properties of $\mathcal{M}(\mathcal{A})$. If we work over a finite field $\mathbb{K} = \mathbb{F}_q$, then V is a discrete set of points of cardinality q^k. The complement $\mathcal{M}(\mathcal{A})$ then consists of a finite set of points.

Theorem 2.1.1.1. *If $\mathcal{A}$ is an arrangement over a finite field $\mathbb{F}_q$, then the cardinality of $\mathcal{M}(\mathcal{A})$ is equal to*

$$\#\mathcal{M}(\mathcal{A}) = \chi_{\mathsf{M}}(q).$$

Proof. We prove it by induction on the number of hyperplanes in $\mathcal{A}$. If the arrangement is empty the statement clearly holds. Otherwise, consider the arrangement $\mathcal{A} \setminus \{\varphi_e\}$ in $\mathbb{K}^{E\setminus\{e\}}$. Then $\#\mathcal{M}(\mathcal{A}\setminus\{\varphi_e\}) = \chi_{\mathsf{M}\setminus e}(q)$. All the corresponding points in $\mathbb{K}^E$ are in $\mathcal{M}(\mathcal{A})$ unless $z_e = 0$. However, the points that satisfy $z_e = 0$ and $z_f \neq 0$ for every $f \neq e$ are exactly the ones in $\mathcal{M}(\mathcal{A}/e)$, which are counted, by induction, by $\chi_{\mathsf{M}/e}(q)$. Therefore,

$$\#\mathcal{M}(\mathcal{A}) = \chi_{\mathsf{M}\setminus e}(q) - \chi_{\mathsf{M}/e}(q) = \chi_{\mathsf{M}}(q),$$

where the last equality comes from the deletion-contraction formula from Theorem 1.1.5.5. □

If the arrangement is over the real numbers, every hyperplane cuts V into two halfspaces. Hence, the complement is a disjoint union of contractible connected components.

Theorem 2.1.1.2 ([Zas75])**.** *If $\mathcal{A}$ is an arrangement over $\mathbb{R}$, then the number of connected components of $\mathcal{M}(\mathcal{A})$ is equal to*

$$\#\{\textit{connected components of } \mathcal{M}(\mathcal{A})\} = (-1)^{\operatorname{rk}\mathsf{M}}\chi_{\mathsf{M}}(-1).$$

The proof relies again on the deletion-contraction formula. One can also obtain similar results for the projectivization $\mathbb{P}(\mathcal{M}(\mathcal{A}))$.

Theorem 2.1.1.3. *If $\mathcal{A}$ is an arrangement over a finite field $\mathbb{F}_q$, then the cardinality of $\mathbb{P}(\mathcal{M}(\mathcal{A}))$ is equal to*

$$\#\mathbb{P}(\mathcal{M}(\mathcal{A})) = \overline{\chi}_{\mathsf{M}}(q).$$

Example 2.1.1.4. The complement of a Boolean arrangement B_n is $\mathcal{M}(\mathsf{B}_n) = (\mathbb{K}^\times)^n$ and therefore if we realize it over a finite field $\mathbb{F}_q$, then $\#\mathcal{M}(\mathsf{B}_n) = (q-1)^n = \chi_{\mathsf{B}_n}(q)$. If instead we realize it over $\mathbb{R}$, the complement has $2^n = (-1)^n(-1-1)^n = (-1)^n\chi_{\mathsf{B}_n}(-1)$ connected components. If we consider the arrangement from Example 2.1.0.1, the characteristic polynomial of $\mathsf{U}_{2,3}$ is $\chi_{\mathsf{U}_{2,3}}(x) = x^2 - 3x + 2$, thus $\overline{\chi}_{\mathsf{U}_{2,3}}(x) = x - 2$. If we work on a finite plane $(\mathbb{F}_q)^2$ (which has q^2 points), each hyperplane is a line with q points. Notice that as a result of removing all the points of the three lines the origin is also removed three times. By adding it back twice we obtain

$$\#\mathcal{M}(\mathcal{A}) = q^2 - 3q + 2.$$

The projectivization of $\mathcal{A}$ is an arrangement of three points on a projective line, thus

$$\#\mathbb{P}(\mathcal{M}(\mathcal{A})) = (q+1) - 3 = q - 2 = \overline{\chi}_{\mathsf{U}_{2,3}}(q).$$

For real arrangements notice that $\mathcal{A}$ cuts $\mathbb{R}^2$ in six regions, and $\chi_{\mathsf{U}_{2,3}}(-1) = 6$.

The situation becomes more interesting when we start to consider arrangements over $\mathbb{C}$. In this case, the complement $\mathcal{M}(\mathcal{A})$ is a smooth variety, which is studied in more details in Section 2.2.1.

2.1.2 Wonderful model

We now proceed to describe the second variety we are interested in, the *De Concini–Procesi wonderful model* $\underline{X}(\mathcal{A})$, first introduced in [DCP95].

Definition 2.1.2.1. For a fixed hyperplane arrangement $\mathcal{A}$ we start by considering the projectivization in $\mathbb{P}(\mathbb{K}^E)$ and the corresponding strata $\mathbb{P}(H_F)$. Then, we proceed by blowing up in $\mathbb{P}(V)$ all the points $\mathbb{P}(H_F)$ corresponding to corank 1 flats, then the strict transforms of $\mathbb{P}(H_F)$ for corank 2 flats, and so on up to rank 1 flats. The resulting variety is the *De Concini–Procesi wonderful model* $\underline{X}_{\mathcal{A}}$.

In $\underline{X}_{\mathcal{A}}$ we have a divisor for every non-empty proper flat of M, and two divisors intersect if and only if the two flats are comparable in $\mathcal{L}(\mathsf{M})$. As usual, far from the blow-up, the variety is unchanged, thus the complement $\mathbb{P}(\mathcal{M}(\mathcal{A}))$ naturally sits inside $\underline{X}_{\mathcal{A}}$.

Example 2.1.2.2. The wonderful model of the Boolean matroid $\underline{X}_{\mathsf{B}_n}$ is known as the *permutohedral variety*. In particular, $\underline{X}_{\mathsf{B}_1}$ is a point. Consider Example 2.1.0.1. The projectivization $\mathbb{P}(V)$ is a 1-dimensional projective space and the three projective strata $\mathbb{P}(H_i)$ are points on $\mathbb{P}^1$. The blow-up of these three points leaves the variety unchanged. This means that

$$\underline{X}_{\mathcal{A}} \cong \mathbb{P}^1(\mathbb{K}).$$

2.1.3 Reciprocal plane

We follow [EPW16]. Taking the reciprocal of all the coordinates of $(\mathbb{K}^\times)^E$ is an automorphism of the complement $\mathcal{M}(\mathcal{A}) \subset (\mathbb{K}^\times)^E$. After applying it, we obtain

$$\mathcal{M}(\mathcal{A})^{-1} = \{\left(\frac{1}{z_i}\right) \mid z_i \neq 0\, \forall i \in E\} \subset (\mathbb{K}^\times)^E.$$

We define the *reciprocal plane* of $\mathcal{A}$ as

$$Y_\emptyset(\mathcal{A}) := \overline{\mathcal{M}(\mathcal{A})^{-1}} \subset \mathbb{K}^E,$$

where we allow some coordinates to be equal to zero. The choice of notation is consistent with Section 2.1.4. By considering the equations of $\mathcal{M}(\mathcal{A})$ given by the dependencies of the arrangement we find the following defining set of equations. If $\sum_{i\in C} a_i\varphi_i = 0$ is a circuit, then

$$\sum_{i\in C} a_i \prod_{j\neq i} z_j = 0.$$

This variety is also stratified,

$$Y_\emptyset(\mathcal{A}) = \bigsqcup_{F\in\mathcal{L}(\mathsf{M})} X_F,$$

with strata indexed by all the flats of the matroid M

$$X_F = \{(z_i) \mid z_i = 0 \Leftrightarrow i \notin F\} \cap Y_\emptyset(\mathcal{A}).$$

Example 2.1.3.1. The reciprocal plane of B_n is $(\mathbb{K}^1)^n$, as we have no circuits. The reciprocal plane of the arrangement in Example 2.1.0.1 is obtained by taking the closure of

$$\mathcal{M}(\mathcal{A})^{-1} = \{(\frac{1}{s}, \frac{1}{s+t}, \frac{1}{t}) \mid s, +t, t \neq 0\}.$$

By using the equations computed in 2.1.0.5, we find the defining set of equations

$$Y_\emptyset(\mathcal{A}) = \{z_2z_3 - z_1z_3 + z_1z_2 = 0\}.$$

The strata are

- $X_\emptyset = \{(0,0,0)\}$,
- $X_1 = \{(\frac{1}{s}, 0, 0), s \neq 0\}$,
- $X_2 = \{(0, \frac{1}{s+t}, 0), s+t \neq 0\}$,
- $X_3 = \{(0,0,\frac{1}{t}), t \neq 0\}$,
- $X_E = \{(\frac{1}{s}\frac{1}{s+t}, \frac{1}{t}), s, s+t, t \neq 0\} = \mathcal{M}(\mathcal{A})^{-1}$.

Observe how the dimension of every stratum is equal to the rank of the corresponding flat. Moreover, the following theorems hold.

Theorem 2.1.3.2. *Given a stratum X_F,*

$$X_F \cong \mathcal{M}(\mathcal{A}_{|F})$$

$$\overline{X_F} = \bigsqcup_{G \leq F} X_G \cong Y_\emptyset(\mathcal{A}_{|F}).$$

Example 2.1.3.3. Consider the arrangement $\mathcal{A}_{|1}$. The corresponding matroid $\mathsf{M}_{|1}$ is isomorphic to a Boolean matroid B_1. The stratum X_1 is

$$X_1 = \{\left(\frac{1}{s}, 0, 0\right), s \neq 0\} \cong \mathbb{K}^\times \cong \mathcal{M}(\mathcal{A}_{|1})^{-1}.$$

When closing the stratum X_1 we add exactly the point zero corresponding to the empty flat,

$$\overline{X_1} = X_1 \sqcup X_\emptyset = \mathbb{K}^1 \cong Y_\emptyset(\mathcal{A}_{|1}).$$

We are also interested in the local geometry of this variety.

Theorem 2.1.3.4. *The reciprocal plane $Y_\emptyset(\mathcal{A}/F)$ associated to the contraction by F is a normal slice to the stratum X_F.*

2.1.4 Schubert variety

We follow [PXY18]. There are several ways of taking a projectivization of the ambient space $\mathbb{K}^E$. We can consider $\mathbb{K}^E = \prod_{i\in E} \mathbb{K}^1$ and then take the projective closure of each of the separate n lines separately. We define the *Schubert variety* $Y(\mathcal{A})$ of the arrangement $\mathcal{A}$ as the closure of the set

$$\{([z_1 : 1], \ldots, [z_n : 1])\} \subset (\mathbb{P}^1)^n.$$

Fix coordinates $([x_i : y_i])$ for this new ambient space. This variety was also studied in [AB16] and [HW17a]. By considering the equations of $\mathcal{M}(\mathcal{A})$ given by the dependencies of the arrangement we find the following defining equation. If $\sum_{i\in C} a_i\varphi_i = 0$, then

$$\sum_{i\in C} a_i x_i \prod_{j\neq i} y_j = 0.$$

This variety is also stratified,

$$Y(\mathcal{A}) = \bigsqcup_{F \in \mathcal{L}(\mathsf{M})} Y_F,$$

with strata indexed by all the flats of the matroid M

$$Y_F = \{([x_i : y_i]) \mid y_i = 0 \Leftrightarrow i \notin F\} \cap Y(\mathcal{A}).$$

Example 2.1.4.1. The Boolean matroid B_n has Schubert variety isomorphic to $(\mathbb{P}^1)^n$. The Schubert variety of the arrangement in Example 2.1.0.1 is obtained by taking the closure of V in $(\mathbb{P}^1)^3$,

$$V = \{([s : 1], [s + t : 1], [t : 1])\} \subset (\mathbb{P}^1)^3.$$

By using the equations computed in 2.1.0.5 we find the defining set of equations

$$Y(\mathcal{A}) = \{x_1 y_2 y_3 - x_2 y_1 y_3 + x_3 y_1 y_2 = 0\}.$$

The strata are

- $Y_\emptyset = \{([1 : 0], [1 : 0], [1 : 0])\}$,
- $Y_1 = \{([x_1 : 1], [1 : 0], [1 : 0])\}$,
- $Y_2 = \{([1 : 0], [x_1 + x_2 : 1], [1 : 0])\}$,
- $Y_3 = \{([1 : 0], [1 : 0], [x_2 : 1])\}$,
- $Y_E = \{([x_1 : 1], [x_1 + x_2 : 1], [x_2 : 1])\} = V$.

Observe how the dimension of every stratum is equal to the rank of the corresponding flat. Moreover, the following theorem holds.

Theorem 2.1.4.2. *Given a stratum Y_F,*

$$Y_F \cong V/H_F$$

$$\overline{Y_F} = \bigsqcup_{G \leq F} Y_G \cong Y(\mathcal{A}_{|F}).$$

Example 2.1.4.3. Consider the arrangement $\mathcal{A}_{|1}$. The stratum Y_1 is

$$Y_1 = \{([x_1 : 1], [1 : 0], [1 : 0])\} \cong \mathbb{K}^1 \cong V/H_1.$$

When closing the stratum Y_1 we add exactly $Y_\emptyset$, which is the point at infinity $[1 : 0]^3$ that gives us $\mathbb{P}^1 = \mathbb{K}^1 \sqcup \{\infty\}$, i.e.

$$\overline{Y_1} = Y_1 \sqcup Y_\emptyset = \mathbb{P}^1 \cong Y(\mathcal{A}_{|1}).$$

We are also interested in the local geometry of this variety. Consider the affine chart $U_\emptyset = \{x_i \neq 0 \text{ for every } i\}$ centered at the point at infinity $Y_\emptyset$. Working on the chart $[1 : \frac{y_i}{x_i}]$, one finds exactly the same points of $Y_\emptyset(\mathcal{A})$. More precisely, the following theorem holds

Theorem 2.1.4.4. *Locally at the point at infinity, the Schubert variety is isomorphic to the reciprocal plane, i.e.*

$$Y(\mathcal{A}) \cap U_\emptyset \cong Y_\emptyset(\mathcal{A}).$$

More generally, locally at a point $x \in Y_F$, the variety is isomorphic to $Y_\emptyset(\mathcal{A}/F) \times Y_F$.

2.1.5 Augmented wonderful model

The last variety we want to construct is the *augmented wonderful model* $X_{\mathcal{A}}$, firstly studied in [BHM+22a].

Definition 2.1.5.1. For a fixed hyperplane arrangement $\mathcal{A}$, we start by considering the ambient space $\mathbb{P}(V \oplus \mathbb{K}) = V \cup \mathbb{P}(V)$. Consider the subspaces $\mathbb{P}(H_F)$ as linear subspaces of $\mathbb{P}(V)$. Then, we proceed by blowing up all the points corresponding to corank 1 flats, then the strict transforms of $\mathbb{P}(H_F)$ for corank 2 flats, and so on up to rank 1 flats. The resulting variety is the *augmented wonderful model* $X_{\mathcal{A}}$.

Since all the blowups are centered in the hyperplane at infinity $\mathbb{P}(V)$, V remains an open subspace of $X_{\mathcal{A}}$. Moreover, the strict transform of the hyperplane at infinity is clearly isomorphic to $\underline{X}_{\mathcal{A}}$.

Remark 2.1.5.2. A different construction of $X_{\mathcal{A}}$ involves the Schubert variety $Y(\mathcal{A})$. The variety $Y(\mathcal{A})$ is singular and it admits $X_{\mathcal{A}}$ as a canonical resolution, obtained first by blowing up $Y_{\emptyset}$, then the strict transforms of all the strata Y_F corresponding to flats of rank 1 and so on.

2.2 Combinatorial cohomology rings

Once we have obtained geometric objects from hyperplane arrangements, we are interested in studying their cohomological properties. As the goal is to produce invariants for the underlying matroid $\mathsf{M} = \mathsf{M}(\mathcal{A})$, the hope is that the answer becomes purely combinatorial, in the sense that varieties coming from arrangements realizing isomorphic matroids will have the same cohomology. Moreover, if the resulting answer is indeed only dependent on the matroid M, we can then try to define these cohomology rings for every matroid (even if there are no varieties associated to them anymore), not just the 0% that is representable (see Theorem 1.1.4.6).

2.2.1 Orlik–Solomon algebra

We now proceed to build the first combinatorial graded vector space, the Orlik–Solomon algebra. We quickly recap the classical construction by Orlik and Solomon in the case of a realizable matroid following [OT92]. Consider a complex arrangement $\mathcal{A}$ and $\mathrm{H}^\bullet(\mathcal{M}(\mathcal{A})) := \mathrm{H}^\bullet(\mathcal{M}(\mathcal{A}), \mathbb{Z})$, the cohomology ring with integer coefficients of its complement. The degree 1 of $\mathrm{H}^1(\mathbb{C}^\times, \mathbb{Z})$ in De Rham cohomology can be generated by the form

$$\frac{1}{2\pi i}\frac{dz}{z}$$

and, more generally, each hyperplane $H_i \in \mathcal{A}$ gives a generator in cohomology given by

$$\omega_i := \frac{1}{2\pi i}\frac{d\varphi_i}{\varphi_i},$$

where φ_i is the form representing the hyperplane H_i. One can show that the set $\{[\omega_i]\}_{i\in E}$ generates $\mathrm{H}^\bullet(\mathcal{M}(\mathcal{A}))$. Denote by $\bigwedge(E)$ the exterior algebra generated in degree 1 by $\{u_i\}_{i\in[n]}$ and with $u_S := \bigwedge_{i\in S} u_i$. For these definitions, we need to identify E with the interval $[n]$, equipped with the natural order inherited by $\mathbb{N}$. The derivation in $\bigwedge(E)$ is given by

$$\partial u_S = \sum_{j=1}^{r}(-1)^{j-r} u_{S\setminus\{u_{i_j}\}}$$

for $S = \{u_{i_1}, \dots, u_{i_r}\}$.

Consider the homomorphism

$$\begin{aligned}\Phi : \bigwedge(E) &\to \mathrm{H}^\bullet(\mathcal{M}(\mathcal{A}))\\ u_i &\to [\omega_i].\end{aligned}$$

We call the kernel of Φ the *Orlik–Solomon ideal* $I_{\mathcal{A}}$. This is described by the following result.

Theorem 2.2.1.1. *The Orlik–Solomon ideal* $I_{\mathcal{A}}$ *is generated by*

$$\{\partial u_C \mid C \in \mathcal{C}(\mathsf{M})\}.$$

We define the *Orlik–Solomon algebra* of the arrangement $\mathcal{A}$ to be the quotient

$$\mathrm{OS}(\mathcal{A}) := \frac{\bigwedge(E)}{I_{\mathcal{A}}}.$$

In [OT92, Section 5.4] we find the proof of the following result.

Theorem 2.2.1.2. *The map*

$$\mathrm{OS}(\mathcal{A}) \to \mathrm{H}^\bullet(\mathcal{M}(\mathcal{A}))$$

is an isomorphism of graded algebras.

One quickly notices that the definition of $\mathrm{OS}(\mathcal{A})$ only relies on the underlying matroid M. This motivates us to define it for every matroid and not just the representable ones.

Definition 2.2.1.3. Let M be a loopless matroid. The *Orlik–Solomon algebra* of M is the quotient

$$\mathrm{OS}(\mathsf{M}) = \frac{\bigwedge(E)}{I_\mathsf{M}},$$

where $\bigwedge(E)$ is the exterior algebra generated in degree 1 by elements $\{u_e\}_{e\in E}$ and I_M is the Orlik–Solomon ideal defined by

$$I_\mathsf{M} = (\partial u_C \mid C \in \mathcal{C}(\mathsf{M}))\,.$$

By construction, since I_M is homogeneous, $\mathrm{OS}(\mathsf{M})$ is graded, with the grading induced by the natural one on $\bigwedge(E)$. The set of generators of I_M given by circuits is minimal; one could indeed consider the ideal generated by $\{\partial u_S\}$ for every dependent set $S \subseteq E$. This implies that $\mathrm{OS}^i(\mathsf{M}) = 0$ for every $i > \operatorname{rk} M$, hence we can write

$$\mathrm{OS}^\bullet(\mathsf{M}) = \bigoplus_{i=0}^{\operatorname{rk}\mathsf{M}} \mathrm{OS}^i(\mathsf{M}).$$

We now present some known results on $\mathrm{OS}(\mathsf{M})$.

Theorem 2.2.1.4 (Brieskorn's Lemma [OT92, Lemma 5.91])**.** *For every matroid* M*,*

$$\mathrm{OS}(\mathsf{M}) = \bigoplus_{i=0}^{\operatorname{rk}\mathsf{M}} \bigoplus_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \operatorname{rk} F = i}} \mathrm{OS}^i(\mathsf{M}_{|F}).$$

Theorem 2.2.1.5 ([OT92, Theorem 5.87])**.** *Consider a matroid* M *and a fixed element* $e \in E$*. Then for every* $i \geq 0$ *the following is a split short exact sequence*

$$0 \to \mathrm{OS}^i(\mathsf{M}\setminus e) \to \mathrm{OS}^i(\mathsf{M}) \to \mathrm{OS}^{i-1}(\mathsf{M}/e) \to 0.$$

Lastly, we produce an explicit additive basis for $\mathrm{OS}(\mathsf{M})$ called the *nbc basis*. This basis is not canonical and depends on a linear order ω of the ground set E. For the rest of this section, we identify E with $\{1,\dots,n\}$ with the natural ordering inherited from $\mathbb{N}$. For every circuit $C \in \mathcal{C}(\mathsf{M})$ we define the associated *broken circuit* to be $\widetilde{C} := C \setminus \min_\omega C$.

Lemma 2.2.1.6. *If a set is dependent, then it contains a broken circuit.*

Proof. Every dependent set contains a circuit C, which in turns contains the broken circuit $\widetilde{C}$. □

Among all the independent sets, we say that $I \in \mathcal{I}(\mathsf{M})$ is a *non-broken-circuit set* (or nbc for short) if it does not contain any broken circuit (for a fixed linear order of the ground set ω).

Consider the grading on $\bigwedge(E)$ given by setting the degree of u_i equal to i. The grading induces an increasing filtration on $\mathrm{OS}(\mathsf{M})$ whose i-th piece is equal to the image of classes of degree $\leq i$ in $\bigwedge(E)$, and the associated graded ring $\operatorname{gr}\mathrm{OS}(\mathsf{M})$ is isomorphic to the quotient

$$\operatorname{gr}\mathrm{OS}(\mathsf{M}) = \frac{\bigwedge(E)}{\langle u_{\widetilde{C}} \mid C \in \mathcal{C}(\mathsf{M})\rangle}.$$

Theorem 2.2.1.7 ([OT92, Theorem 3.43])**.** *Given a matroid* M *and a linear order of its ground set* ω*, the family*

$$\{u_S \mid S \text{ is nbc of cardinality } i\}$$

is a basis for $\operatorname{gr}\mathrm{OS}^i(\mathsf{M})$*. Moreover,*

$$\{u_S + I_{\mathsf{M}} \mid S \text{ is nbc}\}$$

is a basis for $\mathrm{OS}(\mathsf{M})$ *as a graded vector space.*

Example 2.2.1.8. Consider the matroid from Example 2.1.0.1 on the groundset $E = \{1, 2, 3\}$. The family of circuits of this matroid is $\mathcal{C}(\mathsf{M}) = \{123\}$ and therefore the only broken circuit is $\{23\}$. A basis for $\mathrm{OS}^\bullet(\mathsf{U}_{2,3})$ is given by

$$\mathrm{OS}(\mathsf{U}_{2,3}) = \mathbb{Q} \oplus \mathbb{Q}\langle u_1, u_2, u_3\rangle \oplus \mathbb{Q}\langle u_{12}, u_{13}\rangle.$$

2.2.2 Chow rings

As with the Orlik–Solomon algebra $\mathrm{OS}(\mathsf{M})$ from the previous section, one can now build the cohomology ring of the smooth projective varieties $\underline{X}(\mathcal{A})$ and $X(\mathcal{A})$ and then try to generalize this construction to every matroid.

In [FY04] Feichtner and Yuzvinsky introduced the notion of Chow ring for an arbitrary finite atomic lattice. Their construction takes as inputs an atomic lattice $\mathcal{L}$ and a so-called *building set* $\mathcal{G} \subseteq \mathcal{L}$, and returns a ring $D(\mathcal{L}, \mathcal{G})$ that they refer to as *the Chow ring of* $\mathcal{L}$ *with respect to* $\mathcal{G}$. In our setting, the atomic lattice $\mathcal{L}$ will be the lattice of flats of a loopless matroid $\mathcal{L}(\mathsf{M})$ and the building set will be the so-called *maximal* one $\mathcal{G}_{\max}$, which is given by all the non-empty flats. The following result lets us conclude that the Chow ring of $\underline{X}_{\mathcal{A}}$ is completely determined combinatorially.

Theorem 2.2.2.1 ([FY04, Corollary 2])**.** *Let* $\mathcal{L}(\mathsf{M})$ *be the lattice of flats of a representable matroid* M *and* $\mathcal{A}$ *an arrangement that realizes* M*. Then*

$$D^\bullet(\mathcal{L}, \mathcal{G}_{\max}) \cong \mathrm{H}^\bullet(\underline{X}(\mathcal{A}), \mathbb{Z}).$$

By making a slight modification to its presentation, and following the notation of [BHM+22a], we can now introduce the Chow ring of a matroid using the following definition.

Definition 2.2.2.2. Let M be a loopless matroid. The *Chow ring* of M is the quotient

$$\underline{\mathrm{CH}}(\mathsf{M}) = \mathbb{Q}[x_F \mid F \in \mathcal{L}(\mathsf{M}) \setminus \{\emptyset, E\}]/(\underline{I} + \underline{J}),$$

where the ideals $\underline{I}$ and $\underline{J}$ are defined respectively by

$$\underline{I} = \langle x_{F_1} x_{F_2} \mid F_1, F_2 \in \mathcal{L}(\mathsf{M}) \setminus \{\emptyset, E\} \text{ are incomparable}\rangle,$$
$$\underline{J} = \left\langle \sum_{i \in F} x_F - \sum_{j \in F} x_F \mid i, j \in E \right\rangle.$$

In [BHM+22a] and [BHM+22b], Braden, Huh, Matherne, Proudfoot, and Wang introduced an "augmented" version of the Chow ring of a matroid $\mathrm{CH}(\mathsf{M})$ and showed that this is isomorphic to the Chow ring of $X(\mathcal{A})$.

Definition 2.2.2.3. Let M be a loopless matroid. The *augmented Chow ring* of M is the quotient

$$\mathrm{CH}(\mathsf{M}) = \mathbb{Q}[x_F,\, y_e \mid F \in \mathcal{L}(\mathsf{M}) \setminus \{E\} \text{ and } e \in E]/(I + J),$$

where the ideals I and J are defined respectively by

$$I = \left\langle y_e - \sum_{e \notin F} x_F \mid e \in E \right\rangle,$$

$$J = \langle x_{F_1} x_{F_2} \mid F_1, F_2 \in \mathcal{L}(\mathsf{M}) \setminus \{E\} \text{ are incomparable}\rangle + \langle y_e\, x_F \mid F \in \mathcal{L}(\mathsf{M}) \setminus \{E\},\, e \notin F\rangle .$$

The augmented Chow ring can be defined in terms of the construction of Feichtner and Yuzvinsky, namely as the Chow ring of the lattice of flats of the *free coextension* of M with respect to a certain building set; we refer to [EHL22, Lemma 5.14] for more details regarding this perspective. One can also recover the Chow ring $\mathrm{CH}(\mathsf{M})$ from the augmented Chow ring $\mathrm{CH}(\mathsf{M})$ by quotienting by the ideal generated by $\{y_e \mid e \in E\}$.

By construction, since all the ideals are homogeneous, both the Chow ring $\underline{\mathrm{CH}}(\mathsf{M})$ and the augmented Chow ring $\mathrm{CH}(\mathsf{M})$ are graded rings, with the grading induced by the one on the polynomial ring. Moreover, by the incomparability relations given by $\underline{J}$ and J, it is easy to deduce that $\underline{\mathrm{CH}}^i(\mathsf{M}) = 0$ for every $i \geq \operatorname{rk} \mathsf{M}$ and $\mathrm{CH}^i(\mathsf{M}) = 0$ for every $i \geq \operatorname{rk} \mathsf{M} + 1$. Therefore, both rings admit a decomposition of the form

$$\underline{\mathrm{CH}}(\mathsf{M}) = \bigoplus_{i=0}^{\operatorname{rk} \mathsf{M}-1} \underline{\mathrm{CH}}^i(\mathsf{M}), \qquad \mathrm{CH}(\mathsf{M}) = \bigoplus_{i=0}^{\operatorname{rk} \mathsf{M}} \mathrm{CH}^i(\mathsf{M}).$$

We now present some known results on $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$.

One of the main results of Feichtner and Yuzvinsky, [FY04, Corollary 1], simplifies the computational challenge of building $\underline{\mathrm{CH}}(\mathsf{M})$ by providing an explicit Gröbner basis for the Chow ring of an atomic lattice with respect to an arbitrary building set.

Theorem 2.2.2.4. *If M has no loops, then $\underline{\mathrm{CH}}(\mathsf{M})$ has a basis consisting of monomials of the form $x_{F_1}^{m_1} \cdots x_{F_r}^{m_r}$, where $r \in \mathbb{N}$, $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r$, and $0 < m_i < \operatorname{rk} F_i - \operatorname{rk} F_{i-1}$ for all $i \in \{1, \ldots, r\}$.*

The next result provides a counterpart result for the augmented Chow ring. Given a flat F of M, choose any maximal independent set $I \subset F$, and let $y_F := \prod_{e \in I} y_e \in \mathrm{CH}(\mathsf{M})$. The element y_F does not depend on the choice of I [BHM$^+$22a, Lemma 2.11(2)].

Proposition 2.2.2.5. *For any matroid M, the augmented Chow ring $\mathrm{CH}(\mathsf{M})$ has a basis consisting of monomials of the form $y_{F_0} x_{F_1}^{m_1} \cdots x_{F_r}^{m_r}$, where $r \in \mathbb{N}$, $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r$, and $0 < m_i < \operatorname{rk} F_i - \operatorname{rk} F_{i-1}$ for all $i \in \{1, \ldots, r\}$.*

Proof. Let $\mathfrak{m} \subset \mathrm{CH}(\mathsf{M})$ be the ideal generated by $\{y_e \mid e \in E\}$. The argument in the proof of [BHM$^+$22b, Proposition 1.8] shows that we have an isomorphism

$$\operatorname{gr} \mathrm{CH}(\mathsf{M}) := \bigoplus_{i \geq 0} \frac{\mathfrak{m}^i \mathrm{CH}(\mathsf{M})}{\mathfrak{m}^{i+1} \mathrm{CH}(\mathsf{M})} \cong \bigoplus_{F \in \mathcal{L}(\mathsf{M})} \underline{\mathrm{CH}}(\mathsf{M}/F)[-\operatorname{rk} F], \tag{2.1}$$

where $\underline{\mathrm{CH}}(\mathsf{M}/F)[-\operatorname{rk} F]$ embeds into $\operatorname{gr} \mathrm{CH}(\mathsf{M})$ by sending a polynomial η in $\{x_G \mid F \subset G \subset E\}$ to the polynomial $y_F \eta$. We may therefore use the basis for each $\underline{\mathrm{CH}}(\mathsf{M}/F)$ from Theorem 2.2.2.4 to construct a basis for $\operatorname{gr} \mathrm{CH}(\mathsf{M})$, and this lifts to a basis for $\mathrm{CH}(\mathsf{M})$. □

Notice that Theorem 2.2.2.4 also uses the variable x_E, while our definition of $\underline{\mathrm{CH}}(\mathsf{M})$ does not. The two definitions coincide if we introduce the relation

$$x_E = -\sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ e\in F}} x_F.$$

The definition does not depend on the choice of $e \in E$. In [AHK18] this element of the Chow ring was denoted by $\alpha := -x_E$.

Example 2.2.2.6. The Chow ring of the matroid from Example 2.1.0.1 is

$$\underline{\mathrm{CH}}(\mathsf{U}_{2,3}) = \mathbb{Q}\langle 1\rangle \oplus \mathbb{Q}\langle \alpha\rangle.$$

The augmented Chow ring is

$$\mathrm{CH}(\mathsf{U}_{2,3}) = \mathbb{Q}\langle 1\rangle \oplus \mathbb{Q}\langle \alpha, y_1, y_2, y_3\rangle \oplus \mathbb{Q}\langle y_{123}\rangle.$$

2.2.3 Intersection cohomology module

The *graded Möbius algebra* of M is the graded vector space

$$\mathrm{H}(\mathsf{M}) = \bigoplus_{F\in\mathcal{L}(\mathsf{M})} \mathbb{Q}y_F,$$

where y_F is placed in degree $\operatorname{rk} F$. It is made into a graded algebra via the multiplication

$$y_F \cdot y_G = \begin{cases} y_{F\vee G} & \text{if } \operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F \vee G),\\ 0 & \text{if } \operatorname{rk} F + \operatorname{rk} G > \operatorname{rk}(F \vee G).\end{cases}$$

We note that $\mathrm{H}(\mathsf{M})$ is a graded subalgebra of $\mathrm{CH}(\mathsf{M})$ [BHM+22a, Proposition 2.18]; thus, we may view $\mathrm{CH}(\mathsf{M})$ as a graded $\mathrm{H}(\mathsf{M})$-module.

Definition 2.2.3.1. Let M be a matroid. The *intersection cohomology module* of M, denoted by $\mathrm{IH}(\mathsf{M})$, is the unique (up to isomorphism) indecomposable graded $\mathrm{H}(\mathsf{M})$-module direct summand of $\mathrm{CH}(\mathsf{M})$ that contains $\mathrm{H}(\mathsf{M})$. The *stalk at the empty flat* of $\mathrm{IH}(\mathsf{M})$ is the graded vector space $\mathrm{IH}(\mathsf{M})_\emptyset := \mathrm{IH}(\mathsf{M}) \otimes_{\mathrm{H}(\mathsf{M})} \mathbb{Q}$, where $\mathbb{Q}$ is the one-dimensional graded $\mathrm{H}(\mathsf{M})$-module placed in degree zero.

Remark 2.2.3.2. In [BHM+22b, Definition 3.2], a construction of $\mathrm{IH}(\mathsf{M})$ is given as an explicit $\mathrm{H}(\mathsf{M})$-submodule of $\mathrm{CH}(\mathsf{M})$.

This construction mimicks the construction of $Y(\mathcal{A})$ and $Y_\emptyset(\mathcal{A})$ in the realizable case. We follow Section 1.3 in [BHM+22b].

We observed in Remark 2.1.5.2 that the variety $Y(\mathcal{A})$ admits a canonical resolution to $X(\mathcal{A})$. The graded Möbius algebra $\mathrm{H}(\mathsf{M})$ is isomorphic to the rational cohomology ring $\mathrm{H}^\bullet(Y(\mathcal{A}))$ [HW17b, Theorem 14], and the augmented Chow ring $\mathrm{CH}(\mathsf{M})$ is isomorphic to the rational Chow ring of $X(\mathcal{A})$, or equivalently to the rational cohomology ring $\mathrm{H}^\bullet(X(\mathcal{A}))$.

By applying the decomposition theorem to the map from $X(\mathcal{A})$ to $Y(\mathcal{A})$, we find that the intersection cohomology $\mathrm{IH}^\bullet(Y(\mathcal{A}))$ is isomorphic as a graded $\mathrm{H}^\bullet(Y(\mathcal{A}))$-module to a direct summand of $\mathrm{H}^\bullet(X(\mathcal{A}))$ (All of these cohomology rings and intersection cohomology groups of varieties vanish in odd degree, and our isomorphisms double degree. So $\mathrm{H}^1(\mathsf{M}) \cong \mathrm{H}^2(Y(\mathcal{A}))$, $\mathrm{CH}^1(\mathsf{M}) \cong \mathrm{H}^2(X(\mathcal{A}))$, $\mathrm{IH}^1(\mathsf{M}) \cong \mathrm{IH}^2(Y(\mathcal{A}))$, and so on). An extension of an argument of Ginzburg [Gin91] shows that $\mathrm{IH}^\bullet(Y(\mathcal{A}))$ is indecomposable as an $\mathrm{H}^\bullet(Y(\mathcal{A}))$-module, which implies that it coincides with $\mathrm{IH}^\bullet(\mathsf{M})$.

Example 2.2.3.3. The intersection cohomology module of the matroid from Example 2.1.0.1 is given by

$$\mathrm{IH}(\mathsf{M}) := \mathbb{Q}\langle 1\rangle \oplus \mathbb{Q}\langle y_1, y_2, y_3\rangle \oplus \mathbb{Q}\langle y_{123}\rangle.$$

2.3 Hodge Theory

2.3.1 Combinatorial Hodge Theory

Definition 2.3.1.1. We say that a graded Artinian ring

$$A^\bullet = \bigoplus_{i=0}^{r} A^i$$

is a *Poincaré Duality Algebra of dimension* r if

- $A^0 \cong \mathbb{Q}$ and $A^r \cong \mathbb{Q}$,
- $A^i \cong 0$ for $i < 0$ or $i > r$,
- the multiplication in $A^\bullet$ gives isomorphisms

$$A^{r-i} \to \mathrm{Hom}_{\mathbb{Q}}(A^i, A^r).$$

Definition 2.3.1.2. We call *degree map* any isomorphism

$$\deg : A^r \to \mathbb{Q}.$$

Example 2.3.1.3. The graded polynomial ring $R^\bullet = \mathbb{Q}[x]/(x^{r+1})$ trivially satisfies all the properties of a Poincaré duality algebra. A degree map on $R^\bullet$ is given by the evaluation $x^r \mapsto 1$.

Definition 2.3.1.4. Let $\ell \in A^1$, $0 \leq i \leq \frac{r}{2}$

- The *Lefschetz operator* L_ℓ^i *associated to* ℓ *on* A^i is the linear map

$$\begin{aligned} L_\ell^i : A^i &\to A^{r-i} \\ u &\mapsto \ell^{r-2i} u; \end{aligned}$$

- The *Hodge–Riemann form* Q_ℓ^i *associated to* ℓ *on* A^i is the symmetric bilinear form

$$\begin{aligned} Q_\ell^i : A^i \times A^i &\to \mathbb{Q} \\ (u_1, u_2) &\mapsto (-1)^i \deg(u_1 \cdot L_\ell^i u_2); \end{aligned}$$

- The *primitive subspace* P_ℓ^i *of* A^i *associated to* ℓ is

$$P_\ell^i = \{u \in A^i \mid \ell \cdot L_\ell^i(u) = 0\} \subset A^i.$$

Definition 2.3.1.5. Let $A^\bullet$ be a Poincaré Duality Algebra.

- $A^\bullet$ has property $(\mathrm{HL})_\ell$ if L_ℓ^i is an isomorphism, for every $i \leq \frac{r}{2}$.
- $A^\bullet$ has property $(\mathrm{HR})_\ell$ if Q_ℓ^i is positive definite on P_ℓ^i, for every $i \leq \frac{r}{2}$
- If L_ℓ^i is an isomorphism, then

$$A^{i+1} = P_\ell^{i+1} \oplus \ell A^i.$$

  If $A^\bullet$ has property $(\mathrm{HL})_\ell$ we have the *Lefschetz decomposition*

$$A^i = \bigoplus_{j=0}^{i} \ell^j P_\ell^{i-j},$$

  for every $i \leq \frac{r}{2}$. This decomposition is orthogonal with respect to Q_ℓ^i.

If $A^\bullet$ is a Poincaré Duality Algebra that has property $(\mathrm{HL})_\ell$ and $(\mathrm{HR})_\ell$. Then we say that $A^\bullet$ satisfies the *Hodge package*.

Example 2.3.1.6. In Example 2.3.1.3, let $\ell = x$ and $\deg : x^r \mapsto 1$. Then,

- The multiplication by x^{r-2i} is a Lefschetz operator that has property $(\mathrm{HL})_\ell$;
- $P^0 = \mathbb{Q}$ and $P^i_x = \{0\}$ for all $i \neq 0$; hence $(\mathrm{HR})_\ell$ is trivially true for $i \neq 0$ and
$$(-1)^0 \deg(a\, x^{r-2\cdot 0} a) = a^2 > 0,$$
for every $a \neq 0$.
- It follows immediately that the Lefschetz decomposition is trivial for every i, as
$$P^i = x^i P^0 = \mathbb{Q}\langle x^i \rangle.$$

Proposition 2.3.1.7. *Suppose that $A^\bullet$ satisfies the Hodge package. Denote by $p(x)$ its Hilbert–Poincaré series, i.e.*
$$p(x) := \mathrm{Hilb}(A, x) = \sum_{i \geq 0}^{k} \dim A^i\, x^i.$$
Then,

- *Poincaré duality implies that $\dim A^i = \dim A^{k-i}$. In turn, this implies that the polynomial $p(x)$ is palindromic.*
- *The property $(\mathrm{HL})_\ell$ implies that $\dim A^{i-1} \leq \dim A^i$ for every $i \leq \frac{k}{2}$. This is because the multiplication by ℓ injects A^{i-1} in A^i as a direct summand. In turns, this implies that the polynomial $p(x)$ is unimodal.*

2.3.2 Hodge theory for matroids and consequences

A remarkable result in matroid theory is that three of the graded vector spaces defined in Section 2.2 satisfy the Hodge package.

Theorem 2.3.2.1 ([AHK18, Theorem 1.4], [BHM+22a, Theorem 1.6])**.** *For every matroid M, $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ are Poincaré duality algebras, i.e. for every nonnegative integer $i \leq \frac{1}{2}\operatorname{rk}\mathsf{M}$ the bilinear pairing*
$$\underline{\mathrm{CH}}^i(\mathsf{M}) \times \underline{\mathrm{CH}}^{\operatorname{rk}\mathsf{M}-i-1}(\mathsf{M}) \to \mathbb{Q} \qquad (\eta_1, \eta_2) \mapsto \underline{\deg}(\eta_1\eta_2)$$
is non-degenerate. Moreover, let
$$\underline{\ell} = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \emptyset, E}} c_F x_F,$$
with $c_{F_1} + c_{F_2} > c_{F_1 \cup F_2} + c_{F_1 \cap F_2}$ for every incomparable flats F_1 and F_2 and $c_\emptyset = c_E = 0$. Then, $\underline{\mathrm{CH}}(\mathsf{M})$ satisfies $(\mathrm{HL})_{\underline{\ell}}$ and $(\mathrm{HR})_{\underline{\ell}}$. Analogous results hold for $\mathrm{CH}(\mathsf{M})$.

Theorem 2.3.2.2 ([BHM+22b, Theorem 1.6])**.** *For every matroid M, $\mathrm{IH}(\mathsf{M})$ satisfies Poincaré duality, i.e. for every nonnegative integer $i \leq \frac{1}{2}\operatorname{rk}\mathsf{M}$ the bilinear pairing*
$$\mathrm{IH}^i(\mathsf{M}) \times \mathrm{IH}^{\operatorname{rk}\mathsf{M}-i}(\mathsf{M}) \to \mathbb{Q} \qquad (\eta_1, \eta_2) \mapsto \deg(\eta_1\eta_2)$$

is non-degenerate. Moreover, let

$$\ell = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ \operatorname{rk} F = 1}} c_F x_F, \qquad c_F > 0 \text{ for every rank 1 flat } F \text{ of } \mathsf{M}.$$

Then, $\mathrm{IH}(\mathsf{M})$ *satisfies* $(\mathrm{HL})_\ell$ *and* $(\mathrm{HR})_\ell$.

Corollary 2.3.2.3. *For every loopless matroid* M *the following results hold:*

- *The Hilbert–Poincaré series of* $\underline{\mathrm{CH}}(\mathsf{M})$ *and* $\mathrm{CH}(\mathsf{M})$ *are palindromic unimodal polynomials.*
- *The Hilbert–Poincaré series of* $\mathrm{IH}(\mathsf{M})$ *is a palindromic unimodal polynomial.*

Using this combinatorial version of Hodge theory for matroids, Adiprasito, Huh and Katz proved the following in [AHK18].

Theorem 2.3.2.4 (Heron-Rota-Welsh Conjecture)**.** *The characteristic polynomial of a matroid* M *is log-concave.*

In particular, the proof exploits the Hodge–Riemann bilinear relations in degree 1 of $\underline{\mathrm{CH}}(\mathsf{M})$ and an interpretation of the coefficients of $\chi_{\mathsf{M}}(x)$ as products of powers of elements in $\underline{\mathrm{CH}}^1(\mathsf{M})$.

2.3.2.1 Semi-small decomposition

In [BHM+22a] Braden, Huh, Matherne, Proudfoot, and Wang found a *semi-small decomposition* for both the Chow ring and the augmented Chow ring of arbitrary loopless matroids. These decompositions can be used to prove the Hodge package for both of these rings. Before stating this result, let us introduce some useful notations: whenever M is a loopless matroid, for every $e \in E$ we define two special families of flats of M:

$$\underline{\mathcal{S}}_e = \underline{\mathcal{S}}_e(\mathsf{M}) = \{F \in \mathcal{L}(\mathsf{M}) : \emptyset \subsetneq F \subsetneq E \setminus \{e\} \text{ and } F \cup \{e\} \in \mathcal{L}(\mathsf{M})\},$$
$$\mathcal{S}_e = \mathcal{S}_e(\mathsf{M}) = \{F \in \mathcal{L}(\mathsf{M}) : F \subsetneq E \setminus \{e\} \text{ and } F \cup \{e\} \in \mathcal{L}(\mathsf{M})\}.$$

Theorem 2.3.2.5 ([BHM+22a, Theorems 1.2 and 1.5])**.** *Let* M *be a loopless matroid and let* $e \in E$ *be an element that is not a coloop. Then, there is an isomorphism of* $\underline{\mathrm{CH}}(\mathsf{M} \setminus \{e\})$*-modules:*

$$\underline{\mathrm{CH}}(\mathsf{M}) \cong \underline{\mathrm{CH}}(\mathsf{M} \setminus e) \oplus \bigoplus_{F \in \underline{\mathcal{S}}_e} \underline{\mathrm{CH}}\left(\mathsf{M}/(F \cup \{e\})\right) \otimes \underline{\mathrm{CH}}(\mathsf{M}_{|F})[-1].$$

Additionally, there is an isomorphism of $\mathrm{CH}(\mathsf{M} \setminus \{e\})$*-modules:*

$$\mathrm{CH}(\mathsf{M}) \cong \mathrm{CH}(\mathsf{M} \setminus e) \oplus \bigoplus_{F \in \mathcal{S}_e} \underline{\mathrm{CH}}\left(\mathsf{M}/(F \cup \{e\})\right) \otimes \mathrm{CH}(\mathsf{M}_{|F})[-1].$$

If, instead, e is a coloop, then

$$\underline{\mathrm{CH}}(\mathsf{M}) \cong \underline{\mathrm{CH}}(\mathsf{M} \setminus e) \oplus \underline{\mathrm{CH}}(\mathsf{M} \setminus e)[-1] \oplus \bigoplus_{F \in \underline{\mathcal{S}}_e} \underline{\mathrm{CH}}(\mathsf{M}/(F \cup \{e\})) \otimes \underline{\mathrm{CH}}(M_{|F})[-1]$$

and

$$\mathrm{CH}(\mathsf{M}) \cong \mathrm{CH}(\mathsf{M} \setminus e) \oplus \mathrm{CH}(\mathsf{M} \setminus e)[-1] \oplus \bigoplus_{F \in \mathcal{S}_e} \underline{\mathrm{CH}}\left(\mathsf{M}/(F \cup \{e\})\right) \otimes \mathrm{CH}(\mathsf{M}_{|F})[-1].$$

We stress the fact that in the case of augmented Chow rings, the terms appearing in the direct sum depend on the Chow ring (as opposed to *augmented*) of certain contractions.

2.4 Hilbert–Poincaré series

The goal of this section is to take the polynomials that we defined in Section 1.2.4 and give them a geometric interpretation as Hilbert–Poincaré series of graded vector spaces.

2.4.1 Characteristic polynomial

Since $\mathrm{OS}^\bullet(\mathsf{M})$ is graded, it makes sense to compute its Hilbert–Poincaré series.

Definition 2.4.1.1. The *Poincaré polynomial* of a matroid M is the Hilbert–Poincaré series of its Orlik–Solomon algebra, i.e.

$$\pi_{\mathsf{M}}(x) := \mathrm{Hilb}(\mathrm{OS}(\mathsf{M}), x) = \sum_{i\geq 0} \dim(\mathrm{OS}^i(\mathsf{M}))x^i.$$

Using Theorem 2.2.1.5 we are able to obtain a combinatorial formula for $\pi_{\mathsf{M}}(x)$.

Theorem 2.4.1.2. *For every matroid* M*,*

$$\pi_{\mathsf{M}}(x) = (-x)^{\operatorname{rk}\mathsf{M}}\chi_{\mathsf{M}}(-x^{-1}).$$

Proof. If $\operatorname{rk}\mathsf{M} = 0$ the statement is true. Otherwise, Theorem 2.2.1.5 implies that

$$\pi_{\mathsf{M}}(x) = \pi_{\mathsf{M}\setminus e}(x) + x\,\pi_{\mathsf{M}/e}(x)$$

for any $e \in E$. We can now use the induction hypothesis on both polynomials on the right hand side, as they correspond to matroids with smaller rank or ground set. If e is a coloop of M, then $\mathsf{M} = \mathsf{M}' \oplus e$ and

$$\begin{aligned}\pi_{\mathsf{M}}(x) =&(1+x)(-x)^{\operatorname{rk}\mathsf{M}-1}\chi_{\mathsf{M}'}(-x^{-1})\\ =&(-x)\chi_{\mathsf{B}_1}(-x^{-1})(-x)^{\operatorname{rk}\mathsf{M}-1}\chi_{\mathsf{M}'}(-x^{-1})\\ =&(-x)^{\operatorname{rk}\mathsf{M}}\chi_{\mathsf{M}}(-x^{-1}).\end{aligned}$$

If e is not a coloop,

$$\begin{aligned}\pi_{\mathsf{M}}(x) =&(-x)^{\operatorname{rk}\mathsf{M}}\chi_{\mathsf{M}\setminus e}(-x^{-1}) + x(-x)^{\operatorname{rk}\mathsf{M}-1}\chi_{\mathsf{M}/e}(-x^{-1})\\ =&(-x)^{\operatorname{rk}\mathsf{M}}\left(\chi_{\mathsf{M}\setminus e}(-x^{-1}) - \chi_{\mathsf{M}/e}(-x^{-1})\right)\\ =&(-x)^{\operatorname{rk}\mathsf{M}}\chi_{\mathsf{M}}(-x^{-1}).\end{aligned}$$

In both cases we used the deletion-contraction formula from Theorem 1.1.5.5. □

Recall that we also have a notion of projectivization of the arrangement, $\mathbb{P}(\mathcal{A})$. Since the complement of a projective arrangement may be viewed as the complement of an affine arrangement, everything we said can be extended to the projective case using the results for affine arrangements found in [OT92, Section 3.2].

Theorem 2.4.1.3. *If* $\mathbb{P}(\mathcal{A})$ *is a projective arrangement over* $\mathbb{C}$*, then the Hilbert–Poincaré series of the cohomology ring of the complement* $\mathbb{P}(\mathcal{M}(\mathcal{A}))$ *satisfies*

$$(-x)^{\operatorname{rk}\mathsf{M}-1}\overline{\chi}_{\mathsf{M}}(x^{-1}) = \mathrm{Hilb}(\mathrm{H}^\bullet(\mathbb{P}(\mathcal{M}(\mathcal{A}))), x).$$

2.4.2 Chow polynomials

Since both $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ are graded, it makes sense to study the Hilbert–Poincaré series.

Definition 2.4.2.1. For every matroid M, we denote by $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$, respectively,

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) := \mathrm{Hilb}(\underline{\mathrm{CH}}(\mathsf{M}), x) = \sum_{i=0}^{k-1} \dim_{\mathbb{Q}}(\underline{\mathrm{CH}}^i(\mathsf{M}))\, x^i,$$
$$\mathrm{H}_{\mathsf{M}}(x) := \mathrm{Hilb}(\mathrm{CH}(\mathsf{M}), x) = \sum_{i=0}^{k} \dim_{\mathbb{Q}}(\mathrm{CH}^i(\mathsf{M}))\, x^i.$$

Theorem 2.2.2.4, when translated into the setting of Hilbert–Poincaré series, yields the following formula.

Proposition 2.4.2.2. *Let M be a loopless matroid. The Hilbert–Poincaré series of the Chow ring $\underline{\mathrm{CH}}(\mathsf{M})$ is given by*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m} \prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk} F_i - \operatorname{rk} F_{i-1} - 1})}{1 - x}.$$

Here the sum is taken over all the non-empty chains of flats starting at the empty set, i.e. $\emptyset = F_0 \subsetneq \cdots \subsetneq F_m$ in $\mathcal{L}(\mathsf{M})$ for every $0 \leq m \leq \operatorname{rk} \mathsf{M} - 1$.[1]

This is the counterpart for the augmented Chow ring.

Proposition 2.4.2.3. *Let M be a loopless matroid. The Hilbert–Poincaré series of the augmented Chow ring $\mathrm{CH}(\mathsf{M})$ is given by*

$$\mathrm{H}_{\mathsf{M}}(x) = 1 + \sum_{F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m} \frac{x(1 - x^{\operatorname{rk} F_0})}{1 - x} \prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk} F_i - \operatorname{rk} F_{i-1} - 1})}{1 - x}.$$

Here the sum is taken over all the non-empty chains of flats, i.e. $F_0 \subsetneq \cdots \subsetneq F_m$ in $\mathcal{L}(\mathsf{M})$ for every $0 \leq m \leq k - 1$.

Proof. As mentioned before, one can construct the augmented Chow ring of $\mathcal{L}(\mathsf{M})$ by considering the lattice of flats of the free coextension of M and taking a suitable building set on it; this allows one to use the Gröbner basis of Feichtner and Yuzvinsky. This computation was carried out by Mastroeni and McCullough in [MM22, Section 5.1]. In particular, the basis they construct in [MM22, Corollary 5.4] immediately yields our claimed formula. Alternatively, see [EHL22, Lemma 7.8] or [Lia22, Corollary 3.12]. □

We point out that although the formulas of Propositions 2.4.2.2 and 2.4.2.3 can be used to compute the Hilbert–Poincaré series of (augmented) Chow rings of small matroids, a drawback that they have is that they require iterating over all the chains of flats in the matroid. The total number of chains of flats of a matroid on n elements can be as large as $\frac{2n!}{\log_2(n)}$ (see sequence A000670 in the OEIS [Slo18]), so this approach is also considerably slow even for relatively small values of n. The goal of [FMSV22, Section 3.3] is to find clearer and more efficient formulas. The rest of this section is devoted to present those formulas.

The following result, is a first link between these two Hilbert–Poincaré series and the functions in $I(\mathcal{L}(\mathsf{M}))$ defined in Section 1.2.4.

[1] The chain consisting of only the empty flat yields $m = 0$, and the corresponding summand is an empty product, which by convention will be considered as 1.

Proposition 2.4.2.4. *Let* M *be a loopless matroid. Then*

$$\mathrm{H}_{\mathsf{M}}(x) = \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x).$$

In particular, the Hilbert–Poincaré series of the Chow ring and of the augmented Chow ring as defined in 2.4.2.1 satisfy the recursion of Theorem 1.2.4.18.

Proof. From the formula of Proposition 2.4.2.2, by considering each flat $F \neq \emptyset$ as the term F_1 in the chain, we see that

$$\begin{aligned}\underline{\mathrm{H}}_{\mathsf{M}}(x) &= 1 + \sum_{\substack{F \in \mathcal{L}(\mathsf{M})\\ F \neq \emptyset}} \frac{x(1 - x^{\operatorname{rk} F - 1})}{1 - x} \sum_{F = F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_m} \prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk} F_i - \operatorname{rk} F_{i-1} - 1})}{1 - x}\\ &= 1 + \sum_{\substack{F \in \mathcal{L}(\mathsf{M})\\ F \neq \emptyset}} \frac{x(1 - x^{\operatorname{rk} F - 1})}{1 - x} \cdot \underline{\mathrm{H}}_{\mathsf{M}/F}(x). \end{aligned} \tag{2.2}$$

The summand equal to 1 comes from considering the chain that consists of only the flat $F_0 = \emptyset$ separately. The last equation follows from the fact that the lattice of flats of M/F is isomorphic to the interval $[F, E]$ in $\mathcal{L}(\mathsf{M})$. Analogously, by applying the same argument to the formula of Proposition 2.4.2.3, we can fix the flat $F = F_0$ of the chain and write

$$\begin{aligned}\mathrm{H}_{\mathsf{M}}(x) &= 1 + \sum_{F \in \mathcal{L}(\mathsf{M})} \frac{x(1 - x^{\operatorname{rk} F})}{1 - x} \sum_{F \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_m} \prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk} F_i - \operatorname{rk} F_{i-1} - 1})}{1 - x}\\ &= 1 + \sum_{F \in \mathcal{L}(\mathsf{M})} \frac{x(1 - x^{\operatorname{rk} F})}{1 - x} \cdot \underline{\mathrm{H}}_{\mathsf{M}/F}(x)\\ &= 1 + \sum_{\substack{F \in \mathcal{L}(\mathsf{M})\\ F \neq \emptyset}} \frac{x(1 - x^{\operatorname{rk} F})}{1 - x} \cdot \underline{\mathrm{H}}_{\mathsf{M}/F}(x). \end{aligned} \tag{2.3}$$

Observe that for each integer $r \geq 1$, we have that $\frac{x(1-x^r)}{1-x} = \frac{x(1-x^{r-1})}{1-x} + x^r$. In particular, by combining Equations (2.2) and (2.3), we obtain

$$\begin{aligned}\mathrm{H}_{\mathsf{M}}(x) &= 1 + \sum_{\substack{F \in \mathcal{L}(\mathsf{M})\\ F \neq \emptyset}} \frac{x(1 - x^{\operatorname{rk} F - 1})}{1 - x} \cdot \underline{\mathrm{H}}_{\mathsf{M}/F}(x) + \sum_{\substack{F \in \mathcal{L}(\mathsf{M})\\ F \neq \emptyset}} x^{\operatorname{rk} F}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x)\\ &= \underline{\mathrm{H}}_{\mathsf{M}}(x) + \sum_{\substack{F \in \mathcal{L}(\mathsf{M})\\ F \neq \emptyset}} x^{\operatorname{rk} F}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x)\\ &= \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x). \end{aligned}$$

This proves the desired recursion. □

Remark 2.4.2.5. Notice that Proposition 2.4.2.4 can also be obtained by taking Hilbert–Poincaré series in Equation (2.1).

Theorem 2.4.2.6. *Let* M *be a loopless matroid. Then, the Hilbert–Poincaré series of the Chow ring of* M *satisfies*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \begin{cases} 1 & \text{if } \operatorname{rk}\mathsf{M} = 0 \\ \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq\emptyset}} \overline{\chi}_{\mathsf{M}|_F}(x)\,\underline{\mathrm{H}}_{\mathsf{M}/F}(x) & \text{otherwise.} \end{cases} \tag{2.4}$$

In particular, the Chow polynomial and the augmented Chow polynomial defined in Section 1.2.4 *are, respectively, the Hilbert–Poincaré series of the Chow ring and the augmented Chow ring for every matroid* M.

Proof. To establish this result, it suffices to prove that the polynomial in the statement satisfies all the properties of Theorem 1.2.4.18. The first two conditions are immediate to check, and for the last, it will suffice to verify that $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ satisfies the recursion of Remark 1.2.4.21. We have a chain of equalities:

$$\begin{aligned} 1 + x \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq E}} \underline{\mathrm{H}}_{\mathsf{M}/F}(x) &= 1 + x \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq E}} \sum_{\substack{G\in\mathcal{L}(\mathsf{M}/F)\\ G\neq\emptyset}} \overline{\chi}_{(\mathsf{M}/F)|_G}(x)\,\underline{\mathrm{H}}_{(\mathsf{M}/F)/G}(x) \\ &= 1 + x \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq E}} \sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\supsetneq F}} \overline{\chi}_{(\mathsf{M}/F)|_{G\setminus F}}(x)\,\underline{\mathrm{H}}_{\mathsf{M}/G}(x) \\ &= 1 + x \sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\neq\emptyset}} \underline{\mathrm{H}}_{\mathsf{M}/G}(x) \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\subsetneq G}} \overline{\chi}_{(\mathsf{M}/F)|_{G\setminus F}}(x) \\ &= 1 + x \sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\neq\emptyset}} \underline{\mathrm{H}}_{\mathsf{M}/G}(x) \sum_{\substack{F\in\mathcal{L}(\mathsf{M}|_G)\\ F\subsetneq G}} \overline{\chi}_{(\mathsf{M}|_G)/F}(x) \\ &= 1 + x \sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\neq\emptyset}} \underline{\mathrm{H}}_{\mathsf{M}/G}(x) \cdot \frac{1 - x^{\operatorname{rk}\mathsf{M}|_G}}{1-x} \qquad (2.5) \\ &= \mathrm{H}_{\mathsf{M}}(x), \end{aligned}$$

where in the last two equalities we used Lemma 1.2.2.14 and the formula we had obtained in Equation (2.3). ☐

Once we have established this relation between the polynomials, by taking Hilbert–Poincaré series of the decompositions in Theorem 2.3.2.5 one also obtains the following additional relations.

Theorem 2.4.2.7. *Let* M *be a loopless matroid and let* $e \in E$ *be an element that is not a coloop. Then,*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \underline{\mathrm{H}}_{\mathsf{M}\setminus e}(x) + x \sum_{F\in\underline{\mathcal{S}}_e} \underline{\mathrm{H}}_{\mathsf{M}/(F\cup\{e\})}(x)\underline{\mathrm{H}}_{\mathsf{M}|_F}(x)$$

and

$$\mathrm{H}_{\mathsf{M}}(x) = \mathrm{H}_{\mathsf{M}\setminus e}(x) + x \sum_{F\in\mathcal{S}_e} \underline{\mathrm{H}}_{\mathsf{M}/(F\cup\{e\})}(x)\mathrm{H}_{\mathsf{M}|_F}(x).$$

If $e \in E$ *is a coloop, then*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = (1+x)\,\underline{\mathrm{H}}_{\mathsf{M}\setminus e}(x) + x \sum_{F\in\underline{\mathcal{S}}_e} \underline{\mathrm{H}}_{\mathsf{M}/(F\cup\{e\})}(x)\underline{\mathrm{H}}_{\mathsf{M}|_F}(x)$$

and

$$\mathrm{H}_{\mathsf{M}}(x) = (1+x)\,\mathrm{H}_{\mathsf{M}\setminus e}(x) + x\sum_{F\in\mathcal{S}_e} \underline{\mathrm{H}}_{\mathsf{M}/(F\cup\{e\})}(x)\mathrm{H}_{\mathsf{M}|_F}(x).$$

At this point it is also of interest to produce a recursion for $\mathrm{H}_{\mathsf{M}}(x)$ in terms of Hilbert–Poincaré series of augmented Chow rings of contractions $\mathrm{H}_{\mathsf{M}/F}(x)$ for $F \in \mathcal{L}(\mathsf{M})$, but with specific care to not make any references to $\underline{\mathrm{H}}_{\mathsf{M}}(x)$.

Theorem 2.4.2.8. *Let* M *be a loopless matroid. Then the Hilbert–Poincaré series of the augmented Chow ring of* M *satisfies*

$$\mathrm{H}_{\mathsf{M}}(x) = -\sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq\emptyset}} \mu(\emptyset,F)(1+x+\cdots+x^{\operatorname{rk}F})\,\mathrm{H}_{\mathsf{M}/F}(x). \tag{2.6}$$

In particular, the above recursion and the initial condition $\mathrm{H}_{\mathsf{M}}(x) = 1$ *if* M *is empty uniquely define the map associating to each loopless matroid the Hilbert–Poincaré series of its augmented Chow ring.*

Proof. The proof relies on Theorems 1.2.4.18 and 2.4.2.6. Starting from the right-hand side of Equation (2.6) (without the minus sign) and using the recursion of Theorem 1.2.4.18, we have

$$\begin{aligned}
&\sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq\emptyset}} \mu(\emptyset,F)\frac{1-x^{\operatorname{rk}F+1}}{1-x}\,\mathrm{H}_{\mathsf{M}/F}(x)\\
&= \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq\emptyset}} \mu(\emptyset,F)\frac{1-x^{\operatorname{rk}F+1}}{1-x}\sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\supseteq F}} x^{\operatorname{rk}G-\operatorname{rk}F}\underline{\mathrm{H}}_{\mathsf{M}/G}(x)\\
&= \sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\neq\emptyset}} \underline{\mathrm{H}}_{\mathsf{M}/G}(x)\sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \emptyset\neq F\subseteq G}} \mu(\emptyset,F)\,x^{\operatorname{rk}G-\operatorname{rk}F}\frac{1-x^{\operatorname{rk}F+1}}{1-x},
\end{aligned}$$

where in the last step we just interchanged the order of summation. Now, breaking the inner sum into a difference of two sums yields

$$\begin{aligned}
&= \sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\neq\emptyset}} \underline{\mathrm{H}}_{\mathsf{M}/G}(x)\frac{1}{1-x}\left(\sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \emptyset\neq F\subseteq G}} \mu(\emptyset,F)\,x^{\operatorname{rk}G-\operatorname{rk}F} - \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \emptyset\neq F\subseteq G}} \mu(\emptyset,F)x^{\operatorname{rk}G+1}\right)\\
&= \sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\neq\emptyset}} \underline{\mathrm{H}}_{\mathsf{M}/G}(x)\frac{1}{1-x}\left(\left(\chi_{\mathsf{M}|_G}(x) - x^{\operatorname{rk}G}\right) + x^{\operatorname{rk}G+1}\right)\\
&= -\sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\neq\emptyset}} \overline{\chi}_{\mathsf{M}|_G}(x)\,\underline{\mathrm{H}}_{\mathsf{M}/G}(x) - \sum_{\substack{G\in\mathcal{L}(\mathsf{M})\\ G\neq\emptyset}} x^{\operatorname{rk}G}\underline{\mathrm{H}}_{\mathsf{M}/G}(x).
\end{aligned}$$

Using the recursions of Theorem 2.4.2.6 and Theorem 1.2.4.18, we obtain

$$\begin{aligned}
&= -\underline{\mathrm{H}}_{\mathsf{M}}(x) - (\mathrm{H}_{\mathsf{M}}(x) - \underline{\mathrm{H}}_{\mathsf{M}}(x))\\
&= -\mathrm{H}_{\mathsf{M}}(x),
\end{aligned}$$

and therefore the proof is complete. ☐

2.4.3 Kazhdan–Lusztig–Stanley polynomials

Theorem 1.9 in [BHM+22b] lets us describe the Hilbert–Poincaré series of $\mathrm{IH}(\mathsf{M})$ and $\mathrm{IH}(\mathsf{M})_\emptyset$.

Theorem 2.4.3.1 ([BHM+22b, Theorem 1.9]). *Let* M *be a loopless matroid.*

- *The Kazhdan–Lusztig polynomial of* M *coincides with the Hilbert–Poincaré series of the stalk at the empty flat of* $\mathrm{IH}(\mathsf{M})$*. In other words,*

$$P_{\mathsf{M}}(x) = \sum_{i\geq 0} \dim_{\mathbb{Q}}(\mathrm{IH}^i(\mathsf{M})_\emptyset)\, x^i.$$

- *The* Z*-polynomial of* M *coincides with the Hilbert–Poincaré series of the intersection cohomology module of* M*. In other words,*

$$Z_{\mathsf{M}}(x) = \sum_{i\geq 0} \dim_{\mathbb{Q}}(\mathrm{IH}^i(\mathsf{M}))\, x^i.$$

The proof is mostly based on the following results.

Proposition 2.4.3.2 ([BHM+22b, Proposition 1.8, 1.7]). *Let* $\mathfrak{m}$ *be the graded maximal ideal of* $\mathrm{H}(\mathsf{M})$*. Then,*

$$\mathrm{IH}(\mathsf{M})_\emptyset \cong \mathrm{IH}(\mathsf{M})/\mathfrak{m}\mathrm{IH}(\mathsf{M}).$$

This graded vector space vanishes in degree $\geq \frac{1}{2}\operatorname{rk}\mathsf{M}$*. Moreover,*

$$\mathfrak{m}^i\mathrm{IH}(\mathsf{M})/\mathfrak{m}^{i+1}\mathrm{IH}(\mathsf{M}) \cong \bigoplus_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \operatorname{rk}F=i}} \mathrm{IH}(\mathsf{M}/F)[-i].$$

Theorem 2.4.3.1 then follows by taking Hilbert–Poincaré series and observing that these polynomials are the unique polynomials that satisfy Theorem 1.2.4.16.

Remark 2.4.3.3. The similarity between the recurrence linking $P_{\mathsf{M}}(x)$ to $Z_{\mathsf{M}}(x)$ with the one linking $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ to $\mathrm{H}_{\mathsf{M}}(x)$ is also hinted by the fact that $\mathrm{CH}(\mathsf{M})_\emptyset := \mathrm{CH}(\mathsf{M}) \otimes_{\mathrm{H}(\mathsf{M})} \mathbb{Q}$, and that the latter is isomorphic to $\underline{\mathrm{CH}}(\mathsf{M})$ (see [BHM+22a, Remark 1.4]). In other words, in terms of stalks [BHM+22b, Section 5], the Chow ring $\underline{\mathrm{CH}}(\mathsf{M})$ is the stalk at the empty flat of the augmented Chow ring $\mathrm{CH}(\mathsf{M})$.

The following formula was first proved in [PXY18] and gives a way of computing the polynomial $P_{\mathsf{M}}(x)$ as an alternating sum that counts flags of flats.

Given a sequence $i_r, \ldots, i_1$ of positive integers we define the r*-Whitney number* as

$$W_{\mathsf{M}}(i_r, \ldots, i_1) := \#\{(F_r, \ldots, F_1) \mid F_r \leq \ldots \leq F_1,\ F_j \in \mathcal{L}(\mathsf{M}) \text{ and } \operatorname{crk} F_j = i_j\}$$

i.e. the number of flags of flats of prescribed corank in the lattice of flats $\mathcal{L}(\mathsf{M})$.

Given positive integers j and r along with a subset $S \subset [r]$, let

$$t_j(S) := \min\{k \mid k \geq j \text{ and } k \notin S\} \in [r+1].$$

Theorem 2.4.3.4. *Let* M *be a loopless matroid of rank* k *on the ground set* E*. The Kazhdan–Lusztig polynomial* $P_{\mathsf{M}}(x)$ *is equal to*

$$P_{\mathsf{M}}(x) = 1 + \sum_{r=1}^{i} x^i \sum_{S\subset[r]} (-1)^{\#S} \sum_{\substack{0<a_1<\ldots<a_{r+1}\\ a_0=0\\ a_r=i\\ a_{r+1}=\operatorname{rk}-i}} W_{\mathsf{M}}\left(a_{t_r(S)} + a_{r-1}, \ldots, a_{t_1(S)} + a_0\right).$$

We observe that Theorem 1.1.5.5 and Theorem 2.4.2.7 both let us produce formulas for the polynomials $\chi_{\mathsf{M}}(x)$, $\pi_{\mathsf{M}}(x)$, $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ for a matroid M in terms of a deletion $\mathsf{M} \setminus e$. A natural question is whether an analogous statement can be made for $P_{\mathsf{M}}(X)$ and $Z_{\mathsf{M}}(x)$. Before formulating it, we introduce the following notation.

Definition 2.4.3.5. For every matroid M we define

$$\tau(\mathsf{M}) := \begin{cases} [x^{\frac{\operatorname{rk}\mathsf{M}-1}{2}}]P_{\mathsf{M}}(x) & \text{if } \operatorname{rk}\mathsf{M} \text{ is odd,} \\ 0 & \text{if } \operatorname{rk}\mathsf{M} \text{ is even.} \end{cases}$$

Theorem 2.4.3.6 ([BV20, Theorem 2.8]). *Let M be a loopless matroid and $e \in E$ be an element of the ground set that is not a coloop. Then,*

$$P_{\mathsf{M}}(x) = P_{\mathsf{M}\setminus e}(x) - x\, P_{\mathsf{M}/e}(x) + \sum_{F \in \mathcal{S}_e} \tau\left(\mathsf{M}/(F \cup \{e\})\right) x^{\frac{\operatorname{rk}\mathsf{M} - \operatorname{rk}F}{2}} P_{\mathsf{M}_{|F}}(x),$$

$$Z_{\mathsf{M}}(x) = Z_{\mathsf{M}\setminus e}(x) + \sum_{F \in \mathcal{S}_e} \tau\left(\mathsf{M}/(F \cup \{e\})\right) x^{\frac{\operatorname{rk}\mathsf{M} - \operatorname{rk}F}{2}} Z_{\mathsf{M}_{|F}}(x).$$

Observe that a priori this result does not witness any decompositions at the level of (stalks of) intersection cohomology modules for generic matroids, even though the formula is motivated by algebraic geometry and computations on the Schubert variety $Y(\mathcal{A})$ in the case when M is realizable.

Chapter 3

Combinatorics

3.1 Combinatorial properties

We are now interested in developing a combinatorial theory for all the polynomials that arose as Hilbert–Poincaré series in Section 2.4. In particular, one of the main motivations to study their combinatorial properties is the following collection of conjectures on their real-rootedness that we refer to as the "real deal" with matroids.

Conjecture 3.1.0.1. *For every matroid* M*, the following polynomials have only real roots.*

- *([FS22, Conjecture 10.19]) The polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x)$*, i.e. the Hilbert–Poincaré series of the Chow ring* $\underline{\mathrm{CH}}(\mathsf{M})$*.*
- *([Ste21, Conjecture 4.3.3]) The polynomial* $\mathrm{H}_{\mathsf{M}}(x)$*, i.e. the Hilbert–Poincaré series of the augmented Chow ring* $\mathrm{CH}(\mathsf{M})$*.*
- *([PXY18, Conjecture 5.1]) The polynomial* $Z_{\mathsf{M}}(x)$*, i.e. the Hilbert–Poincaré series of the intersection cohomology module* $\mathrm{IH}(\mathsf{M})$*.*
- *([GPY17b, Conjecture 3.2]) The polynomial* $P_{\mathsf{M}}(x)$*, i.e. the Hilbert–Poincaré series of the stalk* $\mathrm{IH}(\mathsf{M})_\emptyset$ *at the empty flat of* $\mathrm{IH}(\mathsf{M})$*.*

Using the formulas we describe in Chapter 4, we are able to provide the following experimental evidence in support of these conjectures.

Proposition 3.1.0.2. *If* M *is a sparse paving matroid with at most* 40 *elements, then* $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ *and* $\mathrm{H}_{\mathsf{M}}(x)$ *are real-rooted. If* M *is a sparse paving matroid with at most* 30 *elements, then* $P_{\mathsf{M}}(x)$ *and* $Z_{\mathsf{M}}(x)$ *are real-rooted.*

Another intriguing collection of conjectures prescribes for which matroids these polynomials have maximal coefficients, for fixed rank k and cardinality of the ground set n.

Conjecture 3.1.0.3. *Uniform matroids maximize the coefficients of Kazhdan–Lusztig polynomials,* Z*-polynomials, Chow polynomials and augmented Chow polynomials among all matroids with fixed rank and cardinality.*

The conjecture for $P_{\mathsf{M}}(x)$ is attributed to Gedeon (unpublished); we show that it holds for all paving matroids later in Corollary 4.1.3.9. In Section 3.1.5, we prove instead that its counterpart for the polynomials $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ holds for all matroids.

The degrees of the polynomials $P_{\mathsf{M}}(x)$, $Q_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ are of much interest in the framework of Kazhdan–Lusztig–Stanley theory for matroids, as they might suggest interlacing properties for their roots. A matroid M is said to be *non-degenerate* if $P_{\mathsf{M}}(x)$ has degree $\lfloor \frac{\operatorname{rk} \mathsf{M}-1}{2} \rfloor$ (this is of course related to the invariant τ introduced in Definition 2.4.3.5). Gedeon, Proudfoot and Young posed the following conjecture.

Conjecture 3.1.0.4 ([GPY17c]). *Every connected regular matroid is non-degenerate.*

This conjecture remains open, but it is important to point out that the class of regular matroids is extremely restrictive (see again Theorem 1.1.4.6).

3.1.1 Properties arising from algebraic geometry

We now list some known combinatorial properties of these polynomials that are a direct consequence of them being Hilbert–Poincaré series of a graded vector space. Then, we highlight which of these properties can be proven in a purely combinatorial setting.

Remark 3.1.1.1. The request of a "fully combinatorial" proof can be misleading at first and one should be careful with interpretating its meaning. We have been using the term geometric whenever the matroid is realizable, i.e. there is a variety that can be attached to the matroid over which we can compute some version of its cohomology. In this sense, the graded vector spaces $\mathrm{OS}(\mathsf{M})$, $\underline{\mathrm{CH}}(\mathsf{M})$, $\mathrm{CH}(\mathsf{M})$, $\mathrm{IH}(\mathsf{M})$ and $\mathrm{IH}(\mathsf{M})_\emptyset$ are indeed combinatorial, as one is able to produce them for every matroid, not just the realizable ones. In this context, we want to look at the polynomials as invariants on the lattice of flats $\mathcal{L}(\mathsf{M})$ as defined in Section 1.2.4. For a proof to be "fully combinatorial" we would like then to be able to verify the statements without mentioning neither the five graded vector spaces we listed above, nor the fact that these polynomials compute their Hilbert–Poincaré series, and, as a consequence, without using the heavy algebro-geometric machinery that this interpretation carries.

- *(Non-negativity)* As their coefficients represent the dimensions of some vector spaces, of course, $\pi_{\mathsf{M}}(x)$, the three Kazhdan–Lusztig–Stanley polynomials, and the two Chow polynomials all have non-negative coefficients. Combinatorially, one can show that $\pi_{\mathsf{M}}(x)$ has non-negative coefficients by using Lemma 1.2.4.3 and the fact that $\pi_{\mathsf{M}}(x) = (-x)^{\operatorname{rk} \mathsf{M}} \chi_{\mathsf{M}}(-x^{-1})$. For the two Chow polynomials, one could use Proposition 2.4.2.4 and Theorem 2.4.2.6 and rederive Propositions 2.4.2.2 and 2.4.2.3, which are explicitly non-negative. A fully combinatorial proof of the non-negativity of the other polynomials is yet to be obtained. Notice that the corresponding formula with flags of flats of Theorem 2.4.3.4 is alternating in sign and therefore not explicitly positive. However, we show in Theorem 4.2.1.1 that the three Kazhdan–Lusztig–Stanley polynomials are non-negative for every sparse paving matroid.
- *(Palindromicity)* As a consequence of Poincaré duality, we have mentioned in Corollary 2.3.2.3 that $\underline{\mathrm{H}}_{\mathsf{M}}(x)$, $\mathrm{H}_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ are palindromic. Combinatorially, we showed this in Theorem 1.2.4.16 and Theorem 1.2.4.18.
- *(Log-concavity)* The only known log-concave polynomial is $\chi_{\mathsf{M}}(x)$, as discussed in Theorem 2.3.2.4. This also implies that it is unimodal as proved in Proposition 1.4.1.7.
- *(Unimodality)* The Z-polynomial and the two Chow polynomials are unimodal as a consequence of the Hard Lefschetz.

- *(γ-positivity)* Section 3.1.3 is devoted to showing the γ-positivity of $Z_{\mathsf{M}}(x)$, $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$. The proof is combinatorial even though for the Chow polynomials it exploits Theorem 2.4.2.7, which is a consequence of the semi-small decompositions described in Theorem 2.3.2.5.

3.1.2 Uniform matroids

It appears evident, from Conjecture 3.1.0.3, that the values of our polynomials for uniform matroids are of great interest. They will also become fundamental in Chapter 4. We collect here the results that were already known in the literature for Kazhdan–Lusztig–Stanley polynomials

Theorem 3.1.2.1 ([GLX+21, Theorem 1.3 and 1.5] [GX21, Theorem 3.3])**.** *For every $k \leq n$ the Kazhdan–Lusztig polynomial, the inverse Kazhdan–Lusztig polynomial and the Z-polynomial of the uniform matroid $\mathsf{U}_{k,n}$ are*

$$P_{\mathsf{U}_{k,n}}(x) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{k-i}\binom{n}{i} \sum_{h=0}^{n-k-1} \binom{k-i+h}{h+i+1}\binom{i-1+h}{h} x^i,$$

$$Q_{\mathsf{U}_{k,n}}(x) = \binom{n}{k} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(n-k)(k-2i)}{(n-k+i)(n-i)}\binom{k}{i} x^i,$$

$$Z_{\mathsf{U}_{k,n}}(x) = \sum_{i=0}^{k} \frac{\binom{n}{n-k+i}\binom{n}{i}}{\binom{n}{k}} \sum_{h=0}^{m-1} \frac{i(h-n+k+1)+n-k}{(h+1)(n-k)}\binom{i-1+h}{h}\binom{k-i+h}{h} x^i.$$

Corollary 3.1.2.2. *In particular, for Boolean matroids*

$$P_{\mathsf{B}_n}(x) = 1,$$

$$Q_{\mathsf{B}_n}(x) = 1,$$

$$Z_{\mathsf{B}_n}(x) = (x+1)^n.$$

The above formulas can be found either by setting $k = n$ in the formula for uniform matroids, or by exploiting the multiplicativity of P, Q and Z under direct sum and by knowing their values on B_1.

One of the goals pursued in [FMSV22] is determining $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ whenever M is an arbitrary uniform matroid. We mention that the polynomials $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ for arbitrary uniform matroids were addressed by Hameister, Rao, and Simpson in [HRS21]. One of their main results [HRS21, Theorem 5.1] is useful to retrieve the following facts.

Theorem 3.1.2.3. *Let $\mathsf{M} \cong \mathsf{B}_n$ be isomorphic to a boolean matroid. Then,*

$$\underline{\mathrm{H}}_{\mathsf{B}_n}(x) = A_n(x),$$

where A_n is the n-th Eulerian polynomial defined in Section 1.4.2. Let $\mathsf{M} \cong \mathsf{U}_{n-1,n}$ be isomorphic to a corank 1 uniform matroid. Then,

$$\underline{\mathrm{H}}_{\mathsf{U}_{n-1,n}}(x) = \frac{1}{x} d_n(x),$$

where $d_n(x)$ denotes the n-th derangement polynomial.

Unfortunately, deducing compact expressions for Chow polynomials of arbitrary uniform matroids via their result seems rather difficult; although their formula is nice in terms of statistics of permutations, it is intricate from a computational point of view.

Without making any references to augmented Chow rings of matroids, the polynomial $\mathrm{H}_{\mathsf{M}}(x)$ for Boolean matroids has been studied in detail recently in [PRW08, Ath20, SW20, HZ19, Han21, BJ22]. To be precise, one has the following.

Theorem 3.1.2.4. *Let $\mathsf{M} \cong \mathsf{B}_n$ be isomorphic to a Boolean matroid. Then,*

$$\mathrm{H}_{\mathsf{B}_n}(x) = \widetilde{A}_n(x),$$

where $\widetilde{A}_n(x)$ denotes the n-th binomial Eulerian polynomial.

A further motivation to study the Hilbert–Poincaré series of Chow rings and augmented Chow rings is that they may be viewed as vast generalizations of the Eulerian, binomial Eulerian, and derangement polynomials, and within this broader framework one can derive interesting new identities relating them.

We now collect the results on uniform matroids as presented in [FMSV22, Section 3.5]. We start by producing more general formulas using the incidence algebra of the lattice of flats $\mathcal{L}(\mathsf{M})$. These new formulas work for arbitrary matroids, but will be of particular use to produce the first concrete expressions for the Hilbert–Poincaré series of both the Chow ring and the augmented Chow ring of arbitrary uniform matroids. Since $\underline{\mathrm{H}}$ and $\overline{\chi}$ are inverse of each other up to a sign in the incidence algebra of $\mathcal{L}(\mathsf{M})$ (see Definition 1.2.2.15), and since left and right inverses in the incidence algebra coincide by Proposition 1.2.1.4, we are able to produce the following alternative convolution.

Proposition 3.1.2.5. *Let M be a loopless matroid on E. The following formula holds:*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq E}} \underline{\mathrm{H}}_{\mathsf{M}|_F}(x)\,\overline{\chi}_{\mathsf{M}/F}(x).$$

This proposition provides a recursion that is particularly useful to compute $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ whenever M is an arbitrary uniform matroid. The reason for this is that the restriction $\mathsf{M}_{|F}$ for $F \in \mathcal{L}(\mathsf{M})$ is always a Boolean matroid whenever F is a proper flat. One of the motivations for this idea came from the Kazhdan–Lusztig theory of matroids, where the inverse Kazhdan–Lusztig polynomial plays a role to compute the Kazhdan–Lusztig polynomial of arbitrary uniform matroids in [GX21, Section 3].

Corollary 3.1.2.6. *The Hilbert–Poincaré series of the Chow ring of a uniform matroid of rank k and cardinality n is given by*

$$\underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x) = \sum_{j=0}^{k-1} \binom{n}{j} A_j(x)\,\overline{\chi}_{\mathsf{U}_{k-j,n-j}}(x).$$

This formula can be made explicit because the reduced characteristic polynomial of a uniform matroid is not difficult to compute (see Lemma 1.1.5.9). Regarding the augmented case, one can use Theorem 2.4.2.8 and carefully work again with inverses in the incidence algebra to produce a formula similar to that of Proposition 3.1.2.5 for $\mathrm{H}_{\mathsf{M}}(x)$.

Proposition 3.1.2.7. *Let M be a loopless matroid on E. The following formula holds:*

$$\mathrm{H}_{\mathsf{M}}(x) = -\sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq E}} \mathrm{H}_{\mathsf{M}|_F}(x)\,\mu(F,E)\left(1 + x + \cdots + x^{\operatorname{rk}(\mathsf{M})-\operatorname{rk}(F)}\right).$$

The above proposition yields a formula for the Hilbert–Poincaré series of the augmented Chow ring of arbitrary uniform matroids, via expressing them in terms of binomial Eulerian polynomials.

Corollary 3.1.2.8. *The Hilbert–Poincaré series of the augmented Chow ring of a uniform matroid of rank k and cardinality n is given by*

$$\mathrm{H}_{\mathsf{U}_{k,n}}(x) = \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{n}{j} \binom{n-1-j}{k-1-j} \widetilde{A}_j(x)\,(1 + x + \cdots + x^{k-j}).$$

Although Corollary 3.1.2.6 and Corollary 3.1.2.8 are explicit expressions, they are both alternating sums, due to basic properties of the Möbius function and the reduced characteristic polynomial. In the remainder of this section we aim to show the positivity of $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ by producing an alternative explicit formula. That result provides the cleanest way we are aware of for computing the Hilbert–Poincaré series of Chow rings and augmented Chow rings of uniform matroids. The proof will be carried out by leveraging the following lemma, whose proof relies on (and later will extend) a recursion found by Juhnke-Kubitzke, Murai, and Sieg [JKMS19] for derangement polynomials.

Lemma 3.1.2.9. *Let $\mathsf{M} = \mathsf{B}_n$ be a Boolean matroid on a ground set E of $n \geq 1$ elements. Then, by considering only chains of flats that end at the top element E of $\mathcal{L}(\mathsf{M})$, we have*

$$d_n(x) = \sum_{\emptyset = F_0 \subsetneq \cdots \subsetneq F_m = E} \prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk} F_i - \operatorname{rk} F_{i-1} - 1})}{1-x},$$

$$xA_n(x) = \sum_{F_0 \subsetneq \cdots \subsetneq F_m = E} \frac{x(1 - x^{\operatorname{rk} F_0})}{1-x} \prod_{i=1}^{m} \frac{x(1 - x^{\operatorname{rk} F_i - \operatorname{rk} F_{i-1} - 1})}{1-x}.$$

Proof. Let us prove the first identity. For each $n \geq 1$, denote by $\mathfrak{d}_n(x)$ the sum on the right-hand side. By considering what the penultimate element of the chain ending in E is, we see that the sequence $\mathfrak{d}_n(x)$ satisfies the recurrence

$$\mathfrak{d}_n(x) = \sum_{j=0}^{n-2} \binom{n}{j} \mathfrak{d}_j(x)(x + x^2 + \cdots + x^{n-j-1}).$$

In [JKMS19, Corollary 4.2], it is proved that this recursion determines the derangement polynomials, and hence we have $\mathfrak{d}_n(x) = d_n(x)$ for each $n \geq 1$, as claimed.

To prove the second identity we rely on the first. Call the right-hand side $\mathfrak{a}_n(x)$. Choosing the set F_0 we obtain

$$\begin{aligned}
\mathfrak{a}_n(x) &= \sum_{j=1}^{n} \binom{n}{j} (x + x^2 + \cdots + x^j) d_{n-j}(x) \\
&= \sum_{i=0}^{n-1} \binom{n}{i} d_i(x)(x + x^2 + \cdots + x^{n-i}) \\
&= xA_n(x),
\end{aligned}$$

where the last equality follows from the locality formula for the h-polynomial of the barycentric subdivision of the boundary of the $(n-1)$-simplex, i.e. from combining [Sta92, Proposition 2.4] and [Sta92, Theorem 3.2]. ∎

Theorem 3.1.2.10. *The Hilbert–Poincaré series of the Chow ring and augmented Chow ring of arbitrary uniform matroids are given by*

$$\underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x) = \sum_{j=0}^{k-1} \binom{n}{j} d_j(x)(1 + x + \cdots + x^{k-1-j}),$$

$$\mathrm{H}_{\mathsf{U}_{k,n}}(x) = 1 + x\sum_{j=0}^{k-1} \binom{n}{j} A_j(x)(1 + x + \cdots + x^{k-1-j}).$$

Proof. Let us apply the formula of Proposition 2.4.2.2 to the uniform matroid $\mathsf{U}_{k,n}$. Each chain of flats $\emptyset = F_0 \subsetneq \cdots \subsetneq F_m$ appearing in the sum either has $F_m \subsetneq E$ or $F_m = E$. Lemma 3.1.2.9 tells us that, for each flat $F \subsetneq E$ of rank j, the sum over all chains finishing at F yields the polynomial $d_j(x)$. On the other hand, by fixing the flat F_{m-1}, Lemma 3.1.2.9 also allows us to calculate the sum of all the summands for which $F_m = E$. This yields

$$\underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x) = \sum_{j=0}^{k-1} \binom{n}{j} d_j(x) + \sum_{j=0}^{k-1} \binom{n}{j} d_j(x) \frac{x(1-x^{k-j-1})}{1-x}.$$

After rearranging, the claimed identity is proved. The formula for the augmented Chow ring follows in a completely analogous way. □

Remark 3.1.2.11. The expression for $\mathrm{H}_{\mathsf{U}_{k,n}}(x)$ derived in the last statement bears an intriguing resemblance to the h-polynomials of the class of polytopes known as *partial permutohedra*, studied recently in [BCC+22]. More precisely, compare our formula with their [BCC+22, Theorem 3.17].

3.1.2.1 The Hameister–Rao–Simpson conjecture

The formula from Corollary 3.1.2.6 essentially resolves a conjecture posed by Hameister, Rao, and Simpson in [HRS21] regarding the face enumeration of the Bergman complex of a matroid (i.e. the order complex of the proper part of the lattice of flats). For a matroid M we denote by $\Delta(\widehat{\mathcal{L}}(\mathsf{M}))$ the order complex of the proper part of the lattice of flats of M; this complex is also known as the *Bergman complex* and has been studied for example in [AK06]. This is a simplicial complex whose simplices correspond to chains of proper non-empty flats of M. One can consider its *f-polynomial*, defined by $f_{\Delta(\widehat{\mathcal{L}}(\mathsf{M}))}(x) := \sum_{i=0}^{d} f_{i-1}\, x^{d-i}$ where each f_i counts the number of i-dimensional faces and $d := \dim \Delta(\widehat{\mathcal{L}}(\mathsf{M})) = \operatorname{rk} \mathsf{M} - 2$. Both the f- and the h-polynomial, which is defined by $h_{\Delta(\widehat{\mathcal{L}}(\mathsf{M}))}(x) := f_{\Delta(\widehat{\mathcal{L}}(\mathsf{M}))}(x-1)$, have non-negative coefficients (the first is clear; for the second we refer to [Bjö92, Section 7.6]); each of them is conjectured to have only real roots [AK23, Conjecture 1.2]. Hameister, Rao, and Simpson observed that the following equation holds for several small cases of k and n:

$$h_{\Delta(\widehat{\mathcal{L}}(\mathsf{U}_{k,n}))}(x) = \sum_{i=1}^{k} \binom{n-i-1}{k-i} \underline{\mathrm{H}}_{\mathsf{U}_{i,n}}(x),$$

and conjectured that this holds for all k and n. Corollary 3.17 in [FMSV22] shows that this conjecture is true.

Corollary 3.1.2.12 ([HRS21, Conjecture 6.2])**.** *Let us denote by $h_{\Delta(\widehat{\mathcal{L}}(\mathsf{U}_{k,n}))}(x)$ the h-polynomial of the Bergman complex of $\mathsf{U}_{k,n}$. Then*

$$h_{\Delta(\widehat{\mathcal{L}}(\mathsf{U}_{k,n}))}(x) = \sum_{i=1}^{k} \binom{n-i-1}{k-i} \underline{\mathrm{H}}_{\mathsf{U}_{i,n}}(x). \tag{3.1}$$

The work of Brenti and Welker [BW08, Theorem 1] provides an explicit formula for the polynomials on the left-hand side in terms of Eulerian polynomials,[1] concretely,

$$h_{\Delta(\widehat{\mathcal{L}}(\mathsf{U}_{k,n}))}(x) = \sum_{j=0}^{k-1} \binom{n}{j} A_j(x)\,(x-1)^{k-1-j}.$$

By applying the principle of inclusion-exclusion and the preceding formula, the conjecture of Hameister, Rao, and Simpson is asserting that

$$\begin{aligned} \underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x) &= \sum_{i=1}^{k} (-1)^{k-i} \binom{n-i-1}{k-i} h_{\Delta(\widehat{\mathcal{L}}(\mathsf{U}_{i,n}))}(x) \\ &= \sum_{i=1}^{k}\sum_{j=0}^{i-1} (-1)^{k-i} \binom{n-i-1}{k-i}\binom{n}{j} A_j(x)\,(x-1)^{i-1-j}. \end{aligned} \tag{3.2}$$

In what follows, we show how one can manipulate the right-hand side of Corollary 3.1.2.6 to prove this equality.

Lemma 3.1.2.13. *The following identity of binomial coefficients holds:*

$$\binom{n+p+q+1}{n} = \sum_{j=0}^{n} \binom{p+j}{j}\binom{q+n-j}{n-j}.$$

A proof of the above identity can be found in Riordan's book [Rio79, p. 148]; alternatively one can prove it just by induction. We are ready to prove the Conjecture.

Proof. Proof of Corollary 3.1.2.12 As we have indicated before, proving the above equality is equivalent to proving the validity of equation (3.2). Let us denote by $(\star)$ the right-hand side of that equation. Notice that we can expand the term $(x-1)^{k-1-j}$ to obtain

$$(\star) = \sum_{i=1}^{k}\sum_{j=0}^{i-1}\sum_{\ell=0}^{i-j-1} (-1)^{k-i+i-j-1-\ell} \binom{n-i-1}{k-i}\binom{n}{j}\binom{i-1-j}{\ell} A_j(x)\,x^{\ell},$$

which after interchanging the first two sums becomes

$$= \sum_{j=0}^{k-1}\sum_{i=j+1}^{k}\sum_{\ell=0}^{i-j-1} (-1)^{k-j-\ell-1} \binom{n-i-1}{k-i}\binom{n}{j}\binom{i-1-j}{\ell} A_j(x)\,x^{\ell},$$

and after interchanging the order of the second and third sum and relabelling,

$$= \sum_{i=0}^{k-1}\sum_{\ell=0}^{k-i-1}\sum_{j=\ell+i+1}^{k} (-1)^{k-i-\ell-1} \binom{n-j-1}{k-j}\binom{n}{i}\binom{j-1-i}{\ell} A_i(x)\,x^{\ell}. \tag{3.3}$$

[1] Using Brenti and Welker's terminology, the displayed formula is explained by considering the simplicial complex given by the $(k-1)$-skeleton of an n-simplex, and using the fact that the barycentric subdivision yields the order complex of the lattice of flats of the matroid $\mathsf{U}_{k,n}$ without the top element.

On the other hand, by combining Corollary 3.1.2.6 and Lemma 1.1.5.9, the Hilbert–Poincaré series of the Chow ring of $\mathsf{U}_{k,n}$ is given by

$$\underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x) = \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}(-1)^j\binom{n}{i}\binom{n-i-1}{j}A_i(x)\,x^{k-i-j-1},$$

and after reindexing the second sum with $\ell = k-i-j-1$,

$$= \sum_{i=0}^{k-1}\sum_{\ell=0}^{k-i-1}(-1)^{k-i-\ell-1}\binom{n}{i}\binom{n-i-1}{k-i-\ell-1}A_i(x)\,x^{\ell}. \tag{3.4}$$

By Lemma 3.1.2.13, after reparameterizing, we have the following equality:

$$\sum_{j=\ell+i+1}^{k}\binom{n-j-1}{k-j}\binom{j-i-1}{\ell} = \binom{n-i-1}{k-i-\ell-1},$$

which allows us to conclude that the expressions of equations (3.3) and (3.4) are equal, and hence $(\star) = \underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x)$, and the proof is complete. □

3.1.2.2 Haglund–Zhang polynomials

In [FMSV22, Section 5.1], we establish the real-rootedness of $\mathrm{H}_{\mathsf{M}}(x)$ when $\mathsf{M} = \mathsf{U}_{k,n}$ is an arbitrary uniform matroid by showing that $\mathrm{H}_{\mathsf{U}_{k,n}}(x)$ is an example of a Haglund-Zhang polynomial (see Section 1.4.2.4).

Based on computational evidence, we first conjectured and then proved that, for convenient choices of the vector $\underline{s}$, one can obtain the Hilbert–Poincaré series of the augmented Chow ring of arbitrary uniform matroids.

Theorem 3.1.2.14 ([FMSV22, Theorem 5.1])**.** *For $\underline{s} = (n-k+2, n-k+3, \dots, n)$, we have*

$$\widetilde{E}^{\underline{s}}_{k-1}(x) = \mathrm{H}_{\mathsf{U}_{k,n}}(x).$$

In particular, $\mathrm{H}_{\mathsf{U}_{k,n}}(x)$ is always a real-rooted polynomial.

Observe that the preceding statement is an extension of the real-rootedness of the binomial Eulerian polynomials. When the uniform matroid is the Boolean matroid B_n by taking $\underline{s} = (2, 3, \dots, n)$ one has $\widetilde{E}^{\underline{s}}_{n-1}(x) = \widetilde{A}_n(x) = \mathrm{H}_{\mathsf{B}_n}(x)$. This particular case was precisely the content of another result of Haglund and Zhang [HZ19, Theorem 3.1].

Let us fix integers k and n such that $2 \leq k \leq n$. We can consider the vector of consecutive integers $\underline{s} := (n-k+2, \dots, n) \in \mathbb{Z}^{k-1}$ and define the set $\mathcal{I}_{k,n} := \mathcal{I}^{\underline{s}}_{k-1}$. In other words, we have

$$\mathcal{I}_{k,n} := \left\{(e_1, \dots, e_{k-1}) \in \mathbb{Z}^{k-1} : 0 \leq e_i < s_i \text{ for } 1 \leq i \leq k-1\right\},$$

where additionally we use the conventions $e_0 = e_k = 0$ and $s_0 = s_k = 1$.

We are interested in proving that the polynomial

$$E_{k,n}(x) := \sum_{\sigma \in \mathcal{I}_{k,n}} (1+x)^{\mathrm{col}(\sigma)} x^{\mathrm{asc}(\sigma)} \tag{3.5}$$

is precisely $\mathrm{H}_{\mathsf{U}_{k,n}}(x)$. Observe that even though the vector $\underline{s} = (n-k+2, \dots, n)$ is not well-defined whenever $k = 0$ or $k = 1$, we can make sense of the definition of the polynomial $E_{k,n}(x)$

for those values of k, just by setting $E_{1,n}(x) := x+1$ and $E_{0,n}(x) := 1$. In particular, note that $E_{1,n}(x)$ is consistent with equation (3.5) by interpreting that there is only one "empty" vector in $\mathcal{I}_{1,n}$ leading to a single collision and no descents after adding the left zero coordinate $e_0 = 0$ and the right zero coordinate $e_1 = 0$.

The first step towards proving Theorem 3.1.2.14 consists of showing the following lemma.

Lemma 3.1.2.15. *The polynomials $E_{k,n}(x)$ satisfy the following recursion*

$$E_{k,n}(x) = E_{k-1,n-1}(x) + x\sum_{j=0}^{k-1}\binom{n-1}{j} A_j(x)\, E_{k-1-j,n-1-j}(x).$$

This, along with the initial conditions $E_{0,n}(x) = 1$ for all $n \geq 0$, determines them uniquely.

Proof. Observe that since the elements of $\underline{s}$ are consecutive integers, a position $i \in [0, k-1]$ is a collision if and only if $e_i = e_{i+1} = 0$. Similarly, $i \in [0, k-1]$ is an ascent if and only if $e_i < e_{i+1}$. Notice that each element $\underline{e} \in \mathcal{I}_{k,n}$ can be thought of as an element of $\mathcal{I}_{n,n}$ by adding zeros to the left. For instance, $(2, 2, 3) \in \mathcal{I}_{4,7}$ can be embedded into $\mathcal{I}_{7,7}$ as $(0, 0, 0, 2, 2, 3)$. In particular, following the bijection $\Theta\colon \mathfrak{S}_n \to \mathcal{I}_{n,n}$ of the proof of [HZ19, Theorem 3.1], which is defined by $\pi \mapsto (t_{n-1}, \ldots, t_1)$ where $t_i = \#\{j > i : \pi_j < \pi_i\}$, we have that the preimage of $\mathcal{I}_{k,n} \hookrightarrow \mathcal{I}_{n,n}$ under Θ are precisely the permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma_n > \sigma_{n-1} > \cdots > \sigma_k$. Let us denote this set of permutations $\mathfrak{S}_{n,k}$, for each k and n. In particular, again reasoning as in the proof of [HZ19, Theorem 3.1], we obtain

$$E_{k,n}(x) = \sum_{\underline{e}\in\mathcal{I}_{k,n}} (1+x)^{\operatorname{col}(\underline{e})} x^{\operatorname{asc}(\underline{e})} = \sum_{\sigma\in\mathfrak{S}_{n,k}} (1+x)^{\operatorname{bad}(\sigma)} x^{\operatorname{des}(\sigma)},$$

where $\operatorname{bad}(\sigma) = \{i \in [n] : \sigma_{i-1} < \sigma_i \text{ and } \sigma_i < \sigma_j \text{ for all } j > i\}$, with the convention that $\sigma_0 = 0$ and $\operatorname{des}(\sigma) = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$.

Notice that if $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_{n,k}$ has the property that $\sigma_1 = 1$, then $\overline{\sigma} := \sigma_2 \cdots \sigma_n$ can be thought of as an element of $\mathfrak{S}_{n-1,k-1}$, and $\operatorname{des}(\sigma) = \operatorname{des}(\overline{\sigma})$, but $\operatorname{bad}(\sigma) = \operatorname{bad}(\overline{\sigma}) + 1$. On the other hand, if $\sigma_j = 1$ for $j > 1$, then the condition that the last $n-k$ elements of the permutation are in increasing order forces $2 \leq j \leq k$. There are $\binom{n-1}{j-1}$ ways of choosing the elements $\sigma_1 \cdots \sigma_{j-1}$ and, for every possible choice, this part will not contain any bad elements; at position $j-1$ we have a descent because $\sigma_{j-1} > \sigma_j = 1$, and the possible permutations $\sigma_{j+1} \cdots \sigma_n$ are in bijection with the elements of $\mathfrak{S}_{n-j,k-j}$. Everything considered, we have

$$E_{k,n}(x) = (1+x)E_{k-1,n-1}(x) + x\sum_{j=2}^{k}\binom{n-1}{j-1} A_{j-1}(x)\, E_{k-j,n-j}(x),$$

and, after reindexing the sum to be from $j = 1$ to $k-1$ and then rearranging, we obtain the desired recursion. $\square$

A proof of the next result can be found, for example, in [Pet15, Theorem 1.5].

Lemma 3.1.2.16. *The Eulerian polynomials satisfy the following recurrence:*

$$A_{n+1}(x) = A_n(x) + x\sum_{j=0}^{n-1}\binom{n}{j} A_j(x)\, A_{n-j}(x).$$

Now we have all the ingredients to prove Theorem 3.1.2.14.

Proof. Clearly, the polynomials on the right match the base cases $E_{0,n}(x) = 1$ for all $n \geq 0$. Hence, it suffices to show that they satisfy the recurrence of Lemma 3.1.2.15. Let us focus only on the sum appearing in that recursion; later we will multiply by x and add the expression corresponding to $E_{k-1,n-1}(x)$. We have:

$$\sum_{j=0}^{k-1} \binom{n-1}{j} A_j(x) \left(1 + x \sum_{i=0}^{k-2-j} \binom{n-1-j}{i} A_i(x)\,(1 + \cdots + x^{k-2-j-i})\right)$$

$$= \sum_{j=0}^{k-1} \binom{n-1}{j} A_j(x) + x \sum_{j=0}^{k-1} \sum_{i=0}^{k-2-j} \binom{n-1}{j} \binom{n-1-j}{i} A_j(x)\, A_i(x)\,(1 + \cdots + x^{k-2-j-i}),$$

noticing that $\binom{n-1}{j}\binom{n-1-j}{i} = \binom{n-1}{i+j}\binom{i+j}{j}$, and making the change of variables $r = i + j$,

$$= \sum_{j=0}^{k-1} \binom{n-1}{j} A_j(x) + x \sum_{j=0}^{k-1} \sum_{r=j}^{k-2} \binom{n-1}{r} \binom{r}{j} A_j(x)\, A_{r-j}(x)\,(1 + \cdots + x^{k-2-r}),$$

interchanging the order of summation,

$$= \sum_{j=0}^{k-1} \binom{n-1}{j} A_j(x) + \sum_{r=0}^{k-2} \binom{n-1}{r} (1 + \cdots + x^{k-2-r})\, x \sum_{j=0}^{r} \binom{r}{j} A_j(x)\, A_{r-j}(x),$$

using Lemma 3.1.2.16,

$$= \sum_{j=0}^{k-1} \binom{n-1}{j} A_j(x) + \sum_{r=0}^{k-2} \binom{n-1}{r} (1 + \cdots + x^{k-2-r}) \left(A_{r+1}(x) + (x-1)A_r(x)\right),$$

splitting the second sum and using that $(1 + \cdots + x^{k-2-r})(x-1) = x^{k-1-r} - 1$,

$$= \sum_{j=0}^{k-1} \binom{n-1}{j} A_j(x) + \sum_{r=0}^{k-2} \binom{n-1}{r} A_{r+1}(x)(1 + \cdots + x^{k-2-r}) + \sum_{r=0}^{k-2} \binom{n-1}{r} (x^{k-1-r} - 1)A_r(x),$$

cancelling terms in common between the first and the third sums above,

$$= \binom{n-1}{k-1} A_{k-1}(x) + \sum_{r=0}^{k-2} \binom{n-1}{r} A_{r+1}(x)(1 + \cdots + x^{k-2-r}) + \sum_{r=0}^{k-2} \binom{n-1}{r} x^{k-1-r} A_r(x),$$

grouping the first term and the last sum,

$$= \sum_{r=0}^{k-2} \binom{n-1}{r} A_{r+1}(x)(1 + \cdots + x^{k-2-r}) + \sum_{r=0}^{k-1} \binom{n-1}{r} x^{k-1-r} A_r(x),$$

reindexing the first sum to start at $r = 1$,

$$= \sum_{r=1}^{k-1} \binom{n-1}{r-1} A_r(x)(1 + \cdots + x^{k-1-r}) + \sum_{r=0}^{k-1} \binom{n-1}{r} x^{k-1-r} A_r(x).$$

Now, to conclude the proof we multiply this expression by x and add the term corresponding to $E_{k-1,n-1}(x)$ to obtain

$$1 + x\sum_{j=0}^{k-2}\binom{n-1}{j}A_j(x)\,(1+\cdots+x^{k-2-j})$$
$$+\,x\left(\sum_{j=1}^{k-1}\binom{n-1}{j-1}A_j(x)(1+\cdots+x^{k-1-j}) + \sum_{j=0}^{k-1}\binom{n-1}{j}x^{k-1-j}A_j(x)\right),$$

in the second summation above (the first of the second line), we can isolate the term corresponding to x^{k-1-j}

$$= 1 + x\sum_{j=0}^{k-2}\binom{n-1}{j}A_j(x)\,(1+\cdots+x^{k-2-j})$$
$$+\,x\left(\sum_{j=1}^{k-2}\binom{n-1}{j-1}A_j(x)(1+\cdots+x^{k-2-j}) + \sum_{j=1}^{k-1}\binom{n-1}{j-1}A_j(x)x^{k-1-j} + \sum_{j=0}^{k-1}\binom{n-1}{j}x^{k-1-j}A_j(x)\right),$$

we separate the $j=0$ term from the first and the fourth sum,

$$= 1 + x\left((1+\cdots+x^{k-2}) + x^{k-1}\right)$$
$$+\,x\left(\sum_{j=1}^{k-2}\binom{n-1}{j}A_j(x)\,(1+\cdots+x^{k-2-j}) + \sum_{j=1}^{k-2}\binom{n-1}{j-1}A_j(x)(1+\cdots+x^{k-2-j})\right)$$
$$+\,x\left(\sum_{j=1}^{k-1}\binom{n-1}{j-1}A_j(x)x^{k-1-j} + \sum_{j=1}^{k-1}\binom{n-1}{j}x^{k-1-j}A_j(x)\right),$$

we use Pascal's identity $\binom{n-1}{j} + \binom{n-1}{j-1} = \binom{n}{j}$ with the first and the second sum and with the third and the fourth sum,

$$= 1 + x\left((1+\cdots+x^{k-1}) + \sum_{j=1}^{k-2}\binom{n}{j}A_j(x)(1+\cdots+x^{k-2-j}) + \sum_{j=1}^{k-1}\binom{n}{j}A_j(x)x^{k-1-j}\right),$$

finally, we can group the two sums, and add the case $j=0$, to obtain the desired expression,

$$= 1 + x\sum_{j=0}^{k-1}\binom{n}{j}A_j(x)(1+\cdots+x^{k-1-j}). \qquad \square$$

Remark 3.1.2.17. Given that the polynomials $\mathrm{H}_{\mathsf{U}_{k,n}}(x)$ are related to the generalized binomial Eulerian polynomials studied in [HZ19], it is natural to ask whether the same holds for arbitrary matroids. The answer is no: if $\mathsf{M} = \mathsf{U}_{3,4}\oplus\mathsf{U}_{1,1}$ we can compute $\mathrm{H}_{\mathsf{M}}(x) = 1+23x+55x^2+23x^3+x^4$, and an exhaustive computer search shows that there is no $\underline{s} \in \mathbb{Z}_{>0}^{\operatorname{rk}\mathsf{M}-1}$ with $\widetilde{E}^{\underline{s}}_{\operatorname{rk}\mathsf{M}-1}(x) = \mathrm{H}_{\mathsf{M}}(x)$.

Remark 3.1.2.18. We do not know of any analogues of Theorem 3.1.2.14 for the usual Chow ring. Recall that $\underline{\mathrm{H}}_{\mathsf{U}_{n-1,n}}(x) = \frac{1}{x}d_n(x)$, and hence it is reasonable to search among known generalizations of the derangement polynomials. Although the work of Gustafsson and Solus [GS20] provides one such generalization via Ehrhart local h^*-vectors, we were not able to produce $\underline{\mathrm{H}}_{\mathsf{U}_{n-2,n}}(x)$ as a particular case of their polynomials. In general, the polynomials $\underline{\mathrm{H}}_{\mathsf{U}_{n-1,n}}(x)$ do not arise as instances of the polynomials studied in [HZ19]: if $\mathsf{M} = \mathsf{U}_{5,6}$, then $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1 + 51x + 161x^2 + 51x^3 + x^4$, and an exhaustive computer search shows that there is no $\underline{s} \in \mathbb{Z}_{>0}^{\operatorname{rk}\mathsf{M}-2}$ with $\widetilde{E}^{\underline{s}}_{\operatorname{rk}\mathsf{M}-2}(x) = \underline{\mathrm{H}}_{\mathsf{M}}(x)$.

3.1.3 γ-positivity

We now seek to close the gap on the real-rootedness conjectures by giving results for *every* matroid, not just the uniform ones as we did in Theorem 3.1.2.14. A nice stepping stone towards showing real-rootedness is given by the notion of γ-positivity (see Proposition 1.4.1.12). These results come from [FMSV22, Section 3.6 and 4.3].

3.1.3.1 Chow polynomials and augmented Chow polynomials

Theorem 3.1.3.1. *Let M be a loopless matroid. The polynomials $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ are γ-positive.*

Proof. We proceed by induction on the size of the ground set of M. If the matroid M has a ground set of cardinality 1, then $\mathsf{M} \cong \mathsf{B}_1$. In this case, $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$ and $\mathrm{H}_{\mathsf{M}}(x) = x + 1$. The associated γ-polynomials are $\gamma_{\underline{\mathrm{H}}_{\mathsf{M}}}(x) = 1$, and $\gamma_{\mathrm{H}_{\mathsf{M}}}(x) = 1$, and hence they are γ-positive.

Assuming that we have proved the validity of the statement for all matroids with cardinality at most $n-1$, let us consider a matroid M having cardinality n. If M is a Boolean matroid, then Theorems 3.1.2.3 and 3.1.2.4 tells us that $\underline{\mathrm{H}}_{\mathsf{M}}(x) = A_n(x)$, the Eulerian polynomial, whereas $\mathrm{H}_{\mathsf{M}}(x) = \widetilde{A}_n(x)$, the binomial Eulerian polynomial. As we mentioned in Section 1.4.2, both of these families of polynomials are known to be real-rooted and hence γ-positive.

If M is not Boolean, then there is at least one element $e \in E$ that is not a coloop. Using Theorem 2.4.2.7 in combination with Lemma 1.4.1.14, we obtain the following two recursions:

$$\gamma_{\underline{\mathrm{H}}_{\mathsf{M}}}(x) = \gamma_{\underline{\mathrm{H}}_{\mathsf{M}\setminus e}}(x) + x \sum_{F \in \underline{\mathcal{S}}_e(\mathsf{M})} \gamma_{\underline{\mathrm{H}}_{\mathsf{M}/(F\cup\{e\})}}(x) \cdot \gamma_{\underline{\mathrm{H}}_{\mathsf{M}|_F}}(x),$$

$$\gamma_{\mathrm{H}_{\mathsf{M}}}(x) = \gamma_{\mathrm{H}_{\mathsf{M}\setminus e}}(x) + x \sum_{F \in \mathcal{S}_e(\mathsf{M})} \gamma_{\underline{\mathrm{H}}_{\mathsf{M}/(F\cup\{e\})}}(x) \cdot \gamma_{\mathrm{H}_{\mathsf{M}|_F}}(x).$$

Observe that the induction hypothesis guarantees that each of the summands on the right-hand side has non-negative coefficients. The proof is now complete. ☐

Remark 3.1.3.2. The preceding proof relies on the γ-positivity of the Eulerian and the binomial Eulerian polynomials. Although these two results are now well-known, since their proofs are not straightforward (see [PRW08, Section 11]), a reasonable question that the reader might ask is whether it is possible to circumvent the base cases of Boolean matroids in the induction, or at least give a self-contained proof including this case. The answer is yes; in fact, the second case of the semi-small decompositions of Theorem 2.4.2.7 lets us deal with deleting coloops. By computing the graded dimensions, one obtains formulas for $\underline{\mathrm{H}}_{\mathsf{M}\oplus\mathsf{B}_1}(x)$ and $\mathrm{H}_{\mathsf{M}\oplus\mathsf{B}_1}(x)$ when deleting the coloop corresponding to the ground set of the direct summand B_1. Therefore, by

Lemma 1.4.1.14, at the level of γ-polynomials one has

$$\gamma_{\underline{\mathrm{H}}_{\mathsf{M}\oplus\mathsf{B}_1}}(x) = \gamma_{\underline{\mathrm{H}}_{\mathsf{M}}}(x) + x \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \emptyset\neq F\neq E}} \gamma_{\underline{\mathrm{H}}_{\mathsf{M}/F}}(x)\,\gamma_{\underline{\mathrm{H}}_{\mathsf{M}|_F}}(x),$$
$$\gamma_{\mathrm{H}_{\mathsf{M}\oplus\mathsf{B}_1}}(x) = \gamma_{\mathrm{H}_{\mathsf{M}}}(x) + x \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ F\neq E}} \gamma_{\underline{\mathrm{H}}_{\mathsf{M}/F}}(x)\,\gamma_{\mathrm{H}_{\mathsf{M}|_F}}(x).$$

In particular, since all the restrictions and contractions in a Boolean matroid are again Boolean, reasoning inductively one proves that the γ-polynomial of $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ have non-negative coefficients for all Boolean matroids. This gives an independent proof of the γ-positivity of the families of Eulerian and binomial Eulerian polynomials.

The γ-positivity phenomenon for Hilbert–Poincaré series of Chow rings of matroids was also observed independently by Botong Wang in private communication with the authors of [FMSV22].

Continuing with our digression about this γ-positivity phenomenon really being a consequence of the decompositions of Theorem 2.3.2.5, we comment about what may happen if one pretends to extend this property to other posets.

Observe that if we consider the Chow functions $\underline{\mathrm{H}}_P(x)$ and $\mathrm{H}_P(x)$ for arbitrary posets, the real-rootedness and the γ-positivity fail. For instance, the poset depicted in Figure 3.1 has

$$\underline{\mathrm{H}}_P(x) = x^4 + 7x^3 + 11x^2 + 7x + 1,$$
$$\mathrm{H}_P(x) = x^5 + 8x^4 + 18x^3 + 18x^2 + 8x + 1.$$

We observe that $\underline{\mathrm{H}}_P(x)$ and $\mathrm{H}_P(x)$ are not real-rooted since they are not even γ-positive. In this case we have $\gamma_{\underline{\mathrm{H}}_P}(x) = \gamma_{\mathrm{H}_P}(x) = -x^2 + 3x + 1$.

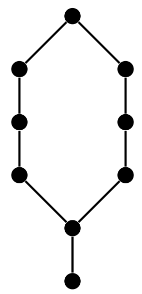

Figure 3.1: A poset P.

We conclude our digression by making one final remark about the practicality of using Theorem 2.3.2.5 to compute Hilbert–Poincaré series. Although the semi-small decomposition does yield a recurrence relation that can be used to compute Hilbert–Poincaré series for arbitrary matroids, making this computation efficient in practice requires memorization in one form or another, together with a fast way of deciding whether two matroids are isomorphic.

3.1.3.2 Z-polynomials

Now our main goal is to prove that the Z-polynomial of a matroid is always γ-positive. To prove this result we are going to use the counterpart of the deletion formula for Chow rings and augmented Chow rings (coming from a semi-small decomposition), due to Braden and Vysogorets that we stated in Theorem 2.4.3.6.

Theorem 3.1.3.3. *For every matroid* M*, the polynomial* $Z_{\mathsf{M}}(x)$ *is* γ*-positive.*

Proof. It suffices to prove the statement only for loopless matroids. We proceed by induction on the cardinality of the ground set of M. If the matroid M is empty, the rank of M is zero and thus the Z-polynomial is $Z_{\mathsf{M}}(x) = 1$, and the associated γ-polynomial is $\gamma_{Z_{\mathsf{M}}}(x) = 1$.

Assuming that we have proved the validity of the statement for all matroids with ground sets of cardinality at most $n-1$, let us consider a matroid M having ground set of cardinality n. If M is a Boolean matroid, then $Z_{\mathsf{M}}(x) = (x+1)^n$. In this case, one obtains $\gamma_{Z_{\mathsf{M}}}(x) = 1$, which has non-negative coefficients.

If M is not Boolean, then there is at least one element $e \in E$ that is not a coloop. Using the result by Braden and Vysogorets, we obtain the following recurrence for the γ-polynomial:

$$\gamma_{Z_{\mathsf{M}}}(x) = \gamma_{Z_{\mathsf{M}\setminus e}}(x) + \sum_{F\in\mathcal{S}_e} \tau\left(\mathsf{M}/(F\cup\{e\})\right) x^{\frac{\operatorname{rk}\mathsf{M}-\operatorname{rk}F}{2}} \gamma_{Z_{\mathsf{M}|_F}}(x).$$

Observe that the induction hypothesis guarantees that each of the summands on the right-hand side has non-negative coefficients. The proof is complete. ☐

Remark 3.1.3.4. We stress the fact that we are using that the τ-invariant is always a non-negative integer. This fact is highly non-trivial and follows from the non-negativity of the coefficients of the Kazhdan–Lusztig polynomials of matroids as proved by Theorem 2.4.3.1. We know of no proof of the non-negativity of $\tau(\mathsf{M})$ that does not rely on that.

Remark 3.1.3.5. Although the Kazhdan–Lusztig polynomial is not palindromic in general, a reasonable question that the reader might ask is whether it is *non-symmetric γ-positive* in the sense of [Ath18, Section 5.1]. In particular, one could ask whether $P_{\mathsf{M}}(x)$ is always right or left γ-positive. Unfortunately it is not the case. As Athanasiadis points out, being right or left γ-positive implies unimodality, and the peak of the coefficients is attained in the middle terms. However, observe that

$$P_{\mathsf{U}_{15,16}}(x) = 1430x^7 + 32032x^6 + \boxed{91728}\,x^5 + 76440x^4 + 23100x^3 + 2640x^2 + 104x + 1$$

and the peak is not in the middle terms (which correspond to degrees 3 and 4). In fact, experimentation suggests that the peak of $P_{\mathsf{U}_{k,n}}(x)$ is always attained approximately at the coefficient of degree $\lfloor \frac{k}{3} \rfloor$.

We are also able to show that the Z-polynomial of every uniform matroid is γ-positive by producing an explicit formula for its coefficients. We record here the result

Theorem 3.1.3.6. *For every uniform matroid* $\mathsf{U}_{k,n}$*, the polynomial* $Z_{\mathsf{U}_{k,n}}(x)$ *is* γ*-positive. Moreover, the constant term is always* 1*, and for* $i > 0$ *the* i*-th coefficient of* $\gamma_{Z_{\mathsf{U}_{k,n}}}(x)$ *is*

$$[x^i]\gamma_{Z_{\mathsf{U}_{k,n}}}(x) = \frac{1}{k-i}\binom{k-i}{i}\sum_{j=i}^{k-1}(k-j)\binom{j-1}{i-1}\binom{n-k+j-1}{j}. \tag{3.6}$$

3.1.4 Valuativity

One last property that will be fundamental in Chapters 4 and 5 is being valuative, as described in Section 1.3.2. We give in details the proof for the Chow polynomials and record the results for the Kazhdan–Lusztig–Stanley polynomials that were already known in the literature.

Theorem 3.1.4.1. *The map* $\underline{\mathrm{H}} : \mathrm{Mat} \to \mathbb{Z}[x]$ *given by* $\mathsf{M} \mapsto \underline{\mathrm{H}}_{\mathsf{M}}(x)$ *is a valuative invariant.*

This result was first proved by Ferroni and Schröter in [FS22, Theorem 10.15] by using a general type of valuation involving arbitrary chains of subsets in a matroid along with Proposition 2.4.2.2. Although the idea of that proof is simple, the details can be quite technical. In [FMSV22] we used Theorem 2.4.2.6 to give a much more compact proof.

Alternative proof of Theorem 3.1.4.1. The map $\underline{\mathrm{H}} : \mathrm{Mat}[\emptyset] \to \mathbb{Z}[x]$ is clearly a valuation, because the only matroid on the empty set is the empty matroid and hence $\underline{\mathrm{H}}$ is constant. Now, assume that $\underline{\mathrm{H}}$ behaves valuatively for matroids with ground sets of size at most $n-1$. Consider the map $\mathrm{V} : \mathrm{Mat} \to \mathbb{Z}[x]$ defined by

$$\mathrm{V}_{\mathsf{M}}(x) = \begin{cases} \underline{\mathrm{H}}_{\mathsf{M}}(x) & \text{if } \mathsf{M} \text{ has a ground set of size at most } n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Since all the matroids involved in a matroid decomposition share a common ground set, the map V is a valuative invariant. Let M be a matroid with a ground set E of size n. If M has a loop e, it means that $\mathcal{P}(\mathsf{M})$ lies on the hyperplane $x_e = 0$ in $\mathbb{R}^E$, and hence so do all the matroid polytopes involved in a decomposition of M. In particular, all of them have loops, and $\underline{\mathrm{H}}$ is identically zero in all of them and hence a valuation. If M is loopless we can write:

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \emptyset}} \overline{\chi}_{\mathsf{M}|_F}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x) = \sum_{\emptyset \neq A \subseteq E} \overline{\chi}_{\mathsf{M}|_A}\, \underline{\mathrm{H}}_{\mathsf{M}/A}(x) = \sum_{A \subseteq E} \overline{\chi}_{\mathsf{M}|_A} \mathrm{V}_{\mathsf{M}/A}(x).$$

The second equality is explained because $\underline{\mathrm{H}}_{\mathsf{M}/A}(x)$ vanishes when M/A has loops, and this happens if and only if A is not a flat of M. Theorem 1.3.3.9 shows that $\underline{\mathrm{H}}$ behaves valuatively, because both the maps $\mathsf{M} \mapsto \overline{\chi}_{\mathsf{M}}$ and $\mathsf{M} \mapsto \mathrm{V}_{\mathsf{M}}(x)$ are valuations. Hence $\underline{\mathrm{H}}$ is a valuation on Mat, and the induction follows. □

Theorem 3.1.4.2. *The map* $\mathrm{H} : \mathrm{Mat} \to \mathbb{Z}[x]$ *given by* $\mathsf{M} \mapsto \mathrm{H}_{\mathsf{M}}(x)$ *is a valuative invariant.*

Proof. As we said in the proof of Theorem 3.1.4.1, if a matroid has loops, all the matroids appearing in a decomposition have the same loops. In particular, for matroids with loops H is a valuation if and only if it is a valuation when removing the common loops. For a loopless matroid M on E we have

$$\mathrm{H}_{\mathsf{M}}(x) = \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x) = \sum_{A \subseteq E} x^{\operatorname{rk} \mathsf{M}_A}\, \underline{\mathrm{H}}_{\mathsf{M}/A}(x),$$

and since both $\mathsf{M} \mapsto x^{\operatorname{rk} \mathsf{M}}$ and $\mathsf{M} \mapsto \underline{\mathrm{H}}_{\mathsf{M}}(x)$ are valuations, we can conclude the valuativeness of $\mathsf{M} \mapsto \mathrm{H}_{\mathsf{M}}(x)$ as well. □

Analogous reasoning shows that the Kazhdan–Lusztig–Stanley polynomials are valuative.

Theorem 3.1.4.3 ([AS23, Theorems 8.8 and 8.9], [FS22, Theorem 11.6])**.** *The maps*

$$\begin{aligned} \mathrm{Mat} &\to \mathbb{Z}[x] \\ P : \mathsf{M} &\mapsto P_{\mathsf{M}}(x) \\ Q : \mathsf{M} &\mapsto Q_{\mathsf{M}}(x) \\ Z : \mathsf{M} &\mapsto Z_{\mathsf{M}}(x) \end{aligned}$$

are valuative invariants.

3.1.5 Monotonicity and weak maps

Theorem 3.1.5.1 ([FMSV22]). *Let* M *be a loopless matroid of rank* k *on a ground set of size* n. *The following inequalities hold:*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) \preceq \underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x),$$
$$\mathrm{H}_{\mathsf{M}}(x) \preceq \mathrm{H}_{\mathsf{U}_{k,n}}(x).$$

In other words, uniform matroids maximize coefficient-wisely the Chow polynomials and augmented Chow polynomials among all matroids with fixed rank and size.

The main ingredients that make the proof possible are the formulas obtained in Propositions 2.4.2.2 and 2.4.2.3. Before stating the main result of this subsection, we start with a useful combinatorial lemma.

Lemma 3.1.5.2. *Let* M *be a loopless matroid on a ground set* E *having size* n *and rank* k. *To each chain* $F_0 \subsetneq \cdots \subsetneq F_m$ *of flats of* M *we can associate injectively a chain* $G_0 \subsetneq \cdots \subsetneq G_m$ *of flats of* $\mathsf{U}_{k,n}$ *in such a way that* $\operatorname{rk}_{\mathsf{M}} F_j = \operatorname{rk}_{\mathsf{U}_{k,n}} G_j$ *for each* $j = 0, \ldots, m$.

Proof. Let us assume that the ground set of M is the set of integers $E = \{1, \ldots, n\}$; the natural total order of the ground set induces a total order for the subsets of $E = [n]$ with fixed cardinality given by comparing lexicographically any pair of sets. Fix a chain of flats $F_0 \subsetneq \cdots \subsetneq F_m$ of M. Let us call $\operatorname{rk}_{\mathsf{M}} F_j = r_j$ for each $j = 0, \ldots, m$. Among all the independent subsets of F_0 that have rank r_0, consider the lexicographically minimum set I_0. Since I_0 is independent in M, it has cardinality at most k, and hence it is independent in $\mathsf{U}_{k,n}$ as well. We define the flat G_0 as the closure of I_0 in $\mathsf{U}_{k,n}$. Notice that

$$\operatorname{rk}_{\mathsf{U}_{k,n}} G_0 = \operatorname{rk}_{\mathsf{U}_{k,n}} I_0 = \# I_0 = \operatorname{rk}_{\mathsf{M}} I_0 = \operatorname{rk}_{\mathsf{M}} F_0.$$

In fact, observe that $G_0 = E$ or $G_0 = I_0$ according to whether $F_0 = E$ or $F_0 \subsetneq E$. Assume we have already constructed $G_0 \subsetneq \cdots \subsetneq G_s$ for $s \geq 0$. To construct the flat G_{s+1} we proceed as follows. First, among all the independent sets of M contained in F_{s+1} that have rank r_{s+1} and contain I_s, consider the lexicographically minimum set I_{s+1}, and now define G_{s+1} as the closure of I_{s+1} in $\mathsf{U}_{k,n}$. Since I_{s+1} was independent in M it is independent in $\mathsf{U}_{k,n}$ as well, and hence $\operatorname{rk}_{\mathsf{M}} G_{s+1} = \operatorname{rk}_{\mathsf{M}} I_{s+1} = \# I_{s+1} = r_{s+1}$. Since $I_{s+1} \supsetneq I_s$, the monotonicity of the closure operator in M guarantees that $G_{s+1} \supsetneq G_s$. Observe that the whole construction is injective, because each flat F_i of the original chain in M can be recovered by taking the closure of G_i in M. □

Proof of Theorem 3.1.5.1. Consider $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$. Observe that the formulas of Proposition 2.4.2.2 and Proposition 2.4.2.3 express them as sums over a set of certain chains of flats of M of polynomials with non-negative coefficients. Using the map of Lemma 3.1.5.2 we can associate injectively a chain of flats in $\mathsf{U}_{k,n}$ in which the flats have the same ranks correspondingly. In other words, each summand appearing in the expressions of $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ (resp. $\mathrm{H}_{\mathsf{M}}(x)$) appears in $\underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x)$ too (resp. in $\mathrm{H}_{\mathsf{U}_{k,n}}(x)$). This proves the desired inequalities. □

With a more general version of Lemma 3.1.5.2, it is possible to extend the preceding statement to a broader setting. Consider a rank-preserving weak map $\mathsf{N} \to \mathsf{M}$ between two matroids M and N. It is reasonable to ask whether it is possible to map injectively chains of flats of M to chains of flats of N preserving the ranks. In fact a straightforward modification of the proof of Lemma 3.1.5.2 yields a map that already appears to work. We note, however, that care is needed, especially because of the injectivity requirement. On the other hand, the case of chains consisting of only one flat is already a non-obvious and challenging statement whose proof can be found in [Luc75, Proposition 5.12]. This can be used to prove the monotonicity under weak maps of the Whitney numbers of the second kind. Observe that it is not true in general that there is an order preserving map from the family of flats of N to the family of flats of M (see the example and the digression in [Luc75, p. 259]).

Chapter 4

Paving matroids

4.1 Relaxation

All the operations from Section 1.1.3 either changed the cardinality of the ground set, the rank of the matroid, or both. However, if one has a matroid $\mathsf{M} = (E, \mathcal{B})$, it is reasonable to ask under which conditions it is possible to add a new member $A \subseteq E$ to the family $\mathcal{B}$ so that $(E, \mathcal{B} \sqcup \{A\})$ is again a matroid.

Theorem 4.1.0.1. *Let* M *be a matroid and* $H \subset E$ *a* circuit-hyperplane*, i.e.* $H \in \mathcal{C} \cap \mathcal{H}$. *Then,*

$$\widetilde{\mathcal{B}} = \mathcal{B} \sqcup \{H\}$$

is the family of bases of a matroid $\widetilde{\mathsf{M}}$.

Proof. See for example [Oxl11, Proposition 1.5.14]. □

The operation of building $\widetilde{\mathsf{M}}$ from M is known as *circuit-hyperplane relaxation*. This operation is among the most basic tools in matroid theory and, according to a result by Truemper [Tru82], this is essentially the *only* way of constructing new matroids from old ones by adjoining exactly one extra basis. The rest of this Section is dedicated to studying a generalization of this operation as presented in [FNV22]. If now we wanted to add a family of subsets $A_1, \ldots, A_s$ to the family $\mathcal{B}$, we can ask what conditions we can impose on them in order to guarantee that $\mathcal{B} \sqcup \{A_1, \ldots, A_s\}$ is again the family of bases of a matroid. In order to extend and generalize this operation, we introduce some terminology.

Definition 4.1.0.2. Let M be a matroid. A hyperplane H of M is said to be *stressed* if all the subsets of H of cardinality $\operatorname{rk} \mathsf{M}$ are circuits. Equivalently, H is stressed if and only if the restriction $\mathsf{M}_{|H}$ is uniform.

Later, in Proposition 4.1.1.5, we will see a prototypical family of matroids having a stressed hyperplane.

Remark 4.1.0.3. A flat that can be obtained as a union of circuits is said to be *cyclic*. A stressed hyperplane of cardinality at least $\operatorname{rk} \mathsf{M}$ is therefore a cyclic hyperplane. The converse is, however, not true.

Example 4.1.0.4. Consider a matroid M with a circuit-hyperplane H; since H is a circuit, $\#H = \operatorname{rk} H + 1$, and since it is a hyperplane, $\operatorname{rk} H = \operatorname{rk} \mathsf{M} - 1$. Hence $\#H = \operatorname{rk} \mathsf{M}$ and the only subset of H of cardinality $\operatorname{rk} \mathsf{M}$ is H itself, which was initially assumed to be a circuit.

In other words, the notion of stressed hyperplanes covers the case of circuit-hyperplanes. Of course, this notion is more general. Its statement introduces some new notation and terminology that will be useful to elaborate the theory that leads to the proofs of our main results.

It turns out that, in general, the presence of a stressed hyperplane H such that $\#H \geq \operatorname{rk}\mathsf{M}$ provides a way of transitioning from the matroid M into another matroid with more bases.

Theorem 4.1.0.5. *Let $\mathsf{M} = (E, \mathcal{B})$ be a matroid. If H is a stressed hyperplane of M, then the set*

$$\widetilde{\mathcal{B}} = \mathcal{B} \sqcup \{S \subseteq H : \#S = \operatorname{rk}\mathsf{M}\},$$

is the family of bases of a matroid $\widetilde{\mathsf{M}} = (E, \widetilde{\mathcal{B}})$.

Proof. Since H is a hyperplane, we have that $\operatorname{rk} H = \operatorname{rk}\mathsf{M} - 1$. If $\#H = \operatorname{rk}\mathsf{M} - 1$ then there is nothing to prove because $\widetilde{\mathcal{B}} = \mathcal{B}$. Let us assume that $\#H \geq \operatorname{rk}\mathsf{M}$. Observe that as H is stressed, if S is a subset of H of cardinality $\operatorname{rk}\mathsf{M}$, it must be a circuit, so we have $\operatorname{rk} S = \operatorname{rk}\mathsf{M} - 1$.

To prove that $\widetilde{\mathcal{B}}$ is the set of bases of a matroid we have to check that it verifies the exchange property. Let us consider two members B_1 and B_2 of $\widetilde{\mathcal{B}}$ and an element $x \in B_1 \setminus B_2$. We have four cases:

- If $B_1, B_2 \in \mathcal{B}$. Here there is nothing to do, because the exchange property between B_1 and B_2 in the matroid M extends to $\widetilde{\mathcal{B}}$.
- If $B_1 \in \mathcal{B}$ and $B_2 \subseteq H$ with $\#B_2 = \operatorname{rk}\mathsf{M}$. Let us call $X = (B_1 \setminus \{x\}) \cup B_2$ and $Y = H$. Observe that $X \cup Y = (B_1 \setminus \{x\}) \cup H$ and $X \cap Y = ((B_1 \setminus \{x\}) \cap H) \cup B_2$. By the semimodularity of the rank function of M, we have the inequality

  $$\operatorname{rk} X + \operatorname{rk} Y \geq \operatorname{rk}(X \cup Y) + \operatorname{rk}(X \cap Y).$$

  - If $B_1 \setminus \{x\} \subseteq H$, then choosing any $y \in B_2 \setminus B_1$, we have that $(B_1 \setminus \{x\}) \cup \{y\}$ is a subset of cardinality $\operatorname{rk}\mathsf{M}$ of H, and thus belongs to $\widetilde{\mathcal{B}}$, and the proof ends.
  - If $B_1 \setminus \{x\} \not\subseteq H$, then $\operatorname{rk}(X \cup Y) = \operatorname{rk}((B_1 \setminus \{x\}) \cup H) = k$, as we are adding a new element to the flat H which initially had rank $\operatorname{rk}\mathsf{M} - 1$. Hence, the inequality above translates into

    $$\operatorname{rk}((B_1 \setminus \{x\}) \cup B_2) + (\operatorname{rk}\mathsf{M} - 1) \geq \operatorname{rk}\mathsf{M} + \operatorname{rk}(X \cap Y),$$

    and since $X \cap Y \supseteq B_2$ in particular its rank is at least $\operatorname{rk}\mathsf{M} - 1$. So

    $$\operatorname{rk}((B_1 \setminus \{x\}) \cup B_2) \geq \operatorname{rk}\mathsf{M}.$$

  Note that this inequality is in fact an equality. Hence, there is a basis B_3 of M contained in $(B_1 \setminus \{x\}) \cup B_2$. Note that $B_3 \neq B_1$ since $x \notin B_2$ by assumption, and so there is an element $y \in B_3 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$ by the exchange property. Note that $B_3 \setminus B_1 \subseteq B_2 \setminus B_1$, and so y is in fact an element of $B_2 \setminus B_1$ as desired.
- If $B_1 \subseteq H$, $\#B_1 = \operatorname{rk}\mathsf{M}$ and $B_2 \in \mathcal{B}$. As $B_1 \setminus \{x\}$ is an independent set of cardinality $\operatorname{rk}\mathsf{M} - 1$ in M, by the third independence axiom $I3$, there exists a $y \in B_2 \setminus (B_1 \setminus \{x\})$ so that $B_3 = (B_1 \setminus \{x\}) \cup \{y\}$ is a basis for M.
- If $B_1, B_2 \subseteq H$ with $\#B_1 = \#B_2 = \operatorname{rk}\mathsf{M}$. In this case, by choosing any $y \in B_2 \setminus B_1$ we can form a set $(B_1 \setminus \{x\}) \cup \{y\} \subseteq H$ which has cardinality $\operatorname{rk}\mathsf{M}$ and thus belongs to $\widetilde{\mathcal{B}}$.

□

This operation of changing circuits contained in hyperplanes into bases will be referred to as *stressed hyperplane relaxation*. If H is a stressed hyperplane in M and $\widetilde{\mathsf{M}}$ is the matroid constructed as above, we will say that we have *relaxed* H and that $\widetilde{\mathsf{M}}$ is a *relaxation* of M.

We make the brief comment that relaxing a stressed hyperplane which is also independent, by definition does not change the matroid M. We can ignore such hyperplanes and focus only on those that have cardinality at least $\operatorname{rk}\mathsf{M}$, in order to guarantee that our matroid indeed changes when we do a relaxation.

Now we give a characterization of the matroids that arise by performing a stressed hyperplane relaxation. In other words, it is possible to describe an intrinsic property of a matroid that reveals that it actually comes from the relaxation of a stressed hyperplane in another matroid.

Proposition 4.1.0.6. *Let $\mathsf{M} = (E, \mathcal{B})$ be a matroid. Assume that A is a subset of E with the following three properties.*

- *$A \neq E$.*
- *The set $\mathcal{B}' = \{B' \subseteq A \mid \#B' = \operatorname{rk}\mathsf{M}\}$ is a proper subset of $\mathcal{B}$.*
- *For every $x \in E \setminus A$ and every $B' \in \mathcal{B}'$, the set $B' \cup \{x\}$ is a circuit of M.*

Then $\mathcal{B} \setminus \mathcal{B}'$ is the set of bases of a matroid N. Moreover, A is a stressed hyperplane in N and $\mathsf{M} = \widetilde{\mathsf{N}}$.

Proof. If we consider any basis $B' \in \mathcal{B}'$ and $x \notin A$ then, by the third assumption, we have that $B' \cup \{x\}$ is a circuit. Thus, removing any other element yields an independent set of rank equal to $\operatorname{rk}\mathsf{M}$ in M. In other words,

$$(B' \setminus \{b'\}) \cup \{x\} \in \mathcal{B} \setminus \mathcal{B}' \tag{4.1}$$

for every $b' \in B'$ and $x \notin A$.

To show that $\mathcal{B}\setminus\mathcal{B}'$ is the family of bases of a matroid, as $\mathcal{B}\setminus\mathcal{B}' \neq \emptyset$ by the second assumption, we only need to prove that the bases exchange property holds. To this end, consider two distinct bases $B_1, B_2 \in \mathcal{B}\setminus\mathcal{B}'$ and an element $a \in B_1 \setminus B_2$. Since B_1 and B_2 are bases in M, by applying the bases exchange property in this matroid, we have that there exists $b' \in B_2 \setminus B_1$ such that

$$(B_1 \setminus \{a\}) \cup \{b'\} \in \mathcal{B}.$$

If $(B_1 \setminus \{a\}) \cup \{b'\} \notin \mathcal{B}'$, then there is nothing to prove. Henceforth, we will assume that $(B_1 \setminus \{a\}) \cup \{b'\} \in \mathcal{B}'$. Observe that this implies that there is some $B' \in \mathcal{B}'$ such that

$$B_1 \setminus \{a\} = B' \setminus \{b'\}. \tag{4.2}$$

Now, since $B_2 \notin \mathcal{B}'$, in particular $B_2 \setminus A \neq \emptyset$, because of the first and the second assumption. Let us pick any $x \in B_2 \setminus A$. By (4.1), it follows that

$$(B' \setminus \{b'\}) \cup \{x\} \in \mathcal{B} \setminus \mathcal{B}'.$$

Combining this with equation (4.2) shows that there exists an $x \in B_2 \setminus B_1$ such that $(B_1 \setminus \{a\}) \cup \{x\} \in \mathcal{B} \setminus \mathcal{B}'$, and hence the bases exchange property holds within the family $\mathcal{B} \setminus \mathcal{B}'$.

To finish, it remains to show that A is a stressed hyperplane in the matroid $\mathsf{N} = (E, \mathcal{B} \setminus \mathcal{B}')$.

- Let us prove that A is a hyperplane of N. Choose any basis $B' \in \mathcal{B}'$. Let us pick any $x \notin A$ and $b' \in B'$. By (4.1), if we call $B = (B' \setminus \{b'\}) \cup \{x\}$, we have $B \in \mathcal{B} \setminus \mathcal{B}'$. In particular, notice that $B' \setminus \{b'\} \subseteq B \in \mathcal{B} \setminus \mathcal{B}'$, and hence it is an independent set in N. In other words, $\operatorname{rk}_{\mathsf{N}}(B' \setminus \{b'\}) = \operatorname{rk} \mathsf{M} - 1$. Since $B' \setminus \{b'\} \subseteq B' \subseteq A$, we obtain that $\operatorname{rk}_{\mathsf{N}} A \geq \operatorname{rk} \mathsf{M} - 1$. The second assumption in the statement of Proposition 4.1.0.6 implies that A contains no basis of N, so $\operatorname{rk}_{\mathsf{N}}(A) = \operatorname{rk} \mathsf{M} - 1$. Also, since $A \cup \{x\} \supseteq B \in \mathcal{B} \setminus \mathcal{B}'$, we have that $\operatorname{rk}_{\mathsf{N}}(A \cup \{x\}) = \operatorname{rk} \mathsf{M}$. Since $x \notin A$ was arbitrary, we get that A is indeed a hyperplane in N.
- Observe that every $B' \in \mathcal{B}'$ is a circuit in N. This is implicit in the preceding paragraph, as for every $b' \in B'$, we know by (4.1) that $B' \setminus \{b'\}$ is contained in a basis of N or, equivalently, is independent.

□

Definition 4.1.0.7. A subset A as in the preceding result, will be called a *free subset* of M.

Assume that M is a matroid having a free subset A of cardinality $\operatorname{rk} \mathsf{M}$. It follows from the second condition that in fact A has to be a basis. We call such subset a *free basis*. In other words, any circuit-hyperplane becomes a free basis after being relaxed. This is used later in Section 4.2.2 to study questions regarding non-degeneracy.

Remark 4.1.0.8. We mention that, after the introduction of this operation, stressed hyperplane relaxation was further generalized in [FS22] to the operation of *stressed subset relaxation* on the class of *split matroids*. However, it is hard to prove numerical results for all matroids in that class, as bounds become too broad and formulas too general. That is why, for the purpose of this work, we decided to state everything only in terms of this operation.

4.1.1 Structural properties

If $\widetilde{\mathsf{M}}$ is a relaxation of M, many of the properties of M are still present in $\widetilde{\mathsf{M}}$. For example, their rank functions differ only on a "small" list of subsets.

Proposition 4.1.1.1. *Let M be a matroid and let H be a stressed hyperplane. If $\widetilde{\mathsf{M}}$ denotes the relaxed matroid, then the rank function $\widetilde{\operatorname{rk}}$ of $\widetilde{\mathsf{M}}$ is given by*

$$\widetilde{\operatorname{rk}} A = \begin{cases} \operatorname{rk} A + 1 & \text{if } A \subseteq H \text{ and } \#A \geq k \\ \operatorname{rk} A & \text{otherwise,} \end{cases}$$

where rk *is the rank function of* M.

Proof. Observe that $\operatorname{rk} A \leq \widetilde{\operatorname{rk}} A$ for each A, as $\widetilde{\mathsf{M}}$ contains all the bases of M. Assume that A is a set with $\operatorname{rk} A < \widetilde{\operatorname{rk}} A$. By using the definition of the rank functions of both matroids, we have

$$\max_{I \in \mathcal{I}(\mathsf{M})} \#(A \cap I) < \max_{\substack{S \subseteq H \\ \#S = \operatorname{rk} \mathsf{M}}} \#(A \cap S).$$

In particular, we can choose $S \subseteq H$ with $\#S = \operatorname{rk} \mathsf{M}$ (and hence S is a circuit of M) such that $\#(A \cap S) > \#(A \cap I)$ for all independent sets I of M. Let us prove that $S \subseteq A$. If we choose any $x \in S$, we have that $S \setminus \{x\}$ is independent. Because of how we chose S, it follows that

$$\#(A \cap (S \setminus \{x\})) < \#(A \cap S).$$

which implies that $x \in A$, and we have $S \subseteq A$ as we claimed. Hence this shows that $\#A \geq \operatorname{rk} \mathsf{M}$ when $\operatorname{rk} A < \widetilde{\operatorname{rk}} A$.

Now we prove that we also have $A \subseteq H$ when $\operatorname{rk} A < \widetilde{\operatorname{rk}} A$. To this end, observe that since $\operatorname{rk} S = \operatorname{rk} \mathsf{M} - 1$ and $S \subseteq A$, we must have $\operatorname{rk} A \geq \operatorname{rk} \mathsf{M} - 1$. Also, since $\operatorname{rk} A < \widetilde{\operatorname{rk}} A \leq \operatorname{rk} \mathsf{M}$, we obtain that $\operatorname{rk} A = \operatorname{rk} \mathsf{M} - 1$ and $\widetilde{\operatorname{rk}} A = \operatorname{rk} \mathsf{M}$. Assume that $A \not\subseteq H$, and take $x \in A \setminus H$. Since $x \notin H$, $\operatorname{rk}(S \cup \{x\}) = \operatorname{rk} \mathsf{M}$, as the flat spanned by $S \cup \{x\}$ is E, because the flat spanned by S is the hyperplane H. Since $S \subseteq A$ and $x \in A$, we obtain that $S \cup \{x\} \subseteq A$ and

$$\operatorname{rk} \mathsf{M} = \operatorname{rk}(S \cup \{x\}) \leq \operatorname{rk} A = \operatorname{rk} \mathsf{M} - 1,$$

which is a contradiction. It follows that $A \subseteq H$. In summary, we have proved that the strict inequality $\operatorname{rk} A < \widetilde{\operatorname{rk}} A$ holds only for the subsets $A \subseteq H$ of cardinality at least $\operatorname{rk} \mathsf{M}$, as was claimed. □

It is natural to ask what the stressed hyperplanes of an already relaxed matroid are. The next results provide a proof that, in fact, after relaxing one stressed hyperplane, the remaining stressed hyperplanes of the original matroid continue to be stressed in the new matroid.

Proposition 4.1.1.2. *Let M be a matroid with two distinct stressed hyperplanes H_1 and H_2. Then $\#(H_1 \cap H_2) \leq \operatorname{rk} \mathsf{M} - 2$.*

Proof. Since H_1 and H_2 are distinct hyperplanes, their intersection $F = H_1 \cap H_2$ is a flat strictly contained in both of them. In particular, $\operatorname{rk} F < \operatorname{rk} H_1 = \operatorname{rk} \mathsf{M} - 1$. Since H_1 is stressed, its subsets of cardinality greater than or equal to $\operatorname{rk} \mathsf{M}$ have rank $\operatorname{rk} \mathsf{M} - 1$. Hence, the only possibility is that $\#(H_1 \cap H_2) = \#F \leq \operatorname{rk} \mathsf{M} - 2$. □

Proposition 4.1.1.3. *Let M be a matroid with two distinct stressed hyperplanes H_1 and H_2. If $\widetilde{\mathsf{M}}$ is the matroid obtained from M after relaxing H_1, then H_2 is a stressed hyperplane in $\widetilde{\mathsf{M}}$.*

Proof. For $i \in \{1, 2\}$ consider $\mathcal{C}_i = \binom{H_i}{\operatorname{rk} \mathsf{M}}$, the subsets of cardinality $\operatorname{rk} \mathsf{M}$ of H_i. That is, $\mathcal{C}_i$ is the set of circuits contained in hyperplane H_i. Observe that

- H_2 is a hyperplane in $\widetilde{\mathsf{M}}$. Since H_2 does not satisfy the conditions of Proposition 4.1.1.1 for its rank to increase in $\widetilde{\mathsf{M}}$, we know that $\widetilde{\operatorname{rk}} H_2 = \operatorname{rk} \mathsf{M} - 1$. Suppose H_2 is not a flat in $\widetilde{\mathsf{M}}$, and so $\widetilde{\operatorname{rk}}(H_2 \cup \{x\}) = \operatorname{rk} \mathsf{M} - 1$ for some $x \notin H_2$. Then this would imply that $\operatorname{rk}(H_2 \cup \{x\}) = \operatorname{rk} \mathsf{M} - 1$ again by Proposition 4.1.1.1 since $H_2 \cup \{x\}$ is not contained in H_1, which contradicts the fact that H_2 is a hyperplane in M.
- The elements of $\mathcal{C}_2$ are circuits in $\widetilde{\mathsf{M}}$. By Proposition 4.1.1.2, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. In particular, we can use Proposition 4.1.1.1 to obtain that the members of $\mathcal{C}_2$ are still circuits in $\widetilde{\mathsf{M}}$, since their ranks do not change, and neither do the rank of their subsets.

In particular, the definition implies that H_2 is in fact stressed in $\widetilde{\mathsf{M}}$, as desired. □

Let us now give a description of how the family of flats of a matroid changes when one applies this operation.

Proposition 4.1.1.4. *Let M be a matroid and let H be a stressed hyperplane. If $\widetilde{\mathsf{M}}$ is the relaxed matroid, then*

$$\mathcal{L}(\widetilde{\mathsf{M}}) = (\mathcal{L}(\mathsf{M}) \setminus \{H\}) \sqcup \{A \subseteq H \mid \#A = \operatorname{rk} \mathsf{M} - 1\}. \tag{4.3}$$

Proof. Let F be a flat of $\widetilde{\mathsf{M}}$ that is not a flat of M. We claim that $\widetilde{\mathrm{rk}}F = \mathrm{rk}\, F$. Indeed, if it was not the case then, by Proposition 4.1.1.1, we would have that $F \subseteq H$ and $\#F \geq \mathrm{rk}\,\mathsf{M}$. Both conditions imply that $\widetilde{\mathrm{rk}}F = \mathrm{rk}\,\mathsf{M}$ since every subset of cardinality $\mathrm{rk}\,\mathsf{M}$ of H is a basis in $\widetilde{\mathsf{M}}$, and since F is a flat, F has to be the ground set, which cannot happen as F was not a flat of M. Now, since F is *not* a flat of M, we know that there exists some $x \notin F$ such that $\mathrm{rk}(F \cup \{x\}) = \mathrm{rk}\, F$. Since F is a flat in $\widetilde{\mathsf{M}}$, it follows that

$$\widetilde{\mathrm{rk}}(F \cup \{x\}) > \widetilde{\mathrm{rk}}F = \mathrm{rk}\, F = \mathrm{rk}(F \cup \{x\}).$$

Using Proposition 4.1.1.1 again, we have that $F \cup \{x\} \subseteq H$ and $\#(F \cup \{x\}) \geq \mathrm{rk}\,\mathsf{M}$. Notice that we must have $\#(F \cup \{x\}) = \mathrm{rk}\,\mathsf{M}$, because otherwise it would be the case that $\#F \geq \mathrm{rk}\,\mathsf{M}$ and also $F \subseteq H$, which yields to a contradiction as in the first paragraph. Hence, F has to be a subset of cardinality $\mathrm{rk}\,\mathsf{M} - 1$ of H. So we have proved the inclusion $\subseteq$ in equation (4.3).

Let us prove the other inclusion. Choose a flat $F \in \mathcal{L}(\mathsf{M}) \setminus \{H\}$. Consider any element $x \notin F$. We have that $\mathrm{rk}\, F < \mathrm{rk}(F \cup \{x\})$. Also,

$$\widetilde{\mathrm{rk}}F \leq \mathrm{rk}(F \cup \{x\}) \leq \widetilde{\mathrm{rk}}(F \cup \{x\}).$$

Assume that $\widetilde{\mathrm{rk}}F = \widetilde{\mathrm{rk}}(F \cup \{x\})$. The double inequality above shows that $\widetilde{\mathrm{rk}}F = \mathrm{rk}(F \cup \{x\}) > \mathrm{rk}\, F$. By Proposition 4.1.1.1, it follows that $F \subseteq H$ and $\#F \geq \mathrm{rk}\,\mathsf{M}$. This is impossible, because the only flat of M contained in H and having cardinality at least $\mathrm{rk}\,\mathsf{M}$ is H itself, and we assumed $F \in \mathcal{L}(\mathsf{M}) \setminus \{H\}$. It follows that $\widetilde{\mathrm{rk}}F < \widetilde{\mathrm{rk}}(F \cup \{x\})$ which, since $x \notin F$ was arbitrary, implies that F is a flat of $\widetilde{\mathsf{M}}$.

Now, choose $F \subseteq H$ such that $\#F = \mathrm{rk}\,\mathsf{M} - 1$. Since all the subsets of cardinality $\mathrm{rk}\,\mathsf{M}$ of H are independent in $\widetilde{\mathsf{M}}$, in particular, $\widetilde{\mathrm{rk}}F = \mathrm{rk}\,\mathsf{M} - 1$. If we choose any element $x \notin F$, we have two cases.

- If $x \in H$, then $F \cup \{x\}$ is a subset of H of cardinality $\mathrm{rk}\,\mathsf{M}$, and is thus independent in $\widetilde{\mathsf{M}}$. This says that $\widetilde{\mathrm{rk}}(F \cup \{x\}) > \widetilde{\mathrm{rk}}F$.
- If $x \notin H$, then $\widetilde{\mathrm{rk}}(H \cup \{x\}) \geq \mathrm{rk}(H \cup \{x\}) = \mathrm{rk}\,\mathsf{M}$, because H is a hyperplane in M. In particular $\widetilde{\mathrm{rk}}(H \cup \{x\}) = \mathrm{rk}\,\mathsf{M}$, and since $\widetilde{\mathrm{rk}}(F) = \mathrm{rk}\,\mathsf{M} - 1$ and $F \subseteq H$. Since the flat spanned by F in M is H and $x \notin H$, we have that $\mathrm{rk}\,\mathsf{M} = \mathrm{rk}(F \cup \{x\}) \leq \widetilde{\mathrm{rk}}(F \cup \{x\})$, so the inequality $\widetilde{\mathrm{rk}}(F \cup \{x\}) > \widetilde{\mathrm{rk}}F$ holds, as $\mathrm{rk}\,\mathsf{M} > \mathrm{rk}\,\mathsf{M} - 1$.

It follows that in either case $\widetilde{\mathrm{rk}}(F \cup \{x\}) > \widetilde{\mathrm{rk}}F$, which proves that F is a flat of $\widetilde{\mathsf{M}}$ and the proof is complete. ☐

We end this section with a prototypical class of matroids with a stressed hyperplane.

Proposition 4.1.1.5. *The matroid* $\Pi_{k,h,n} = \mathsf{U}_{k-1,h} \oplus \mathsf{U}_{1,n-h}$ *is a matroid of rank* k*, cardinality* n *having a stressed hyperplane of cardinality* h*. Also, the relaxed matroid* $\Lambda_{k,h,n} := \widetilde{\Pi_{k,h,n}}$ *has the following property*

$$\mathrm{si}\,(\Lambda_{k,h,n}) \cong \mathsf{U}_{k,h+1}.$$

Proof. Notice that the considerations on the rank and the cardinality of $\Pi_{k,h,n}$ are consequences of the definition of the direct sum of matroids. Now, let us label the ground set of $\Pi_{k,h,n}$ as $E = \{1, \ldots, n\}$ such that $E_1 = \{1, \ldots, h\}$ is the ground set of $\mathsf{U}_{k-1,h}$ and $E_2 = \{h+1, \ldots, n\}$ is the ground set of $\mathsf{U}_{1,n-h}$.

We claim that E_1 is a stressed hyperplane. This follows readily from the fact that it is a flat of rank $k-1$ and any subset $S \subseteq E_1$ of cardinality k is a circuit when considered as a subset of $\mathsf{U}_{k-1,h}$.

Now, to prove that the simplification of the matroid $\Lambda_{k,h,n}$ is isomorphic to the uniform matroid $\mathsf{U}_{k,h+1}$ we have to look at the flats first.

The flats of $\Pi_{k,h,n}$ are exactly the disjoint unions of a flat of $\mathsf{U}_{k-1,h}$ and a flat of $\mathsf{U}_{1,n-h}$. In other words, F is a flat of $\Pi_{k,h,n}$ if and only if

$$\#(F \cap E_1) \in \{0, 1, \dots, k-2, h\} \quad \text{and} \quad \#(F \cap E_2) \in \{0, n-h\}.$$

Thus, by Proposition 4.1.1.4, the flats $\widetilde{F} \in \mathcal{L}(\Lambda_{k,h,n})$ have to satisfy either

$$\#(\widetilde{F} \cap E_1) \in \{0, 1, \dots, k-2, k-1\} \quad \text{and} \quad \#(\widetilde{F} \cap E_2) = 0,$$

$$\text{or } \#(\widetilde{F} \cap E_1) \in \{0, 1, \dots, k-2, h\} \quad \text{and} \quad \#(\widetilde{F} \cap E_2) = n-k.$$

Notice that the set E_2 is an atom of this lattice of flats. The remaining h atoms are the elements of E_1. Moreover, if we label the elements of E_1 as $\overline{1}, \dots, \overline{h}$ and label the atom E_2 as $\overline{h+1}$, we can construct an order-preserving bijection from the lattice of flats of $\Lambda_{k,h,n}$ to the family of subsets of $\{\overline{1}, \dots, \overline{h+1}\}$ having cardinalities in $\{0, \dots, k-1, h+1\}$. The latter is just isomorphic to the lattice of flats of $\mathsf{U}_{k,h+1}$, which implies that the simplification of $\Lambda_{k,h,n}$ is isomorphic to $\mathsf{U}_{k,h+1}$, as desired. □

Remark 4.1.1.6. In [HMM+22], an alternative presentation for the matroid $\Lambda_{k,h,n}$ is achieved by a description as a lattice path matroid. They also provide several formulas and results regarding the Ehrhart polynomial for paving matroids and $\Lambda_{k,h,n}$.

In light of this new operation we can redefine the classes of matroids that we introduced in 1.1.2.4 in the following way.

Proposition 4.1.1.7. *A matroid* M *is*

- paving *if and only if it can be relaxed to the uniform matroid with a series of stressed-hyperplane relaxations.*
- sparse paving *if and only if it can be relaxed to the uniform matroid with a series of circuit-hyperplane relaxations.*

The previous proposition also highlights the containment of the classes

$$\{\text{sparse paving matroids}\} \subsetneq \{\text{paving matroids}\} \subsetneq \{\text{all matroids}\}$$

Proof. First, let us show that every hyperplane in a paving matroid is stressed. Observe that a hyperplane of cardinality less than $\operatorname{rk}\mathsf{M}$ is tautologically stressed. Consider a hyperplane H of cardinality at least $\operatorname{rk}\mathsf{M}$ in M. A subset $S \subseteq H$ has rank $\operatorname{rk} S \leq \operatorname{rk} H = \operatorname{rk}\mathsf{M} - 1$, and if we choose S so that $\#S = \operatorname{rk}\mathsf{M}$, then $\operatorname{rk} S \geq \operatorname{rk}\mathsf{M} - 1$ because M is paving. It follows that $\operatorname{rk} S = \operatorname{rk}\mathsf{M} - 1$. Again, since M is paving, any proper subset of S is independent, so that in particular S is a circuit, and as S is arbitrary, it follows that H is stressed.

We also observe that the relaxation of a paving matroid is still paving. In fact, if $\widetilde{\mathsf{M}}$ is obtained from M via relaxing a stressed hyperplane H, then as we added only a few bases when we passed from M to $\widetilde{\mathsf{M}}$, we have $\mathcal{I}(\mathsf{M}) \subseteq \mathcal{I}(\widetilde{\mathsf{M}})$. As $\mathcal{I}(\mathsf{M})$ already contained all the subsets of cardinality $\operatorname{rk}\mathsf{M} - 1$ of the ground set, it follows that so does $\widetilde{\mathsf{M}}$, and hence it is paving as well. Lastly, if we relax all the hyperplanes of M of cardinality at least $\operatorname{rk}\mathsf{M}$, we end up obtaining a paving matroid of rank $\operatorname{rk}\mathsf{M}$ such that all of its hyperplanes have cardinality $\operatorname{rk}\mathsf{M} - 1$, i.e. a uniform matroid.

In sparse paving matroids we observe that no hyperplane can have size larger than the rank of the matroid. The statement then follows after observing that stressed hyperplanes in sparse paving matroids coincide with circuit-hyperplanes. □

4.1.2 Polytopal interpretation

The reader might object to the introduction of the parameter n in the definition of $\Pi_{k,h,n}$ in Proposition 4.1.1.5, since in the end, $\Lambda_{k,h,n}$ is just the uniform matroid $\mathsf{U}_{k,h+1}$ with some extra (parallel) elements. However, from a geometric point of view, the matroids $\Lambda_{k,h,n}$ are the "pieces" that one is gluing to the base polytope of a matroid M of rank k and cardinality n when relaxing a stressed hyperplane of cardinality h.

Let M be a matroid with a stressed hyperplane H and $\widetilde{\mathsf{M}}$ be its relaxation. Consider the matroid

$$\Pi_{k,H,E} := \mathsf{U}_{1,E\setminus H} \oplus \mathsf{U}_{k-1,H},$$

and let $\Lambda_{k,H,E}$ denote the relaxation of $\Pi_{k,H,E}$ with respect to the stressed hyperplane H of $\Pi_{k,H,E}$. The base polytope of $\Pi_{k,H,E}$ is a face of the base polytopes of both $\Lambda_{r,k,E,F}$ and M. For $\Lambda_{k,H,E}$, it is the facet on which the linear functional $x \mapsto \sum_{e\notin H} x_e$ is maximized. For M it is the face on which the linear functional $x \mapsto \sum_{e\in H} x_e$ is maximized, and it is a facet unless $\mathsf{M} = \Pi_{k,H,E}$. Let $\mathcal{N}$ be the collection of matroids consisting of M, $\Lambda_{k,H,E}$, $\Pi_{k,H,E}$, and all of their faces. The following theorem is proved in [FS22, Theorem 5.5].

Theorem 4.1.2.1. *The collection $\mathcal{N}$ is a decomposition of $\widetilde{\mathsf{M}}$. If $\mathsf{M} = \Pi_{k,H,E}$, then $\mathcal{N}$ is the trivial decomposition of $\Lambda_{k,H,E}$. If not, then the only internal faces of $\mathcal{N}$ are M, $\Lambda_{k,H,E}$, and $\Pi_{k,H,E}$.*

A paving matroid is obtained by starting with a hypersimplex $\Delta_{k,n}$ and then cutting out pieces corresponding to $\mathcal{P}(\Lambda_{k,h,n})$ for suitable subsets $H \subset E$.

4.1.3 Polynomial invariants

The polytopal interpretation given in 4.1.2 and Proposition 4.1.1.7 together give us a great tool to compute matroid invariants for paving matroids.

Theorem 4.1.3.1. *Let f be a valuative invariant and let M be a paving matroid of rank k and cardinality n. Suppose M has exactly λ_h stressed hyperplanes of cardinality h. Then*

$$f(\mathsf{M}) = f(\mathsf{U}_{k,n}) - \sum_{h\geq k} \lambda_h \left(f(\Lambda_{k,h,n}) - f(\Pi_{k,h,n})\right).$$

Moreover, if f is invariant up to simplification,

$$f(\mathsf{M}) = f(\mathsf{U}_{k,n}) - \sum_{h\geq k} \lambda_h \left(f(\mathsf{U}_{k,h+1}) - f(\mathsf{U}_{k-1,h} \oplus \mathsf{B}_1)\right).$$

Lastly, if f is multiplicative under direct sum, this reduces to

$$f(\mathsf{M}) = f(\mathsf{U}_{k,n}) - \sum_{h\geq k} \lambda_h \left(f(\mathsf{U}_{k,h+1}) - f(\mathsf{U}_{k-1,h})\, f(\mathsf{B}_1)\right).$$

Since in Section 1.3.3 we observed that the Tutte polynomial $T_{\mathsf{M}}(x,y)$, the characteristic polynomial $\chi_{\mathsf{M}}(x)$, the reduced characteristic polynomial $\overline{\chi}_{\mathsf{M}}(x)$, the Kazhdan–Lusztig–Stanley polynomials $P_{\mathsf{M}}(x)$, $Q_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$, the Chow polynomial $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and the augmented Chow polynomial $\mathrm{H}_{\mathsf{M}}(x)$ are all valuative, the previous result gives us a fast non-recursive formula to compute all these polynomials for every paving matroid only in terms of uniform matroids, for which we have closed formulas.

4.1.3.1 An alternative proof for Kazhdan–Lusztig–Stanley polynomials

Without mentioning the valuativity property, an alternative proof of the relaxation formula for the three Kazhdan–Lusztig–Stanley polynomials is possible. This is the content of [FNV22, Section 4]. The reason to include it is that in Section 5.1 we generalize the relaxation operation to equivariant polynomials and that proof follows this line of thoughts.

The main idea is to think of the definition of these polynomials on the lattice of flats. Since the relaxation changes $\mathcal{L}(\mathsf{M})$ only at the top (see Proposition 4.1.1.4), there is hope that performing a relaxation changes the invariants in a controlled way. Here is what happens for the Tutte polynomial.

Proposition 4.1.3.2. *Let M be a matroid of rank k having a stressed hyperplane H of cardinality h. The Tutte polynomial of the relaxed matroid $\widetilde{\mathsf{M}}$ is given by*

$$T_{\widetilde{\mathsf{M}}}(x,y) = T_{\mathsf{M}}(x,y) + (x+y-xy)\sum_{j=k}^{h}\binom{h}{j}(y-1)^{j-k}.$$

Proof. If $\widetilde{\operatorname{rk}}$ is the rank function on $\widetilde{\mathsf{M}}$ and rk is the rank function on M, by Proposition 4.1.1.1 these two functions agree everywhere except on the sets of cardinality at least k contained in H. Hence, we can manipulate the Tutte polynomial for $\widetilde{\mathsf{M}}$ in the following way.

$$\begin{aligned} T_{\widetilde{\mathsf{M}}}(x,y) - T_{\mathsf{M}}(x,y) &= \sum_{\substack{A\subseteq H\\ \#A\geq k}} (x-1)^{\widetilde{\operatorname{rk}}E-\widetilde{\operatorname{rk}}A}(y-1)^{\#A-\widetilde{\operatorname{rk}}A} \\ &\quad - \sum_{\substack{A\subseteq H\\ \#A\geq k}} (x-1)^{\operatorname{rk}E-\operatorname{rk}A}(y-1)^{\#A-\operatorname{rk}A} \\ &= \sum_{\substack{A\subseteq H\\ \#A\geq k}} (x-1)^{0}(y-1)^{\#A-k} - \sum_{\substack{A\subseteq H\\ \#A\geq k}} (x-1)^{1}(y-1)^{\#A-k+1} \\ &= (1-(x-1)(y-1))\cdot \sum_{\substack{A\subseteq H\\ \#A\geq k}} (y-1)^{\#A-k} \\ &= (x+y-xy)\cdot\sum_{j=k}^{h}\binom{h}{j}(y-1)^{j-k}. \end{aligned}$$

□

Corollary 4.1.3.3. *If M is a matroid having a stressed hyperplane H such that $\#H \geq \operatorname{rk}\mathsf{M}$, then the relaxed matroid $\widetilde{\mathsf{M}}$ is connected.*

Proof. Assume that $\operatorname{rk} \mathsf{M} = k$ and that $\#H = h$. By Proposition 4.1.3.2, we have that

$$\begin{aligned}\beta(\widetilde{\mathsf{M}}) &= [x^1y^0]T_{\widetilde{\mathsf{M}}}(x,y)\\ &= [x^1y^0]\left(T_{\mathsf{M}}(x,y) + (x+y-xy)\cdot\sum_{j=k}^{h}\binom{h}{j}(y-1)^{j-k}\right)\\ &= \beta(\mathsf{M}) + [y^0]\sum_{j=k}^{h}\binom{h}{j}(y-1)^{j-k}\\ &= \beta(\mathsf{M}) + \sum_{j=k}^{h}(-1)^{j-k}\binom{h}{j}\\ &= \beta(\mathsf{M}) + \binom{h-1}{k-1},\end{aligned}$$

where in the last step we used the identity $\binom{h-1}{k-1} = \sum_{j=k}^{h}(-1)^{j-k}\binom{h}{j}$ which can be proved by induction on k. In particular, since $\beta(\mathsf{M}) \geq 0$ and $\binom{h-1}{k-1} > 0$, it follows that $\beta(\widetilde{\mathsf{M}}) > 0$, which proves that $\widetilde{\mathsf{M}}$ is connected. □

An equivalent rewording of the preceding result is that every matroid having a free subset is connected.

Lastly, we show that the characteristic polynomial is also well-behaved with respect to the operation of relaxation.

Corollary 4.1.3.4. *Let M be a matroid of rank k having a stressed hyperplane of cardinality h. The characteristic polynomial of the relaxation $\widetilde{\mathsf{M}}$ is given by*

$$\chi_{\widetilde{\mathsf{M}}}(x) = \chi_{\mathsf{M}}(x) + (-1)^k(1-x)\binom{h-1}{k-1}.$$

Moreover,

$$\overline{\chi}_{\widetilde{\mathsf{M}}}(x) = \overline{\chi}_{\mathsf{M}}(x) - (-1)^k\binom{h-1}{k-1}.$$

Proof. Using Proposition 4.1.3.2, we have

$$\begin{aligned}\chi_{\widetilde{\mathsf{M}}}(x) &= (-1)^kT_{\widetilde{\mathsf{M}}}(1-x,0)\\ &= (-1)^k\left(T_{\mathsf{M}}(1-x,0) + (1-x)\cdot\sum_{j=k}^{h}\binom{h}{j}(-1)^{j-k}\right)\\ &= \chi_{\mathsf{M}}(x) + (-1)^k(1-x)\binom{h-1}{k-1},\end{aligned}$$

where in the last step we used again the identity $\sum_{j=k}^{h}(-1)^{j-k}\binom{h}{j} = \binom{h-1}{k-1}$. □

Now that we know that the relaxation of stressed hyperplanes has nice consequences for the Tutte polynomial, the characteristic polynomial and the lattice of flats, it is natural to ask if there are consequences on further invariants of matroids. In this section we will see that it is the case for the Kazhdan–Lusztig–Stanley framework. The following is the fundamental result.

Theorem 4.1.3.5. *For every pair of integers $k, h \geq 1$ there exist polynomials $p_{k,h}(x)$, $q_{k,h}(x)$ and $z_{k,h}(x)$ with integer coefficients, having the following property: for every matroid M of rank k having a stressed hyperplane of cardinality h,*

$$P_{\widetilde{\mathsf{M}}}(x) = P_{\mathsf{M}}(x) + p_{k,h}(x),$$
$$Q_{\widetilde{\mathsf{M}}}(x) = Q_{\mathsf{M}}(x) + q_{k,h}(x),$$
$$Z_{\widetilde{\mathsf{M}}}(x) = Z_{\mathsf{M}}(x) + z_{k,h}(x),$$

where $\widetilde{\mathsf{M}}$ denotes the corresponding relaxation of M.

Proof. Observe that the matroids M and $\widetilde{\mathsf{M}}$ always have the same rank. We proceed by induction on the rank of the matroids, k. For a matroid M of rank $k = 1$ and cardinality n, having a stressed hyperplane of cardinality h means it contains exactly $h \geq 1$ loops. This implies that $P_{\mathsf{M}}(x) = 0$. When we relax this stressed hyperplane, we obtain the matroid $\widetilde{\mathsf{M}} = \mathsf{U}_{1,n}$, hence $P_{\widetilde{\mathsf{M}}}(x) = 1$. This means that $p_{k,h}(x) = 1$.

Now, let us write down the defining relations for the Kazhdan–Lusztig polynomials of $\widetilde{\mathsf{M}}$ and M:

$$x^k P_{\widetilde{\mathsf{M}}}(x^{-1}) - P_{\widetilde{\mathsf{M}}}(x) = \sum_{\substack{F \in \mathcal{L}(\widetilde{\mathsf{M}}) \\ F \neq \emptyset}} \chi_{\widetilde{\mathsf{M}}_{|F}}(x) P_{\widetilde{\mathsf{M}}/F}(x)$$

and

$$x^k P_{\mathsf{M}}(x^{-1}) - P_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \emptyset}} \chi_{\mathsf{M}_{|F}}(x) P_{\mathsf{M}/F}(x).$$

Subtracting the right-hand-side of the second equation from the right-hand-side of the first, we get an expression consisting on four terms:

$$\underbrace{\sum_{\substack{F \subseteq H \\ \#F = k-1}} \chi_{\widetilde{\mathsf{M}}_{|F}}(x) P_{\widetilde{\mathsf{M}}/F}(x)}_{(1)} - \underbrace{\chi_{\mathsf{M}_{|H}}(x) P_{\mathsf{M}/H}(x)}_{(2)} + \underbrace{\chi_{\widetilde{\mathsf{M}}}(x) - \chi_{\mathsf{M}}(x)}_{(3)}$$
$$+ \underbrace{\sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \emptyset, H, E}} \left(\chi_{\widetilde{\mathsf{M}}_{|F}}(x) P_{\widetilde{\mathsf{M}}/F}(x) - \chi_{\mathsf{M}_{|F}}(x) P_{\mathsf{M}/F}(x) \right)}_{(4)}.$$

Let us show that each of these terms does not depend on M, and only depends on h and k. The items below correspond to the labeled terms above. In what follows, we will take advantage of the fact that as H is stressed, every subset of cardinality at most $k-1$ of H is independent.

1. F is independent in $\widetilde{\mathsf{M}}$ since $\#F = k-1$, and so $\mathcal{L}(\widetilde{\mathsf{M}}_{|F})$ is isomorphic to the Boolean matroid on $k-1$ elements. On the other hand, $\widetilde{\mathsf{M}}/F$ is a rank 1 matroid since F is independent on $\widetilde{\mathsf{M}}$ of cardinality $k-1$, so $P_{\widetilde{\mathsf{M}}/F}(x) = 1$.

2. Since $\operatorname{rk} H = k-1$, it follows that $\operatorname{rk}(\mathsf{M}/H) = 1$ and so $P_{\mathsf{M}/H}(x) = 1$. Also, $\mathsf{M}_{|H} \cong \mathsf{U}_{k-1,h}$.

3. By Corollary 4.1.3.4, $\chi_{\widetilde{\mathsf{M}}}(x) - \chi_{\mathsf{M}}(x) = (1-x)(-1)^k \binom{h-1}{k-1}$.

4. In this case, note that $\widetilde{\mathsf{M}}_{|F} = \mathsf{M}_{|F}$, since any flat of cardinality at most $k-1$ in M is already in $\widetilde{\mathsf{M}}$, by Proposition 4.1.1.4. If $F \nsubseteq H$, then $\widetilde{\mathsf{M}}/F = \mathsf{M}/F$, and so terms in this sum where $F \nsubseteq H$ vanish. Otherwise, if $F \subseteq H$, note that $\mathsf{M}_{|F}$ is the Boolean matroid on $\#F$ elements and $\widetilde{\mathsf{M}}/F$ is obtained via relaxing $H \setminus F$ in M/F. Hence, the terms where $F \subseteq H$ may be rewritten as $\chi_{\mathsf{B}_{\#F}}(x) \cdot (P_{\widetilde{\mathsf{N}}}(x) - P_{\mathsf{N}}(x))$, where $\mathsf{N} = \mathsf{M}/F$, and $\widetilde{\mathsf{N}}$ is the relaxation of $H \setminus F$ in M/F. Because $F \neq \emptyset$, we have $\operatorname{rk}\mathsf{N} < \operatorname{rk}\mathsf{M}$, and so by induction $P_{\widetilde{\mathsf{N}}}(x) - P_{\mathsf{N}}(x)$ is a polynomial only depending on h and k.

The proof for $Q_{\mathsf{M}}(x)$ is very similar. Let us write the defining recursion for M

$$(-x)^k Q_{\mathsf{M}}(x^{-1}) = \sum_{F \in \mathcal{L}(\mathsf{M})} (-1)^{\operatorname{rk} F} Q_{\mathsf{M}_{|F}}(x) x^{k-\operatorname{rk} F} \chi_{\mathsf{M}/F}(x^{-1}),$$

and the analogue for $\widetilde{\mathsf{M}}$. Subtracting the second equation from the first we get

$$\sum_{\substack{F \subseteq H \\ \#F = k-1}} (-1)^{k-1} x Q_{\widetilde{\mathsf{M}}_{|F}}(x) \chi_{\widetilde{\mathsf{M}}/F}(x^{-1}) - (-1)^{k-1} x Q_{\mathsf{M}_{|H}}(x) \chi_{\mathsf{M}/H}(x^{-1}) + x^k \left(\chi_{\widetilde{\mathsf{M}}}(x^{-1}) - \chi_{\mathsf{M}}(x^{-1})\right)$$
$$+ \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \emptyset, H, E}} \left((-1)^{\operatorname{rk} F} x^{k-\operatorname{rk} F} (Q_{\widetilde{\mathsf{M}}_{|F}}(x) \chi_{\widetilde{\mathsf{M}}/F}(x^{-1}) - Q_{\mathsf{M}_{|F}}(x) \chi_{\mathsf{M}/F}(x^{-1}))\right).$$

Similar observations to the ones made for $p_{k,h}(x)$ let us show the independence from M and n.

Finally, we address the Z-polynomial by writing

$$Z_{\mathsf{M}}(x) = \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F} P_{\mathsf{M}/F}(x)$$

and the analogue defining recursion for $\widetilde{\mathsf{M}}$. Subtracting the second equation from the first, on the right-hand-side we obtain

$$\sum_{\substack{F \subseteq H \\ \#F = k-1}} x^{k-1} P_{\widetilde{\mathsf{M}}/F}(x) - x^{k-1} P_{\mathsf{M}/H}(x) + \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq H}} x^{\operatorname{rk} F} \left(P_{\widetilde{\mathsf{M}}/F}(x) - P_{\mathsf{M}/F}(x)\right).$$

From here, with observations similar to the ones made before, we deduce the independence from M and n. □

Since now all three polynomials do not depend on the matroid we start from, as long as they satisfy the conditions of having rank k and a stressed hyperplane of cardinality h, we can take advantage of the example we explored in Proposition 4.1.1.5.

Corollary 4.1.3.6. *The polynomials $p_{k,h}(x)$, $q_{k,h}(x)$ and $z_{k,h}(x)$ in Theorem 4.1.3.5 are given by*

$$p_{k,h}(x) = P_{\mathsf{U}_{k,h+1}}(x) - P_{\mathsf{U}_{k-1,h}}(x),$$
$$q_{k,h}(x) = Q_{\mathsf{U}_{k,h+1}}(x) - Q_{\mathsf{U}_{k-1,h}}(x),$$
$$z_{k,h}(x) = Z_{\mathsf{U}_{k,h+1}}(x) - (1+x) Z_{\mathsf{U}_{k-1,h}}(x).$$

Proof. By Proposition 4.1.1.5, the matroid $\Pi_{k,h,n}$ has rank k, cardinality n and a stressed hyperplane of cardinality h. Also, the simplification of the relaxation $\Lambda_{k,h,n}$ is isomorphic to the uniform matroid $\mathsf{U}_{k,h+1}$. In particular, using this and Theorem 4.1.3.5, we obtain

$$\begin{aligned} P_{\mathsf{U}_{k,h+1}}(x) &= P_{\Lambda_{k,h,n}}(x) \\ &= P_{\Pi_{k,h,n}}(x) + p_{k,h}(x) \\ &= P_{\mathsf{U}_{k-1,h}}(x) \cdot P_{\mathsf{U}_{1,n-h}}(x) + p_{k,h}(x) \\ &= P_{\mathsf{U}_{k-1,h}}(x) + p_{k,h}(x), \end{aligned}$$

where we used that $P_{\mathsf{M}_1 \oplus \mathsf{M}_2}(x) = P_{\mathsf{M}_1}(x) \cdot P_{\mathsf{M}_2}(x)$ for all matroids. The proof for $q_{k,h}(x)$ is entirely analogous. For the Z-polynomial, we have to change slightly the last step, as $Z_{\mathsf{U}_{1,n-h}}(x)$ is equal to $x+1$ for every rank 1 uniform matroid. ☐

In Section 4.2 all these results will be specialized to sparse paving matroids, where $h = k$ and H is a circuit-hyperplane.

Remark 4.1.3.7. In Corollary 4.1.4.5, Corollary 4.1.4.8 and Proposition 4.1.4.11, we will give a combinatorial interpretation for $p_{k,h}(x)$, $q_{k,h}(x)$ and $z_{k,h}(x)$ by looking at some Young tableaux and skew Young tableaux. As a consequence of that, we will show that, for every $k \leq h$, the polynomials $p_{k,h}(x)$, $q_{k,h}(x)$ and $z_{k,h}(x)$ have non-negative coefficients and their degrees are, respectively, $\deg p_{k,h}(x) = \deg q_{k,h}(x) = \lfloor \frac{k-1}{2} \rfloor$ and $\deg z_{k,h}(x) = k-1$.

Since paving matroids are particularly well-behaved under the stressed hyperplane relaxation, as a consequence of Corollary 4.1.1.7, we obtain formulas for the Kazhdan–Lusztig polynomial, the inverse Kazhdan–Lusztig polynomial and the Z-polynomial of paving matroids. Specifically, the formulas depend only on the cardinality of the ground set, the rank and the number of hyperplanes of each cardinality it has. These formulas coincide with the ones from Theorem 4.1.3.1.

Theorem 4.1.3.8. *Let* M *be a paving matroid of rank* k *and cardinality* n*. Suppose* M *has exactly* λ_h *(stressed) hyperplanes of cardinality* h*. Then*

$$\begin{aligned} P_{\mathsf{M}}(x) &= P_{\mathsf{U}_{k,n}}(x) - \sum_{h \geq k} \lambda_h \cdot p_{k,h}(x), \\ Q_{\mathsf{M}}(x) &= Q_{\mathsf{U}_{k,n}}(x) - \sum_{h \geq k} \lambda_h \cdot q_{k,h}(x), \\ Z_{\mathsf{M}}(x) &= Z_{\mathsf{U}_{k,n}}(x) - \sum_{h \geq k} \lambda_h \cdot z_{k,h}(x). \end{aligned}$$

Proof. Since M is paving, according to Corollary 4.1.1.7, after relaxing all the hyperplanes of cardinality at least k, we obtain the uniform matroid $\mathsf{U}_{k,n}$. In particular,

$$P_{\mathsf{M}}(x) + \sum_{h \geq k} \lambda_h \cdot p_{k,h}(x) = P_{\mathsf{U}_{k,n}}(x),$$

from which the result follows for $P_{\mathsf{M}}(x)$. An entirely analogous proof shows the corresponding statement for $Q_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$. ☐

To see the formulas "explicitly", it is enough to remark once again that $P_{\mathsf{M}}(x)$, $Q_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ admit closed expressions for all uniform matroids, as shown in Theorem 3.1.2.1. As we saw above, $p_{k,h}(x)$, $q_{k,h}(x)$ and $z_{k,h}(x)$ can be obtained from them.

The preceding result supports Conjecture 3.1.0.3.

Corollary 4.1.3.9. *If* M *is a paving matroid of rank* k *and cardinality* n*, then* $P_{\mathsf{M}}(x)$*,* $Q_{\mathsf{M}}(x)$ *and* $Z_{\mathsf{M}}(x)$ *are coefficient-wise smaller than* $P_{\mathsf{U}_{k,n}}(x)$*,* $Q_{\mathsf{U}_{k,n}}(x)$ *and* $Z_{\mathsf{U}_{k,n}}(x)$*, respectively.*

Proof. This is now a direct consequence of Remark 4.1.3.7 and Theorem 4.1.3.8. □

4.1.4 Skew and Standard Young Tableaux

4.1.4.1 The main tableaux

In [FNV22, Section 6] we define tableaux-inspired objects, following the pace of [LNR20, Section 2], and the notation of [LNR21]. They will be used to provide combinatorial interpretations of the coefficients of some of the polynomials that have appeared so far. First, consider the Young diagram depicted in Figure 4.1.

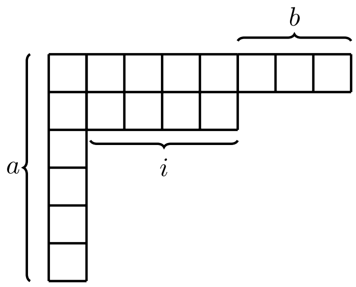

Figure 4.1: The "Syt" shape.

Let $\operatorname{Syt}(a,i,b)$ be the set of standard Young tableaux of the above shape. Notice that the total number of boxes is $a+2i+b$. In other words, in each diagram, we are placing the numbers in $\{1,2,\dots,a+2i+b\}$ into the above diagram so that the rows and the columns strictly increase from left to right and from top to bottom, respectively. We let $\operatorname{syt}(a,i,b) := \#\operatorname{Syt}(a,i,b)$, that is, $\operatorname{syt}(a,i,b)$ is the number of Young tableaux with the above shape. We also define $\overline{\operatorname{Syt}}(a,i,b)$, a special subset of $\operatorname{Syt}(a,i,b)$ where the maximum entry is either at the bottom of the first or $(i+1)$-th column. Let $\overline{\operatorname{syt}}(a,i,b) := \#\overline{\operatorname{Syt}}(a,i,b)$.

Now, we turn our attention to a different (but related) object. Consider the skew Young diagram in Figure 4.2.

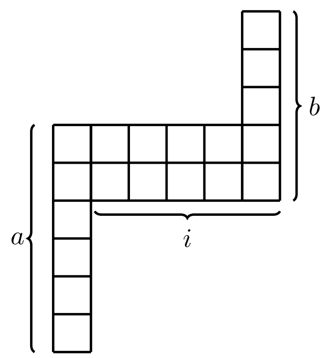

Figure 4.2: The "Skyt" shape.

Observe that the total number of boxes is exactly $a+2i+b-2$. We define a *legal filling* of the above shape as a filling of the boxes using all the integers from $\{1,2,\dots,a+2i+b-2\}$

in such a way that the values in the rows (respectively columns) strictly increase from left to right (respectively, from top to bottom). Note that this is the same restriction on the entries as mentioned above. We denote by $\mathrm{Skyt}(a,i,b)$ the set of all such legal fillings, and we denote $\mathrm{skyt}(a,i,b) := \#\,\mathrm{Skyt}(a,i,b)$. That is, $\mathrm{Skyt}(a,i,b)$ is the collection of fillings for the above skew Young diagram, and $\mathrm{skyt}(a,i,b)$ is the number of these tableaux.

For our skew tableaux to be well-defined, we require $a, b \geq 2$ and $i \geq 1$. To avoid undefined scenarios, we use the following conventions:

- If $i = 0$, then $\mathrm{skyt}(a,i,b) = 1$.
- If $i > 0$ and at least one of a or b is less than 2, then $\mathrm{skyt}(a,i,b) = 0$.

In analogy with what we did for the first shape we introduced, we now consider a subclass of the preceding skew Young tableaux, which we will denote $\overline{\mathrm{Skyt}}(a,i,b)$. This set is the subset of $\mathrm{Skyt}(a,i,b)$ so that 1 is always the entry at the top of the left-most column. The size of $\overline{\mathrm{Skyt}}(a,i,b)$ is denoted $\overline{\mathrm{skyt}}(a,i,b)$. By convention, $\overline{\mathrm{skyt}}(a,i,b) = 0$ if $i = 0$.

4.1.4.2 Enumeration of tableaux and identities

We now give two identities that will be used later to give combinatorial interpretations for the polynomials $p_{k,h}(x)$, $q_{k,h}(x)$ and $z_{k,h}(x)$. First, we have a lemma relating the fillings of the two diagrams mentioned above.

Lemma 4.1.4.1. *We have*

$$\mathrm{syt}(a,i,b-2i-1) = \sum_{j=0}^{b} (-1)^{j+1} \binom{a+b-1}{b-j} \mathrm{skyt}(a,i,j-2i+1).$$

Proof. This result follows from [LNR20, Lemma 21]. □

Now we provide two different results giving formulas for $\overline{\mathrm{skyt}}(a,i,b)$ and $\overline{\mathrm{syt}}(a,i,b)$ in terms of $\mathrm{skyt}(a,i,b)$ and $\mathrm{syt}(a,i,b)$, respectively.

Proposition 4.1.4.2.

$$\overline{\mathrm{skyt}}(a,i,b) = \mathrm{skyt}(a,i,b) - \mathrm{skyt}(a,i,b-1).$$

Proof. Note that for every skew Young tableau in $\mathrm{Skyt}(a,i,b)$, the number 1 is either

(Case 1) at the top of the left-most column, or

(Case 2) at the top of the right-most column.

In Case 1, these are exactly the members of $\overline{\mathrm{Skyt}}(a,i,b)$. In Case 2, observe that these are in bijection with the members of $\mathrm{Skyt}(a,i,b-1)$. Given a tableaux $\lambda \in \mathrm{Skyt}(a,i,b-1)$, we construct a tableaux $\widetilde{\lambda} \in \mathrm{Skyt}(a,i,b)$ satisfying Case 2 above. First, add 1 to each value in λ. Then, add a cell to the top of the right-most column and place the number 1 there. This gives the desired $\widetilde{\lambda}$, and hence, we have shown the desired result. □

Proposition 4.1.4.3.

$$\overline{\mathrm{syt}}(a,i,b) = \mathrm{syt}(a,i,b) - \mathrm{syt}(a,i,b-1).$$

Proof. Note that for every Young tableaux in $\mathrm{Syt}(a,i,b)$, the largest number, $a+2i+b$, is either

(Case 1) at the bottom of the first column,

(Case 2) at the bottom of the $(i+1)$-th column, or

(Case 3) in the right-most cell of the first row.

Note that Cases 1 and 2 make up the members of $\overline{\mathrm{Syt}}(a,i,b)$. In Case 3, observe that these are in bijection with the members of $\mathrm{Syt}(a,i,b-1)$. Given a tableaux $\lambda \in \mathrm{Syt}(a,i,b-1)$, we construct a tableaux $\widetilde{\lambda} \in \mathrm{Syt}(a,i,b)$ satisfying Case 3. Add a cell at the right end of the first row in λ, and place $a+2i+b$ in this cell. This gives the desired $\widetilde{\lambda}$, and hence, we have shown the desired result. □

4.1.4.3 Interpreting the Kazhdan–Lusztig coefficients

One of the main results of [LNR20], is the following.

Theorem 4.1.4.4. *[LNR20, Theorem 2] The Kazhdan–Lusztig polynomial for $\mathsf{U}_{k,n}$ is*

$$P_{\mathsf{U}_{k,n}}(x) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \mathrm{skyt}(n-k+1, i, k-2i+1)\, x^i.$$

In other words, the coefficients of the Kazhdan–Lusztig polynomial of all uniform matroids can be interpreted using the skew tableaux we introduced above. As a consequence of this statement, we obtain the following combinatorial description of the polynomial $p_{k,h}(x)$ appearing in Theorem 4.1.3.5.

Corollary 4.1.4.5. *For every $k, h \geq 1$, we have*

$$p_{k,h}(x) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \overline{\mathrm{skyt}}(h-k+2, i, k-2i+1)\, x^i.$$

In particular, $p_{k,h}(x)$ is a polynomial with non-negative coefficients of degree $\lfloor \frac{k-1}{2} \rfloor$.

Proof. Observe that

$$\begin{aligned} p_{k,h}(x) &= P_{\mathsf{U}_{k,h+1}}(x) - P_{\mathsf{U}_{k-1,h}}(x) \\ &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \mathrm{skyt}(h-k+2, i, k-2i+1)\, x^i - \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \mathrm{skyt}(h-k+2, i, k-2i)\, x^i \end{aligned}$$

where the first equality uses Corollary 4.1.3.6, and the second uses Theorem 4.1.4.4.

Now, we claim we can change the bounds of the two summations to make them match. When k is even, note that $i < \frac{k-1}{2}$ if and only if $i < \frac{k}{2}$. When k is odd, note that substituting $i = \frac{k-1}{2}$ into $\mathrm{skyt}(h-k+2, i, k-2i)$ gives $\mathrm{skyt}(h-k+2, \frac{k-1}{2}, 1) = 0$.

Hence, regardless of k we have

$$\begin{aligned}
&\sum_{i=0}^{\frac{k-1}{2}} \mathrm{skyt}(h-k+2,i,k-2i+1)\,x^i - \sum_{i=0}^{\frac{k-2}{2}} \mathrm{skyt}(h-k+2,i,k-2i)\,x^i \\
&= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \mathrm{skyt}(h-k+2,i,k-2i+1)\,x^i - \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \mathrm{skyt}(h-k+2,i,k-2i)\,x^i \\
&= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \overline{\mathrm{skyt}}(h-k+2,i,k-2i+1)\,x^i,
\end{aligned}$$

by Proposition 4.1.4.2. □

Remark 4.1.4.6. It is worth pointing out something subtle that occurs with the last equality in the prior proof in the case where k is odd and $i = \frac{k-1}{2}$. Recall that for this i we have $\mathrm{skyt}(h-k+2,i,k-2i) = 0$. However, note that in this case $\mathrm{skyt}(h-k+2,i,k-2i+1)$ equals $\mathrm{skyt}(h-k+2,\frac{k-1}{2},2)$. Observe that for tableaux in $\mathrm{Skyt}(h-k+2,\frac{k-1}{2},2)$, the only possible location for the value 1 is at the top of the left-most column, since the top entry of the last column is the last entry of the first row. So $\mathrm{Skyt}(h-k+2,\frac{k-1}{2},2) = \overline{\mathrm{Skyt}}(h-k+2,\frac{k-1}{2},2)$, and hence $\mathrm{skyt}(h-k+2,\frac{k-1}{2},2) = \overline{\mathrm{skyt}}(h-k+2,\frac{k-1}{2},2)$.

Now, let us turn our attention to the inverse Kazhdan–Lusztig polynomial. We are able to get nice formulas for this polynomial as well. The first step is to state an interpretation for the coefficients of $Q_{\mathsf{U}_{k,n}}(x)$.

Theorem 4.1.4.7. *The inverse Kazhdan–Lusztig polynomial of the uniform matroid* $\mathsf{U}_{k,n}$ *is*

$$Q_{\mathsf{U}_{k,n}}(x) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \mathrm{syt}(n-k+1,i,k-2i-1)\,x^i.$$

Proof. Firstly, we use [GX21, Theorem 1.3] to write

$$Q_{\mathsf{U}_{k,n}}(x) = -\sum_{F \neq E} (-1)^{\mathrm{rk}\,\mathsf{M} - \mathrm{rk}\,F} Q_{(\mathsf{U}_{k,n})_{|F}}(x) P_{(\mathsf{U}_{k,n})/F}(x).$$

Since F can never be the ground set of $\mathsf{U}_{k,n}$, this means that $(\mathsf{U}_{k,n})_{|F}$ is a boolean matroid for any F. Thus, combining similar terms, we have

$$Q_{\mathsf{U}_{k,n}}(x) = \sum_{j=1}^{k} (-1)^{j+1} \binom{n}{k-j} P_{\mathsf{U}_{j,n-k+j}}(x),$$

where j ranges over flats so that $j = \mathrm{rk}\,\mathsf{M} - \mathrm{rk}\,F$, that is, the flats of rank $k-j$.

Looking now at the coefficient $[x^i]Q_{\mathsf{U}_{k,n}}(x)$, using Theorem 4.1.4.4 we obtain that

$$[x^i]Q_{\mathsf{U}_{k,n}}(x) = \sum_{j=0}^{k} (-1)^{j+1} \binom{n}{k-j} \mathrm{skyt}(n-k+1,i,j-2i+1).$$

Note we may allow the index j to start at 0 since in this case $\mathrm{skyt}(n-k+1,i,j-2i+1) = 0$. By Lemma 4.1.4.1 with $a = n-k+1$ and $b = k$, we get

$$[x^i]Q_{\mathsf{U}_{k,n}}(x) = \mathrm{syt}(n-k+1,i,k-2i-1),$$

and the result follows. □

We point out that a different proof of the preceding result can be given, along the lines of [GXY22, Theorem 3.2]. On the other hand, in analogy with what we did for $p_{k,h}(x)$, we obtain an interpretation for $q_{k,h}(x)$.

Corollary 4.1.4.8. *For every $k, h \geq 1$,*

$$q_{k,h}(x) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \overline{\text{syt}}(h-k+2, i, k-2i-1) x^i.$$

In particular, $q_{k,h}(x)$ is a polynomial with non-negative coefficients of degree $\lfloor \frac{k-1}{2} \rfloor$.

Proof. The proof is equivalent to that of Corollary 4.1.4.5 by using Corollary 4.1.3.6 and Proposition 4.1.4.3. □

One can use the skew tableaux also to get a combinatorial formula for the Z-polynomial.

Corollary 4.1.4.9.

$$Z_{\mathsf{U}_{k,n}}(x) = x^k + \sum_{j=0}^{k-1} \sum_{i=0}^{\lfloor \frac{k-j}{2} \rfloor} \binom{n}{j} \operatorname{skyt}(n-k+1, i, k-j-2i+1)\, x^{i+j}.$$

Proof. Recall that by definition we have

$$Z_{\mathsf{M}}(x) = \sum_{F \in \mathcal{L}(\mathsf{M})} x^{\operatorname{rk} F} P_{\mathsf{M}/F}(x).$$

Also recall that if $\mathsf{M} = \mathsf{U}_{k,n}$, the flats of rank r for $r \leq k-1$ are the subsets of cardinality r. For this M, we also have that $\mathsf{M}/F \cong \mathsf{U}_{k-\#F, n-\#F}$ for every flat F. Hence, using Theorem 4.1.4.4, we have

$$\begin{aligned} Z_{\mathsf{U}_{k,n}}(x) &= x^k + \sum_{j=0}^{k-1} \binom{n}{j} x^j P_{\mathsf{U}_{k-j,n-j}}(x) \\ &= x^k + \sum_{j=0}^{k-1} \binom{n}{j} x^j \sum_{i=0}^{\lfloor \frac{k-j}{2} \rfloor} \operatorname{skyt}(n-k+1, i, k-j-2i+1)\, x^i \\ &= x^k + \sum_{j=0}^{k-1} \sum_{i=0}^{\lfloor \frac{k-j}{2} \rfloor} \binom{n}{j} \operatorname{skyt}(n-k+1, i, k-j-2i+1)\, x^{i+j}. \end{aligned}$$ □

Remark 4.1.4.10. As with $Q_{\mathsf{U}_{k,n}}(x)$ and $P_{\mathsf{U}_{k,n}}(x)$, it is desirable to find an interpretation for the coefficients of $Z_{\mathsf{U}_{k,n}}(x)$ that corresponds to the number of Young tableaux of some shape. Unfortunately, we have not been able to find such an interpretation. However, we can provide one way of understanding the coefficients as counting a collection of skew tableaux with varying diagram shapes. Observe that if $i < k$, then

$$\begin{aligned} [x^i] Z_{\mathsf{U}_{k,n}}(x) &= \sum_{j=0}^{k-1} \binom{n}{j} [x^{i-j}] P_{\mathsf{U}_{k-j,n-j}}(x) \\ &= \sum_{j=0}^{k-1} \binom{n}{n-j} \operatorname{skyt}(n-k+1, i-j, k-2i+j+1). \end{aligned}$$

Note that $\mathrm{skyt}(n-k+1, i-j, k-2i+j+1)$ has $n-j$ entries. Hence, one can interpret the term $\binom{n}{n-j}\mathrm{skyt}(n-k+1, i-j, k-2i+j+1)$ as counting the number of ways of filling skew Young diagrams of the following shape with entries from $\{1,\dots,n\}$ so that rows increase from left to right and columns increase from top to bottom.

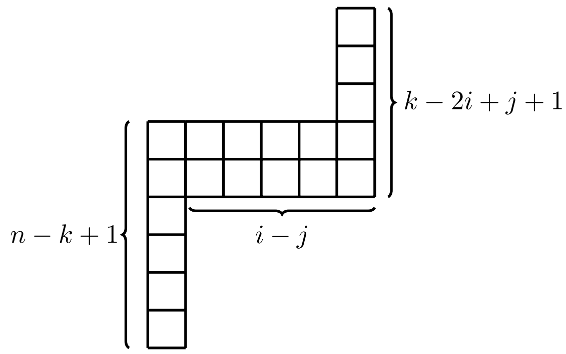

Figure 4.3: One of the diagrams related to the coefficients of $Z_{\mathsf{U}_{k,n}}(x)$.

Hence, the i-th coefficient of $Z_{\mathsf{U}_{k,n}}(x)$ counts the number of such fillings for all diagrams as above, varying in all possible values of j. This is what makes finding a single object that this coefficient counts challenging—this coefficient counts fillings for diagrams of different sizes.

Proposition 4.1.4.11. *For every $k, h \geq 1$,*

$$z_{k,h}(x) = \left[\binom{h}{k-1} - 1\right] x^{k-1} + \sum_{j=0}^{k-2}\sum_{i=1}^{\lfloor\frac{k-j}{2}\rfloor}\binom{h}{j}\overline{\mathrm{skyt}}(h-k+2, i, k-j-2i+1)\, x^{i+j}.$$

This implies that $z_{k,h}(x)$ is a polynomial with non-negative coefficients of degree $k-1$.

Proof. Let us write

$$z_{k,h}(x) = Z_{\mathsf{U}_{k,h+1}}(x) - (1+x)Z_{\mathsf{U}_{k-1,h}}(x).$$

We use the theorem above to make the three terms more explicit.

$$Z_{\mathsf{U}_{k,h+1}}(x) = x^k + \sum_{j=0}^{k-1}\sum_{i=1}^{\lfloor\frac{k-j}{2}\rfloor}\binom{h+1}{j}\mathrm{skyt}(h-k+2, i, k-2i+1)\, x^{i+j}$$

$$x\, Z_{\mathsf{U}_{k-1,h}}(x) = x^k + \sum_{j=0}^{k-2}\sum_{i=1}^{\lfloor\frac{k-j}{2}\rfloor}\binom{h}{j}\mathrm{skyt}(h-k+2, i, k-j-2i)\, x^{i+j+1}$$

$$= x^k + \sum_{j=1}^{k-1}\sum_{i=1}^{\lfloor\frac{k-j}{2}\rfloor}\binom{h}{j-1}\mathrm{skyt}(h-k+2, i, k-j-2i+1)\, x^{i+j}$$

$$Z_{\mathsf{U}_{k-1,h}}(x) = x^{k-1} + \sum_{j=0}^{k-2}\sum_{i=1}^{\lfloor\frac{k-j-1}{2}\rfloor}\binom{h}{j}\mathrm{skyt}(h-k+2, i, k-j-2i)\, x^{i+j}.$$

We proceed by subtracting the first two quantities. The degree-k terms cancel out and we separate from the first sum the terms for $j = 0$ (which do not have a corresponding term in the second sum). After using the known combinatorial fact that $\binom{h+1}{j} - \binom{h}{j-1} = \binom{h}{j}$, this leaves us with

$$\begin{aligned}
Z_{\mathsf{U}_{k,h+1}}(x) - x\, Z_{\mathsf{U}_{k-1,h}}(x) =& \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \operatorname{skyt}(h-k+2, i, k-2i+1)\, x^i \\
&+ \sum_{j=1}^{k-1} \sum_{i=1}^{\lfloor \frac{k-j}{2} \rfloor} \binom{h}{j} \operatorname{skyt}(h-k+2, i, k-j-2i+1)\, x^{i+j} \\
=& \sum_{j=0}^{k-1} \sum_{i=1}^{\lfloor \frac{k-j}{2} \rfloor} \binom{h}{j} \operatorname{skyt}(h-k+2, i, k-j-2i+1)\, x^{i+j}.
\end{aligned}$$

Now we want to subtract from what we obtained the quantity $Z_{\mathsf{U}_{k-1,h}}(x)$. This gives us

$$\begin{aligned}
z_{k,h}(x) =& \sum_{j=0}^{k-1} \sum_{i=1}^{\lfloor \frac{k-j}{2} \rfloor} \binom{h}{j} \operatorname{skyt}(h-k+2, i, k-j-2i+1)\, x^{i+j} \;-\; x^{k-1} \\
&- \sum_{j=0}^{k-2} \sum_{i=1}^{\lfloor \frac{k-j-1}{2} \rfloor} \binom{h}{j} \operatorname{skyt}(h-k+2, i, k-j-2i)\, x^{i+j} \\
=& \binom{h}{k-1} \operatorname{skyt}(h-k+2, 0, 2) x^{k-1} - x^{k-1} \\
&+ \sum_{j=0}^{k-2} \sum_{i=1}^{\lfloor \frac{k-j}{2} \rfloor} \binom{h}{j} \overline{\operatorname{skyt}}(h-k+2, i, k-j-2i+1)\, x^{i+j},
\end{aligned}$$

which gives us the desired result. □

4.1.5 Non-degeneracy

Lastly, we can answer some questions on the degrees of the Kazhdan–Lusztig–Stanley polynomials, and thus on the non-degeneracy of a matroid M. As a consequence of Theorem 4.1.3.5 and Corollary 4.1.4.5, we obtain the following result.

Corollary 4.1.5.1. *If a matroid* M *has a free subset, then it is non-degenerate.*

Proof. By Proposition 4.1.0.6, we know that a matroid M of rank k having a free subset of cardinality h is obtained after relaxing a stressed hyperplane of cardinality h in another matroid N of rank k. We know that the coefficients of $P_{\mathsf{N}}(x)$ are non-negative by Theorem 2.4.3.1. Also, by Remark 4.1.3.7, the polynomial $p_{k,h}(x)$ has degree $\lfloor \frac{k-1}{2} \rfloor$, hence the degree of $P_{\mathsf{M}}(x)$ has to be $\lfloor \frac{k-1}{2} \rfloor$. □

Remark 4.1.5.2. We speculate that almost all matroids have a free subset. This conjecture is weaker than Conjecture 1.1.2.5. Such belief is supported also by [PvdP15, Section 4.1]. It is also interesting to understand how many regular matroids have a free subset, that is how many cases of Conjecture 3.1.0.4 are covered by our results. This will be further investigated in Section 4.2.3.

4.2 Sparse paving matroids

In this section, we specialize our results from Section 4.1 to sparse paving matroids, which were studied in [FV22]. Chronologically, the process happened in reverse, as the notion of stressed hyperplane relaxation was introduced in a second time to generalize to paving matroids the well-known operation of circuit-hyperplanes, which lets us deal with sparse paving matroids.

Remark 4.2.0.1. For $k = h$, the matroids from Proposition 4.1.1.5 become

$$\Pi_{k,k,n} = \mathsf{U}_{k-1,k} \oplus \mathsf{U}_{1,n-k}$$

and

$$\Lambda_{k,k,n} =: \mathsf{T}_{k,n}.$$

The latter is known as the *minimal matroid* which was studied, for example, in [Fer22b]. The minimal matroid $\mathsf{T}_{k,n}$ is graphic, and is obtained from a cycle graph with k edges by adding $n-k$ parallel edges to one of its edges.

One possible reason to approach this class and not the one of paving matroids comes from the fact that, in a sparse paving matroid, the family of stressed hyperplanes and the one of circuit-hyperplanes coincide. In particular, we are able to leverage some well-known upper bounds for the maximum number of circuit-hyperplanes.

Lemma 4.2.0.2 ([MNRInVF12, Theorem 4.8])**.** *Let M be a sparse paving matroid of rank k having n elements. Then, the number of circuit-hyperplanes λ of M satisfies:*

$$\lambda \leq \binom{n}{k} \min\left\{\frac{1}{k+1}, \frac{1}{n-k+1}\right\}.$$

Remark 4.2.0.3. This bound was used to efficiently compute our families of polynomials as mentioned in Proposition 3.1.0.2. When M is sparse paving, their coefficients only depend on the number of circuit-hyperplanes, hence can be computed without using the recursive formula. To achieve this, we used Theorem 4.1.3.1 (in particular, observe that the only non-zero contribution from the sum is given by $h = k$) and the known closed formulas for uniform matroids.

Remark 4.2.0.4. There exist tighter bounds for the number of circuit-hyperplanes of a sparse paving matroid of rank k and cardinality n for some particular values of k and n. In fact, this quantity coincides with the independence number of the Johnson graph $J(n,k)$, and with the maximum number of words that a binary code with word-length n and constant weight k can have, under the constraint of minimal distance 4. Also, Lemma 4.2.0.2 is a weaker version of what in the coding theory literature is called the "Johnson Bound" (see [Joh62]). The exact computation of this maximum is a difficult problem, and precise values are in fact known only for few particular cases.

We will write $\lambda_{k,n}$ to denote the expression $\binom{n}{k} \min\left\{\frac{1}{k+1}, \frac{1}{n-k+1}\right\}$.

4.2.1 Non-negativity

Kazhdan–Lusztig–Stanley polynomials are already known to be non-negative, thanks to their geometric interpretation as Hilbert–Poincaré series. However, it is still interesting to see if this property can be recovered "combinatorially" (see Remark 3.1.1.1).

Theorem 4.2.1.1. *If M is a sparse paving matroid then $P_{\mathsf{M}}(x)$, $Q_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ have non-negative coefficients.*

Proof. We assume that M has rank k, cardinality n, and exactly λ circuit-hyperplanes. For the non-negativity of $P_{\mathsf{M}}(x)$ we refer to [LNR21, Theorem 1], although it is possible to give an alternative proof using our formula. By using the results for uniform matroids listed in Theorem 3.1.2.1 and Theorem 4.1.3.8 we can write

$$\begin{aligned} Z_{\mathsf{M}}(x) &= Z_{\mathsf{U}_{k,n}}(x) - \lambda z_{k,k}(x) \\ &= x^k + \left(\binom{n}{k-1} - \lambda(k-1)\right) x^{k-1} \\ &+ \sum_{j=2}^{k} \left(\binom{n}{k-j} P_{\mathsf{U}_{j,n-k+j}}(x) - \lambda\binom{k}{j} p_{j,j}(x)\right) x^{k-j}. \end{aligned}$$

We will prove that each summand in the last expression is a polynomial with non-negative coefficients. Observe that the second summand has a non-negative coefficient, since $\lambda \leq \lambda_{k,n}$ by Lemma 4.2.0.2, and:

$$\lambda_{k,n}(k-1) \leq \frac{1}{n-k+1}\binom{n}{k}(k-1) = \frac{k-1}{k}\binom{n}{k-1} \leq \binom{n}{k-1}.$$

Now, if we use that $P_{\mathsf{U}_{j,n-k+j}}(x) - \lambda_{j,n-k+j} p_{j,j}(x)$ has positive coefficients (which is the first statement in this theorem), it just suffices to verify the following inequality:

$$\binom{n}{k-j}\lambda_{j,n-k+j} \geq \lambda_{k,n}\binom{k}{j},$$

which is just:

$$\binom{n}{k-j}\binom{n-k+j}{j} \min\left\{\tfrac{1}{j+1}, \tfrac{1}{n-k+1}\right\} \geq \binom{n}{k}\binom{k}{j} \min\left\{\tfrac{1}{k+1}, \tfrac{1}{n-k+1}\right\}.$$

Since it is easy to verify the identity $\binom{n}{k-j}\binom{n-k+j}{j} = \binom{n}{k}\binom{k}{j}$, it suffices to show only that:

$$\min\left\{\tfrac{1}{j+1}, \tfrac{1}{n-k+1}\right\} \geq \min\left\{\tfrac{1}{k+1}, \tfrac{1}{n-k+1}\right\},$$

which holds trivially since $j \leq k$.

The proof for $Q_{\mathsf{M}}(x)$ is a very cumbersome (and uninteresting) computation that we therefore omit. □

4.2.2 Free bases and regularity

Conjecture 3.1.0.4 asserts that connected matroids that are regular are non-degenerate. Although there is good evidence that almost all matroids are expected to possess a free basis (see the discussion in [BPvdP15, Section 7.2]), a natural question that may arise at this point is which of these are regular and connected.

Since almost all matroids are non-representable (see Theorem 1.1.4.6), in particular almost all matroids are non-regular. However, although the family of matroids with a free basis is expected to be asymptotically predominant, the family of regular matroids with a free basis is almost negligible among the whole family of regular matroids.

Proposition 4.2.2.1. *Let* M *be a regular matroid with a free basis. Then* M *is graphic, and is obtained from a cycle graph with at least two edges by repeatedly adding a possibly empty set of parallel edges to one of the edges of the cycle.*

Proof. Since M is regular, in particular M is binary. Let us call B the basis of M such that $B \cup \{e\}$ is a circuit for every $e \notin B$. If $E \setminus B$ consists of only one element, then the conclusion follows. Suppose then that we can pick two distinct elements y and z not in B. Since M is binary, by [Oxl11, Theorem 9.1.2] we have that the circuits $C_1 = B \cup \{y\}$ and $C_2 = B \cup \{z\}$ are such that the symmetric difference $C_1 \triangle C_2 = \{y, z\}$ is a disjoint union of circuits. Since both $B \cup \{y\}$ and $B \cup \{z\}$ are circuits, it cannot happen that either $\{y\}$ nor $\{z\}$ are circuits. The only possibility is that $\{y, z\}$ is itself a circuit. From this, it follows that the elements of $E \setminus B$ are parallel to each other, and the proof is complete. □

Remark 4.2.2.2. In other words, the only matroids that are regular and contain a free basis are the matroids $\mathsf{T}_{k,n}$, where $1 \leq k \leq n-1$, obtained by the circuit-hyperplane relaxation of $\mathsf{U}_{k-1,k} \oplus \mathsf{U}_{1,n-k}$. Observe also that in Proposition 4.2.2.1 we can change the word "regular" for "binary" and the conclusion still holds.

One might ask how many of the cases from Conjecture 3.1.0.4 are covered by Corollary 4.1.5.1, i.e. by replacing free bases with free subsets. What causes the class of regular matroids with a free basis to be small is that regular matroids are binary, and hence the family of circuits must satisfy properties that are too restrictive (see [Oxl11, Theorem 9.1.2]). Unfortunately, even if the relaxation of stressed hyperplanes is more general than the circuit-hyperplane relaxation, it still does not behave well with the property of being regular (in particular, binary). To be precise, one has the following result.

Proposition 4.2.2.3. *Let $\mathsf{M} = (E, \mathscr{B})$ be a regular matroid having a free subset. Then M is graphic, and is obtained from a cycle graph with at least two edges by repeatedly adding a possibly empty set of parallel edges to one of the edges of the cycle, i.e. $\mathsf{M} \cong \Lambda_{k,k,n}$ for some k and n.*

Proof. Assume that $\#E = n$ and that A is a free subset of cardinality h. Notice that the matroid $\mathsf{M}_{|A}$ is isomorphic to $\mathsf{U}_{k,h}$. Also, M is connected, according to Corollary 4.1.3.3. By [Oxl11, Theorem 10.1.1], as M is assumed to be regular, $\mathsf{U}_{2,4}$ cannot be a minor of M. In particular $\mathsf{U}_{2,4}$ cannot be a minor of $\mathsf{U}_{k,h}$. Hence, we must have $k \in \{0, 1, h-1, h\}$.

- If $k = 0$, then $\mathscr{B} = \{\emptyset\}$. It is impossible for a matroid of rank 0 to contain a free subset, so we discard this case.
- If $k = 1$, as M is connected (and hence does not contain loops), we automatically have that all the subsets of cardinality 1 of E are independent, and that every pair of them is parallel. In other words, we just have $\mathsf{M} \cong \mathsf{U}_{1,n}$, and $n \geq 2$ as B_1 does not have free subsets. Such a matroid is as described in the statement.
- If $k = h - 1$, let us call $\mathscr{B}' = \binom{A}{k}$. We have that A itself is a circuit, as the removal of any of its elements yields an independent set (a basis, actually). We claim that all the bases $B \in \mathscr{B}'$ are free bases. Let us pick any such B, and call x the only element such that $B \cup \{x\} = A$. Observe that for every element not in B we have that it is either x or it lies in the complement of A. In the first case, we already know that $B \cup \{x\} = A$ is a circuit, whereas in the second, as A is a free subset, we have that adding any element not in A to B gives a circuit. In particular, we have that M has free bases and the result follows from Proposition 4.2.2.1.
- If $k = h$, then A is a free basis, and the conclusion follows again by Proposition 4.2.2.1.

□

Remark 4.2.2.4. The previous result tells us that the class of regular matroids with a free subset is very small. To be more explicit, what the preceding proposition says is that this class coincides with the class of regular matroids having a free basis.

4.2.3 Modularity and non-degeneracy

In light of Corollary 4.1.5.1 it is reasonable to expect now that degenerate matroids are a very restrictive class of matroids.

So far, computational experiments and partial results have yielded some examples of degenerate matroids, but up to this point they all seem to share one particular property: in some sense they are very close to being modular.

In a preliminary version of [FV22], we left the following question.

Question 4.2.3.1 (Settled by N. Proudfoot)**.** Is the following assertion true?

$$\mathsf{M} \text{ is connected, simple and degenerate} \iff \mathsf{M} \text{ is a projective geometry of rank } k \geq 3$$

It is possible to prove that the implication $\Leftarrow$ is true by noticing that projective geometries are modular, and Elias, Proudfoot and Wakefield proved that modular matroids are degenerate [EPW16, Proposition 2.14]. However, the implication $\Rightarrow$ is not true, as can be shown by the following example by Nicholas Proudfoot.

Example 4.2.3.2. Let M be the projective geometry representable over the field $\mathbb{F}_2$ of rank 5. Since M is modular, we know that $P_{\mathsf{M}}(x) = 1$, see [EPW16]. According to `Sage` [SD20], we have that $P_{\mathsf{M}\setminus e}(x) = x + 1$ for any element $e \in E(\mathsf{M})$. Also, $\mathsf{M} \setminus e$ is a connected, simple matroid with 30 elements and rank 5 that is not a projective geometry (it is not modular) but is still degenerate.

Chapter 5

Beyond polynomial invariants

Once we have established nice properties of our polynomial invariants, a natural thing to do is trying to generalize them. In this chapter we study two natural developments of our theory. Firstly, if we work with a polynomial with non-negative integer coefficients, we can always try to replace it with a graded vector space whose graded dimensions match the coefficients of the polynomial. This process is called *abelian categorification* and in some senses is the exact inverse of what we did when we took Hilbert–Poincaré series of graded vector spaces. A reason for doing so is that some results on polynomials may be seen as a shadow of a result on vector spaces. For example, one can regard Theorem 1.1.5.5 as a consequence of Theorem 2.2.1.5. In fact, the deletion-contraction formula

$$[x^i]\chi_{\mathsf{M}}(x) = [x^i]\chi_{\mathsf{M}\setminus e}(x) - [x^i]\chi_{\mathsf{M}/e}(x),$$

follows directly from taking the Euler characteristic of this short exact sequence

$$0 \to \mathrm{OS}^i(\mathsf{M} \setminus e) \to \mathrm{OS}^i(\mathsf{M}) \to \mathrm{OS}^{i-1}(\mathsf{M}/e) \to 0,$$

together with the fact that

$$\dim \mathrm{OS}^i(\mathsf{M}) = (-1)^{\operatorname{rk}\mathsf{M}-i}[x^{\operatorname{rk}\mathsf{M}-i}]\chi_{\mathsf{M}}(x).$$

In the same sense, one can consider Theorem 2.2.1.4 as a categorification of the definition of $\chi_{\mathsf{M}}(x)$ given in Theorem 1.2.4.1 In particular, since

$$\dim \mathrm{OS}^{\operatorname{rk}\mathsf{M}}(\mathsf{M}) = \mu(\emptyset, E),$$

we know that

$$\dim \mathrm{OS}^i(\mathsf{M}) = \sum_{\substack{F\in\mathcal{L}(\mathsf{M})\\ \operatorname{rk}F=i}} \mu(\emptyset, F)$$

or, in other words,

$$\chi_{\mathsf{M}}(x) = \sum_{F\in\mathcal{L}(\mathsf{M})} \mu(\emptyset, F)x^{\operatorname{rk}\mathsf{M}-\operatorname{rk}F}.$$

If the polynomials are invariants associated to some geometric objects (in our case, matroids up to isomorphism) it is desirable to build these graded vector spaces from the objects. Again, this is our case, as we showed in Section 2.2. Moreover, if the objects have a non trivial symmetry group W, a natural thing to do is see how this group acts on each graded component of the vector space. What we obtain is an *equivariant polynomial*, i.e. a graded representation of W or, in other terms, a polynomial whose coefficients are given by representations of the group W. If the polynomial has integer coefficients, it might be necessary to work in the *ring of virtual representations* $\mathrm{VRep}(W)$, which is the Grothendieck group of the monoid $\mathrm{Rep}(W)$ or, informally, a ring in which we consider formal integer combinations of representations of W.

In Section 5.1 we recall the equivariant version of $\chi_{\mathsf{M}}(x)$ and the Kazhdan–Lusztig–Stanley polynomials and, in analogy with those, we introduce the *equivariant Chow polynomials.* As an application we show that the formulas for paving matroids we described in Chapter 4 still hold and use them to compute concretely some examples of these polynomials. In Section 5.2 we extend the notion of valuativity we introduced in Section 1.3.2 to categorical invariants. As an application we show that the Orlik–Solomon algebra is valuative in this new sense and, as a consequence, we recover the valuativity of the characteristic polynomial.

Remark 5.0.0.1. Once the theory of categorical valuative invariants is fully developed, with due care, one would be able to upgrade the notion of valuativity to equivariant polynomials. This is the content of a project not yet published, joint with Ben Elias, Dane Miyata and Nicholas Proudfoot. Since these results would generalize the results for paving matroids from Section 5.1.2.2 to general decompositions one could argue that the two sections should be swapped with the theory on equivariant paving matroids presented as an application of a more general theory of valuativity for equivariant matroid decompositions. However, we prefer to keep this presentation more faithful to the chronological order in which the results were developed, to highlight the fact that the results on equivariant paving matroids were proved independently of the broader theory and hinted the possibility of the existence of a notion of categorical valuative invariants, not vice versa.

5.1 Equivariant polynomials

In this section we list some known results on the equivariant versions of our polynomials. Most of its content can be seen as a generalization of the work on the incidence algebra of $\mathcal{L}(\mathsf{M})$. The details can be found in [Pro21]. We recall that $W \curvearrowright \mathsf{M}$ is an *equivariant matroid* whenever $W \subset \mathfrak{S}_E$ is a group acting by permutation of the elements of the ground set E that preserves the matroid M.

For every equivariant matroid $W \curvearrowright \mathsf{M}$ we define its *equivariant characteristic polynomial* $\chi^W_{\mathsf{M}}(x)$ as

$$\chi^W_{\mathsf{M}}(x) := \sum_{i=0}^{\operatorname{rk}\mathsf{M}} (-1)^i \left(W \curvearrowright \mathrm{OS}^i(\mathsf{M})\right) x^{\operatorname{rk}\mathsf{M}-i} \in \operatorname{gr}\mathrm{VRep}(W)$$

Lemma 5.1.0.1 ([GPY17a, Lemma 2.1])**.** *For every equivariant matroid $W \curvearrowright \mathsf{M}$, $\chi^W_{\mathsf{M}}(1) = 0 \in \operatorname{gr}\mathrm{VRep}(W)$.*

This also lets us define the *equivariant reduced characteristic polynomial* $\overline{\chi}^W_{\mathsf{M}}(x)$ as

$$\overline{\chi}^W_{\mathsf{M}}(x) := \frac{\chi^W_{\mathsf{M}}(x)}{(x-1)}.$$

Consider now an equivariant matroid $W \curvearrowright \mathsf{M}$ and a subset of its ground set $S \subset E$. Both $\mathsf{M}_{|S}$ and M/S admit actions of the stabilizer group $W_S \subset W$. We denote the trivial representation of W by τ_W.

The *equivariant Kazhdan–Lusztig polynomial* $P_{\mathsf{M}}^W(x)$ and the *equivariant Z-polynomial* $Z_{\mathsf{M}}^W(x)$ are characterized by the following conditions ([BHM+22b, Corollary A.5]):

(i) If $E = \emptyset$, then $P_{\mathsf{M}}^W(x) = Z_{\mathsf{M}}^W(x) = \tau_W$.

(ii) If $E \neq \emptyset$, then $\deg P_{\mathsf{M}}^W(x) < \frac{1}{2} \operatorname{rk} \mathsf{M}$.

(iii) The polynomial $Z_{\mathsf{M}}^W(x)$ is palindromic, with degree equal to the rank of M:

$$x^{\operatorname{rk} \mathsf{M}} Z_{\mathsf{M}}^W(x^{-1}) = Z_{\mathsf{M}}^W(x).$$

(iv) For every M,

$$Z_{\mathsf{M}}^W(x) \quad = \sum_{[S]\in 2^E/W} x^{\operatorname{rk} S} \operatorname{Ind}_{W_S}^W P_{\mathsf{M}_S}^{W_S}(x). \tag{5.1}$$

Remark 5.1.0.2. To be more explicit about how this works, let M be a matroid of rank k on a non-empty ground set E, and assume that equivariant Kazhdan–Lusztig polynomials have been defined for all matroids whose ground sets are proper subsets of E. Let

$$R_{\mathsf{M}}^W(x) := \sum_{[S]\in(2^E\smallsetminus\{\emptyset\})/W} x^{\operatorname{rk} S} \operatorname{Ind}_{W_S}^W P_{\mathsf{M}/S}^{W_S}(x).$$

Then $P_{\mathsf{M}}^W(x)$ is the unique polynomial of degree strictly less than $\frac{1}{2}k$ with the property that $Z_{\mathsf{M}}^W(x) := P_{\mathsf{M}}^W(x) + R_{\mathsf{M}}^W(x)$ is palindromic of degree k.

Remark 5.1.0.3. We would like to remark that this is the categorification of the definition of $P_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ given by Theorem 1.2.4.16. The first definition of $P_{\mathsf{M}}^W(x)$ was given in terms of $\chi_{\mathsf{M}}^W(x)$ as its right Kazhdan–Lusztig–Stanley function in the equivariant incidence algebra of $\mathcal{L}(\mathsf{M})$, i.e. as the graded virtual representation of W that satisfies,

(i) If $E = \emptyset$, then $P_{\mathsf{M}}^W(x) = \tau_W$.

(ii) If $E \neq \emptyset$, then $\deg P_{\mathsf{M}}^W(x) < \frac{1}{2} \operatorname{rk} \mathsf{M}$.

(iii) For every M

$$x^{\operatorname{rk} \mathsf{M}} P_{\mathsf{M}}^W(x^{-1}) := \sum_{[F]\in\mathcal{L}(\mathsf{M})/W} \operatorname{Ind}_{W_F}^W \left(\chi_{\mathsf{M}_{|F}}^{W_F}(x) \otimes P_{\mathsf{M}/F}^{W_F}(x)\right).$$

The *equivariant inverse Kazhdan–Lusztig polynomial* $Q_{\mathsf{M}}^W(x)$ is characterized by the following two conditions [Pro21, Proposition 4.6]:

(i) If $E = \emptyset$, then $Q_{\mathsf{M}}^W(x) = \tau_W$.

(ii) If $E \neq \emptyset$, then

$$\sum_{[S]\in 2^E/W} (-1)^{\operatorname{rk} S} \operatorname{Ind}_{W_S}^W \left(Q_{\mathsf{M}_{|S}}^{W_S}(x) \otimes P_{\mathsf{M}/S}^{W_S}(x)\right) = 0. \tag{5.2}$$

Remark 5.1.0.4. The original definition of the (ordinary or equivariant) Kazhdan–Lusztig polynomial of M and inverse Kazhdan–Lusztig polynomial of M applied only to loopless matroids. With this definition, one can prove inductively that $P_{\mathsf{M}}^W(x) = 0 = Q_{\mathsf{M}}^W(x)$ whenever M has a loop. In contrast, the polynomials $Z_{\mathsf{M}}^W(x)$ are unchanged when we replace M with its simplification. These are the most natural definitions from the geometric point of view.

Remark 5.1.0.5. The contraction M/S is loopless if and only if S is a flat, so Remark 5.1.0.4 implies that we may replace the sums in equations (5.1) and (5.2) with sums over W orbits in $\mathcal{L}(\mathsf{M})$. However, it will be more convenient for our purposes to work with the sum over all subsets.

Remark 5.1.0.6. With these definitions, the equivariant Kazhdan–Lusztig–Stanley polynomials are, a priori, virtual representations. This is the same issue that occured with their non-equivariant version, as the characteristic polynomial has coefficients that alternate in sign. However, [BHM+22b, Theorem 1.3] proves that these graded representations are honest.

5.1.1 Equivariant deletion formula

By shadowing step-by-step the proof of the deletion formula by Braden and Vysogorets from Theorem 2.4.3.6, we can provide the following equivariant deletion formula for the equivariant Kazhdan–Lusztig polynomial and the equivariant Z-polynomial, which were first proved in [FMSV22, Section 4.5]. As it was already pointed out in [BV20, Remark 2.10], the action $W \curvearrowright \mathsf{M}$ is not an action on the deletion $\mathsf{M} \setminus e$. Therefore, our formula is, at best, with respect to W_e, the stabilizer of the flat $\{e\}$.

Theorem 5.1.1.1. *Let* M *be a loopless matroid and let* $e \in E$ *be an element that is not a coloop. Then,*

$$\begin{aligned} P_{\mathsf{M}}^{W_e}(x) &= P_{\mathsf{M}\setminus e}^{W_e}(x) - x\, P_{\mathsf{M}/e}^{W_e}(x) \\ &\quad + \sum_{[F]\in\mathcal{S}_e/W_e} x^{\frac{\operatorname{rk}\mathsf{M}-\operatorname{rk}F}{2}} \operatorname{Ind}_{W_F\cap W_e}^{W_e}\left(\tau(\mathsf{M}/(F\cup\{e\}))^{W_F\cap W_e} \boxtimes P_{\mathsf{M}_{|F}}^{W_F\cap W_e}(x)\right), \\ Z_{\mathsf{M}}^{W_e}(x) &= Z_{\mathsf{M}\setminus e}^{W_e}(x) + \sum_{[F]\in\mathcal{S}_e/W_e} x^{\frac{\operatorname{rk}\mathsf{M}-\operatorname{rk}F}{2}} \operatorname{Ind}_{W_F\cap W_e}^{W_e}\left(\tau(\mathsf{M}/(F\cup\{e\}))^{W_F\cap W_e} \boxtimes Z_{\mathsf{M}_{|F}}^{W_F\cap W_e}(x)\right). \end{aligned}$$

Here, $\tau(\mathsf{M})^W$ *is defined analogously to the non-equivariant case.*

Proof. We sketch how to define the proper tools that are needed for the proof. Let $\operatorname{gr}_{\mathbb{Z}} \operatorname{VRep}(W) = \operatorname{VRep}(W)[x^{\pm 1}]$ denote the ring of $\mathbb{Z}$-graded virtual representations, i.e. the ring of Laurent polynomials over the ring of virtual representations, and let $\mathcal{H}^W(\mathsf{M})$ be the free module over $\operatorname{gr}_{\mathbb{Z}} \operatorname{VRep}(W)$ with basis indexed by $\mathcal{L}(\mathsf{M})/W$. Define also

$$\zeta[E]^W = \sum_{[G]\in\mathcal{L}(\mathsf{M})/W} \left(\zeta_G^E\right)^W [G],$$

where $\left(\zeta_G^E\right)^W := x^{\operatorname{rk}_{\mathsf{M}} F - \operatorname{rk}_{\mathsf{M}} G} \operatorname{Ind}_{W_G}^W \left(P_{\mathsf{M}/G}^{W_G}(x^{-2})\right)$. For a proper flat F, define $\zeta[F]^{W_F}$ similarly in $\mathcal{H}^{W_F}(\mathsf{M}_{|F})$. Lastly, define a morphism by letting

$$\begin{aligned} \Delta^{W_e} : \mathcal{H}^{W_e}(\mathsf{M}) &\to \mathcal{H}^{W_e}(\mathsf{M}\setminus e) \\ [F] &\mapsto x^{\operatorname{rk}_{\mathsf{M}\setminus e}(F\setminus\{e\}) - \operatorname{rk}_{\mathsf{M}} F}[F\setminus\{e\}] \end{aligned}$$

and extending $\operatorname{gr}_{\mathbb{Z}} \operatorname{VRep}(W_i)$-linearly. Now, the element $\Delta^{W_e}(\zeta[E]^{W_e}) \in \mathcal{H}^{W_e}(\mathsf{M}\setminus e)$ can be written as

$$\Delta^{W_e}(\zeta[E]^{W_e}) = \sum_{[F]\in\mathcal{L}(\mathsf{M})/W_e} \left(\zeta_F^E\right)^{W_e} x^{\operatorname{rk}_{\mathsf{M}\setminus e}(F\setminus\{e\}) - \operatorname{rk}_{\mathsf{M}} F}[F\setminus\{e\}].$$

Therefore, the coefficient corresponding to $[\emptyset]$ is

$$[\emptyset]\,\Delta^{W_e}(\zeta[E]^{W_e}) = \left(\zeta^E_\emptyset\right)^{W_e} + x^{-1}\left(\zeta^E_{\{e\}}\right)^{W_e} = x^{\operatorname{rk}\mathsf{M}}\left(P^{W_e}_{\mathsf{M}}(x^{-2}) + x^{-2}P^{W_e}_{\mathsf{M}/e}(x^{-2})\right).$$

Similarly, we can write

$$\Delta^{W_e}(\zeta[E]^{W_e}) = \zeta[E\setminus\{e\}]^{W_e} + \sum_{\substack{[F]\in\mathcal{S}_e/W_e\\ F\neq E\setminus\{e\}}} \operatorname{Ind}^{W_e}_{W_F\cap W_e}\left(\tau(\mathsf{M}/(F\cup\{e\}))^{W_F\cap W_e}\boxtimes\zeta[F]^{W_F\cap W_e}\right),$$

and taking again the coefficient of $[\emptyset]$ we obtain

$$[\emptyset]\Delta^{W_e}(\zeta[E]^{W_e})$$
$$= x^{\operatorname{rk}\mathsf{M}}\left(P^{W_e}_{\mathsf{M}\setminus e}(x^{-2}) + \sum_{[F]\in\mathcal{S}_e/W_e} x^{-(\operatorname{rk}\mathsf{M}-\operatorname{rk}F)}\operatorname{Ind}^{W_e}_{W_F\cap W_e}\left(\tau(\mathsf{M}/(F\cup\{e\}))^{W_F\cap W_e}\boxtimes P^{W_F\cap W_e}_{\mathsf{M}_{|F}}(x^{-2})\right)\right).$$

Dividing by $x^{\operatorname{rk}\mathsf{M}}$ yields the first result after a change of variable, with x in place of x^{-2}. The proof for $Z^{W_e}_{\mathsf{M}}(x)$ is entirely analogous and relies on the definition of the $\operatorname{gr}_{\mathbb{Z}}\operatorname{VRep}(W_e)$-module map $\Phi^{W_e}_{\mathsf{M}}\colon \mathcal{H}^{W_e}(\mathsf{M})\to\operatorname{gr}_{\mathbb{Z}}\operatorname{VRep}(W_e)$ given by

$$\sum_{[F]\in\mathcal{L}(\mathsf{M})/W_e}\alpha_F\,[F] \overset{\Phi^{W_e}_{\mathsf{M}}}{\longmapsto} \sum_{[F]\in\mathcal{L}(\mathsf{M})/W_e} x^{-\operatorname{rk}_{\mathsf{M}}F}\alpha_F.$$

□

5.1.2 Equivariant theory for paving matroids

Given a matroid M with a group action W and a stressed hyperplane H, let $\mathcal{H}$ be the W-orbit of H. Let $W_H\subset W$ denote the stabilizer of H, so that $\mathcal{H}\cong W/W_H$. Let $\widetilde{\mathsf{M}}$ be the matroid obtained by relaxing every hyperplane in $\mathcal{H}$, and note that the W-action on M induces a W-action on $\widetilde{\mathsf{M}}$. The group W_H acts on H, inducing a homomorphism from W_H to the permutation group $\mathfrak{S}_H$. If $h=\#H$ and we fix an ordering of H, then we can identify $\mathfrak{S}_H$ with the symmetric group $\mathfrak{S}_h$. For any representation V of $\mathfrak{S}_h$, we will write $\operatorname{Res}^{\mathfrak{S}_h}_{W_H}V$ to denote the pullback of V to a representation of W_H (even though the homomorphism from W_H to $\mathfrak{S}_h$ need not be an inclusion). Given a partition λ of h, we write V_λ to denote the corresponding irreducible representation of $\mathfrak{S}_h$ over the rational numbers, which is called the *Specht module* associated with λ. More generally, given a pair of partitions λ and μ with $|\lambda|-|\mu|=h$, we write $V_{\lambda/\mu}$ to denote the corresponding *skew Specht module*, which is characterized by the property that the multiplicity of V_ν in $V_{\lambda/\mu}$ is equal to the multiplicity of V_λ in

$$\operatorname{Ind}^{\mathfrak{S}_{|\lambda|}}_{\mathfrak{S}_{|\mu|}\times\mathfrak{S}_h}\Big(V_\mu\boxtimes V_\nu\Big).$$

5.1.2.1 Equivariant characteristic polynomial

Proposition 5.1.2.1. *We have the following identity of equivariant characteristic polynomials:*

$$\chi^W_{\widetilde{\mathsf{M}}}(x) = \chi^W_{\mathsf{M}}(x) + (-1)^k(1-x)\operatorname{Ind}^W_{W_H}\operatorname{Res}^{\mathfrak{S}_h}_{W_H}V_{[h-k+1,1^{k-1}]}.$$

Proof. We start by decomposing $\mathrm{OS}(\mathsf{M})$ using Theorem 2.2.1.4

$$\mathrm{OS}(\mathsf{M}) \cong \bigoplus_{F \in \mathcal{L}(\mathsf{M})} \mathrm{OS}^{\operatorname{rk} F}(\mathsf{M}_{|F}),$$

and similarly for $\mathrm{OS}(\widetilde{\mathsf{M}})$. For each $J \in \mathcal{H}$, let

$$\mathcal{K}_J := \{K \subset J \mid \#K = k-1\},$$

and put

$$\mathcal{K} := \bigcup_{J \in \mathcal{H}} \mathcal{K}_J.$$

Then

$$\mathcal{L}(\widetilde{\mathsf{M}}) = \mathcal{L}(\mathsf{M}) \cup \mathcal{K} \smallsetminus \mathcal{H},$$

and we obtain a similar decomposition

$$\mathrm{OS}(\widetilde{\mathsf{M}}) \cong \bigoplus_{F \in \mathcal{L}(\widetilde{\mathsf{M}})} \mathrm{OS}^{\operatorname{rk} F}(\widetilde{\mathsf{M}}_{|F}).$$

The canonical surjection

$$\varphi : \mathrm{OS}(\widetilde{\mathsf{M}}) \to \mathrm{OS}(\mathsf{M})$$

restricts to an isomorphism

$$\mathrm{OS}^{\operatorname{rk} F}(\widetilde{\mathsf{M}}_{|F}) \to \mathrm{OS}^{\operatorname{rk} F}(\mathsf{M}_{|F})$$

for all flats $F \in \mathcal{L}(\mathsf{M}) \setminus (\mathcal{H} \cup \{E\})$, a surjection

$$\varphi_J^{k-1} : \bigoplus_{K \in \mathcal{K}_J} \mathrm{OS}^{k-1}(\widetilde{\mathsf{M}}_{|K}) \to \mathrm{OS}^{k-1}(\mathsf{M}_{|J})$$

for each $J \in \mathcal{H}$, and a surjection

$$\varphi^k : \mathrm{OS}^k(\widetilde{\mathsf{M}}) \to \mathrm{OS}^k(\mathsf{M}).$$

Let us compute the kernel of φ_H^{k-1}. The domain of φ_H^{k-1} is isomorphic to the degree $k-1$ part of the Orlik–Solomon algebra of the Boolean matroid on the ground set H, which is isomorphic to

$$V[h-k+1, 1^{k-1}] \oplus V[h-k+2, 1^{k-2}]$$

as a representation of $\mathfrak{S}_h \cong \mathfrak{S}_H$. On the other hand, the codomain of φ_H^{k-1} is isomorphic to the degree $k-1$ part of the Orlik–Solomon algebra of the uniform matroid of rank $k-1$ on the ground set H, which is isomorphic to $V[h-k+2, 1^{k-2}]$. Thus, the kernel of φ_H^{k-1} is isomorphic to $\operatorname{Res}^{\mathfrak{S}_h}_{W_H} V[h-k+1, 1^{k-1}]$. We then have

$$\operatorname{Ker}(\varphi^{k-1}) = \bigoplus_{J \in \mathcal{H}} \operatorname{Ker}(\varphi_J^{k-1}) \cong \operatorname{Ind}^W_{W_H} \operatorname{Ker}(\varphi_H^{k-1}) \cong \operatorname{Ind}^W_{W_H} \operatorname{Res}^{\mathfrak{S}_h}_{W_H} V_{[h-k+1, 1^{k-1}]}.$$

So far, this allows us to conclude that $\chi^W_{\widetilde{\mathsf{M}}}(x) - \chi^W_{\mathsf{M}}(x)$ vanishes in degrees greater than 1 and is equal to $(-1)^{k-1}\, x \operatorname{Ind}^W_{W_H} \operatorname{Res}^{\mathfrak{S}_h}_{W_H} V_{[h-k+1,1^{k-1}]}$ in degree 1. The proof of the proposition then follows from Lemma 5.1.0.1. □

5.1.2.2 Equivariant Kazhdan–Lusztig–Stanley polynomials

The content of this Section comes from [KNPV23]. As in the non-equivariant case, to make computations on paving matroids, we need the values of these polynomials for uniform matroids. We record here the known formulas.

Theorem 5.1.2.2 ([GPY17a, Theorem 3.1],[GXY22, Theorem 3.7])**.** *The equivariant uniform matroid $\mathfrak{S}_n \curvearrowright \mathsf{U}_{k,n}$ has the following equivariant Kazhdan–Lusztig polynomial*

$$P^{\mathfrak{S}_n}_{\mathsf{U}_{k,n}}(x) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{h=0}^{\min(n-k,k-2i)} V_{[n-2i-h+1,h+1,2^{i-1}]}\, x^i,$$

where each coefficient is expressed as a sum of Specht modules indexed by Young tableaux. *Alternatively, one can write each coefficient as a skew Specht module to obtain*

$$P^{\mathfrak{S}_n}_{\mathsf{U}_{k,n}}(x) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} V_{[n-2i,(k-2i+1)^i]/[(k-2i-1)^i]}\, x^i.$$

Theorem 5.1.2.3 ([GXY22, Thoerem 3.2])**.** *The equivariant uniform matroid $\mathfrak{S}_n \curvearrowright \mathsf{U}_{k,n}$ has the following equivariant inverse Kazhdan–Lusztig polynomial*

$$Q^{\mathfrak{S}_n}_{\mathsf{U}_{k,n}}(x) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} V_{[n-k+1,2^i,1^{k-2i-1}]}\, x^i.$$

This is the equivariant version of Theorem 4.1.3.5.

Theorem 5.1.2.4. *Fix integers $h \geq k \geq 1$. There exist polynomials $p^{\mathfrak{S}_h}_{k,h}(x)$, $q^{\mathfrak{S}_h}_{k,h}(x)$, and $z^{\mathfrak{S}_h}_{k,h}(x)$, each with isomorphism classes of $\mathfrak{S}_h$-representations as coefficients, such that for any matroid M of rank k, any group W of symmetries of M, and any stressed hyperplane H of cardinality h, the following identities hold:*

$$\begin{aligned} P^W_{\widetilde{\mathsf{M}}}(x) &= P^W_{\mathsf{M}}(x) + \operatorname{Ind}^W_{W_H} \operatorname{Res}^{\mathfrak{S}_h}_{W_H} p^{\mathfrak{S}_h}_{k,h}(x) \\ Q^W_{\widetilde{\mathsf{M}}}(x) &= Q^W_{\mathsf{M}}(x) + \operatorname{Ind}^W_{W_H} \operatorname{Res}^{\mathfrak{S}_h}_{W_H} q^{\mathfrak{S}_h}_{k,h}(x) \\ Z^W_{\widetilde{\mathsf{M}}}(x) &= Z^W_{\mathsf{M}}(x) + \operatorname{Ind}^W_{W_H} \operatorname{Res}^{\mathfrak{S}_h}_{W_H} z^{\mathfrak{S}_h}_{k,h}(x). \end{aligned}$$

Let $S \subset H$ be a non-empty proper subset, and let $W_S \subset W$ be its stabilizer, which acts on the contraction M/S. The matroid $\widetilde{\mathsf{M}}/S := (\widetilde{\mathsf{M}})/S$ can be obtained from M either by first relaxing all of the stressed hyperplanes in $\mathcal{H}$ and then contracting S, or by first contracting S and then relaxing a bunch of stressed hyperplanes in M/S, namely $J \setminus S$ for all $J \in \mathcal{H}$ such that J contains S. This is not necessarily a single W_S-orbit, but rather a collection of W_S-orbits of stressed hyperplanes.

More precisely, let $L(S,H) := \{w \in W \mid wS \subset H\}$. This set admits an action by W_H via left multiplication, as well as a commuting action of W_S via right multiplication. The quotients by these actions can be described as follows:

$$\begin{aligned} C(S,H) &:= \{wS \mid w \in W \text{ and } wS \subset H\} \cong L(S,H)/W_S \\ D(S,H) &:= \{J \in \mathcal{H} \mid S \subset J\} \cong W_H \backslash L(S,H), \end{aligned}$$

where the second isomorphism takes $J = wH$ to the coset $W_H w^{-1}$. The double quotient

$$W_H \backslash C(S,H) \cong W_H \backslash L(S,H)/W_S \cong D(S,H)/W_S$$

may be regarded as a set of W_S-orbits of stressed hyperplanes of the matroid M/S, and relaxing all of these orbits yields the matroid $\widetilde{\mathsf{M}}/S$. Let $\widetilde{\mathsf{M}/S}$ be the matroid obtained by relaxing only one of those orbits, namely the one containing the stressed hyperplane $H \setminus S$.

We now state and prove a lemma that will be a crucial ingredient in the inductive proof of Theorem 5.1.2.4.

Lemma 5.1.2.5. *Suppose that Theorem 5.1.2.4 holds for the matroids of rank equal to the rank of* M/S. *Then we have*

$$\operatorname{Ind}_{W_S}^{W}\left(P^{W_S}_{\widetilde{\mathsf{M}}/S}(x) - P^{W_S}_{\mathsf{M}/S}(x)\right) = \sum_{[T]\in W_H\backslash C(S,H)} \operatorname{Ind}_{W_T}^{W}\left(P^{W_T}_{\widetilde{\mathsf{M}/T}}(x) - P^{W_T}_{\mathsf{M}/T}(x)\right).$$

Proof. Let $i = \#S$. Theorem 5.1.2.4 for the action of W_S on M/S tells us that

$$P^{W_S}_{\widetilde{\mathsf{M}}/S}(x) - P^{W_S}_{\mathsf{M}/S}(x) = \sum_{[J]\in D(S,H)/W_S} \operatorname{Ind}^{W_S}_{W_J\cap W_S} \operatorname{Res}^{\mathfrak{S}_{h-i}}_{W_J\cap W_S} p^{\mathfrak{S}_{h-i}}_{k-i,h-i}(x).$$

Theorem 5.1.2.4 for the action of W_T on M/T tells us that

$$P^{W_T}_{\widetilde{\mathsf{M}/T}}(x) - P^{W_T}_{\mathsf{M}/T}(x) = \operatorname{Ind}^{W_T}_{W_H\cap W_T} \operatorname{Res}^{\mathfrak{S}_{h-i}}_{W_H\cap W_T} p^{\mathfrak{S}_{h-i}}_{k-i,h-i}(x).$$

The lemma now follows from the identification of $D(S,H)/W_S$ with $W_H\backslash C(S,H)$. □

Proof of Theorem 5.1.2.4. We proceed by induction on the rank k of our matroid. If $k = 1$, then H is necessarily the set of all loops in M, and $W_H = W$. In this case, Remark 5.1.0.4 implies that we can take $p^{\mathfrak{S}_h}_{1,h}(x) = V_{[h]} = q^{\mathfrak{S}_h}_{1,h}(x)$ and $z^{\mathfrak{S}_h}_{1,h}(x) = 0$.

For the induction step, we will prove only the statements about $P^W_{\widetilde{\mathsf{M}}}(x)$ and $Z^W_{\widetilde{\mathsf{M}}}(x)$; the proof of the statement about $Q^W_{\widetilde{\mathsf{M}}}(x)$ is nearly identical. By Remark 5.1.0.2, it will be sufficient to prove that there is a polynomial $r^{\mathfrak{S}_h}_{k,h}(x)$ such that

$$R^W_{\widetilde{\mathsf{M}}}(x) = R^W_M(x) + \operatorname{Ind}^W_{W_H} \operatorname{Res}^{\mathfrak{S}_h}_{W_H} r^{\mathfrak{S}_h}_{k,h}(x).$$

The polynomials $p^{\mathfrak{S}_h}_{k,h}(x)$ and $z^{\mathfrak{S}_h}_{k,h}(x)$ can be obtained from $r^{\mathfrak{S}_h}_{k,h}(x)$ in the same way that we obtain $P^W_{\mathsf{M}}(x)$ and $Z^W_{\mathsf{M}}(x)$ from $R^W_{\mathsf{M}}(x)$. Assume $k > 1$, and consider the difference

$$R^W_{\widetilde{\mathsf{M}}}(x) - R^W_{\mathsf{M}}(x) = \sum_{[S]\in(2^E\backslash\{\emptyset\})/W} x^{\operatorname{rk} S} \operatorname{Ind}^W_{W_S}\left(P^{W_S}_{\widetilde{\mathsf{M}}/S}(x) - P^{W_S}_{\mathsf{M}/S}(x)\right).$$

We break the sum into three different parts and analyze each part individually.

- Suppose S is a subset of E that is not contained in any element of $\mathcal{H}$. In this case, $\widetilde{\mathsf{M}}/S = \mathsf{M}/S$, so the summand indexed by $[S]$ vanishes.
- The set $S = H$ is a flat of M but not of $\widetilde{\mathsf{M}}$, and therefore $P^{W_H}_{\widetilde{\mathsf{M}}_H}(x) = 0$ by Remark 5.1.0.4. The contraction M/H is uniform of rank 1, so $P^{W_H}_{\mathsf{M}/H}(x) = \tau_{W_H}$. Thus the summand indexed by $[H]$ is equal to

$$-\operatorname{Ind}^W_{W_H} \tau_{W_H} = -\operatorname{Ind}^W_{W_H} \operatorname{Res}^{\mathfrak{S}_h}_{W_H} \tau_{\mathfrak{S}_h}.$$

- Suppose that $\emptyset \subsetneq S \subsetneq H$. Our inductive hypothesis and Lemma 5.1.2.5 tell us that the contribution indexed by $[S]$ is equal to

$$x^{\operatorname{rk} S} \sum_{[T] \in W_H \backslash C(S,H)} \operatorname{Ind}_{W_T}^{W} \left(P_{\widetilde{\mathsf{M}/T}}^{W_T}(x) - P_{\mathsf{M}/T}^{W}(x) \right).$$

If we take the sum over all such $[S]$, we get

$$x^{\operatorname{rk} S} \sum_{[T] \in (2^H \setminus \{\emptyset, H\})/W_H} \operatorname{Ind}_{W_T}^{W} \left(P_{\widetilde{\mathsf{M}/T}}^{W_T}(x) - P_{\mathsf{M}/T}^{W}(x) \right).$$

Our inductive hypothesis tells us that

$$P_{\widetilde{\mathsf{M}/T}}^{W_T}(x) - P_{\mathsf{M}/T}^{W_T}(x) = \operatorname{Ind}_{W_T \cap W_H}^{W_T} \operatorname{Res}_{W_T \cap W_H}^{\mathfrak{S}_{h-\#T}} p_{k-\#T, h-\#T}^{\mathfrak{S}_{h-\#T}}(x),$$

and therefore that

$$\begin{aligned}
\operatorname{Ind}_{W_T}^{W} \left(P_{\widetilde{\mathsf{M}/T}}^{W_T}(x) - P_{\mathsf{M}/T}^{W_T}(x) \right) &= \operatorname{Ind}_{W_T}^{W} \operatorname{Ind}_{W_T \cap W_H}^{W_T} \operatorname{Res}_{W_T \cap W_H}^{\mathfrak{S}_{h-\#T}} p_{k-\#T, h-\#T}^{\mathfrak{S}_{h-\#T}}(x) \\
&= \operatorname{Ind}_{W_T \cap W_H}^{W} \operatorname{Res}_{W_T \cap W_H}^{\mathfrak{S}_{h-\#T}} p_{k-\#T, h-\#T}^{\mathfrak{S}_{h-\#T}}(x) \\
&= \operatorname{Ind}_{W_H}^{W} \operatorname{Ind}_{W_T \cap W_H}^{W_H} \operatorname{Res}_{W_T \cap W_H}^{\mathfrak{S}_{h-\#T}} p_{k-\#T, h-\#T}^{\mathfrak{S}_{h-\#T}}(x).
\end{aligned}$$

Taking the sum over all $[T] \in (2^H \setminus \{\emptyset, H\})/W_H$, we get

$$\begin{aligned}
&\sum_{[T] \in (2^H \setminus \{\emptyset, H\})/W_H} \operatorname{Ind}_{W_H}^{W} \operatorname{Ind}_{W_T \cap W_H}^{W_H} \operatorname{Res}_{W_T \cap W_H}^{\mathfrak{S}_{h-\#T}} p_{k-\#T, h-\#T}^{\mathfrak{S}_{h-\#T}}(x) \\
&= \operatorname{Ind}_{W_H}^{W} \left(\sum_{[T] \in (2^H \setminus \{\emptyset, H\})/W_H} \operatorname{Ind}_{W_T \cap W_H}^{W_H} \operatorname{Res}_{W_T \cap W_H}^{\mathfrak{S}_{h-\#T}} p_{k-\#T, h-\#T}^{\mathfrak{S}_{h-\#T}}(x) \right) \\
&= \operatorname{Ind}_{W_H}^{W} \left(\sum_{\emptyset \subsetneq T \subsetneq H} \operatorname{Res}_{W_T \cap W_H}^{\mathfrak{S}_{h-\#T}} p_{k-\#T, h-\#T}^{\mathfrak{S}_{h-\#T}}(x) \right),
\end{aligned}$$

where the second equality is a standard fact about induced representations; see for example [Pro21, Lemma 2.7]. (Note that the individual terms in the sum are not representations of W_H, but rather of $W_H \cap W_T$. An element $w \in W_H$ takes the term indexed by T to the term indexed by wT). We may rewrite this expression as

$$\operatorname{Ind}_{W_H}^{W} \operatorname{Res}_{W_H}^{\mathfrak{S}_h} \left(\sum_{\emptyset \subsetneq T \subsetneq [h]} p_{k-\#T, h-\#T}^{\mathfrak{S}_{h-\#T}}(x) \right),$$

where now the individual terms in the sum are representations of $\mathfrak{S}_{[h] \setminus T} \cong \mathfrak{S}_{h-\#T}$, and the entire sum is a representation of $\mathfrak{S}_h$. Finally, we once again employ the same standard fact about induced representations, this time using the action of $\mathfrak{S}_h$, to rewrite our expression as

$$\operatorname{Ind}_{W_H}^{W} \operatorname{Res}_{W_H}^{\mathfrak{S}_h} \sum_{i=1}^{h-1} \operatorname{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_{h-i}}^{\mathfrak{S}_h} \left(\tau_{\mathfrak{S}_i} \boxtimes p_{k-i, h-i}^{\mathfrak{S}_{h-i}}(x) \right),$$

which is manifestly of the desired form.

Putting the four parts together, we may take

$$r^{\mathfrak{S}_h}_{k,h}(x) = -\tau_{\mathfrak{S}_h} + \sum_{i=1}^{h-1} \mathrm{Ind}^{\mathfrak{S}_h}_{\mathfrak{S}_i \times \mathfrak{S}_{h-i}} \left(\tau_{\mathfrak{S}_i} \boxtimes p^{\mathfrak{S}_{h-i}}_{k-i,h-i}(x) \right).$$

This completes the proof. □

Next, we give explicit formulas for two of the three $\mathfrak{S}_h$-equivariant polynomials appearing in Theorem 5.1.2.4. This is the equivariant version of Corollaries 4.1.4.5 and 4.1.4.8.

Theorem 5.1.2.6. *When $k = 1$, we have $p^{\mathfrak{S}_h}_{1,h}(x) = V_{[h]} = q^{\mathfrak{S}_h}_{1,h}(x)$ and $z^{\mathfrak{S}_h}_{1,h}(x) = 0$. When $k > 1$, we have the following explicit formulas:*[1]

$$p^{\mathfrak{S}_h}_{k,h}(x) = \sum_{0<i<k/2} V_{[h-2i+1,(k-2i+1)^i]/[k-2i,(k-2i-1)^{i-1}]}\, x^i,$$

$$q^{\mathfrak{S}_h}_{k,h}(x) = \sum_{0\le i<k/2} \left(V_{[h-k+2,2^{i-1},1^{k-2i}]} + V_{[h-k+1,2^i,1^{k-2i-1}]} \right) x^i.$$

Remark 5.1.2.7. One can use similar methods to obtain an explicit formula for $z^{\mathfrak{S}_h}_{k,h}(x)$, but since this formula is considerably less elegant, we omit it.

We prove Theorem 5.1.2.6 by first examining a single example. Consider again the matroid defined in Proposition 4.1.1.5

$$\Pi_{k,h,h+1} := \mathsf{U}_{k-1,h} \oplus \mathsf{B}_1.$$

The group $\mathfrak{S}_h$ acts on the first summand, which is a stressed hyperplane of cardinality h. The relaxation $\Lambda_{k,h,h+1}$ is isomorphic to $\mathsf{U}_{k,h+1}$. We have the equalities

$$p^{\mathfrak{S}_h}_{k,h}(x) = P^{\mathfrak{S}_h}_{\Lambda_{k,h,h+1}}(x) - P^{\mathfrak{S}_h}_{\Pi_{k,h,h+1}}(x)$$
$$q^{\mathfrak{S}_h}_{k,h}(x) = Q^{\mathfrak{S}_h}_{\Lambda_{k,h,h+1}}(x) - Q^{\mathfrak{S}_h}_{\Pi_{k,h,h+1}}(x),$$

so it will suffice to compute the four polynomials on the right-hand sides of the two equations.

We begin with the polynomials associated with the matroid $\Pi_{k,h,h+1}$. We have

$$P^{\mathfrak{S}_h}_{\mathsf{B}_1}(x) = Q^{\mathfrak{S}_h}_{\mathsf{B}_1}(x) = V_{[h]},$$

and each of our three polynomials is multiplicative with respect to direct sums. By Theorem 5.1.2.2, we have

$$P^{\mathfrak{S}_h}_{\Pi_{k,h,h+1}}(x) = P^{\mathfrak{S}_h}_{\mathsf{U}_{k-1,h}}(x) = \sum_{i<(k-1)/2} V_{[h-2i,(k-2i)^i]/[(k-2i-2)^i]}\, x^i. \tag{5.3}$$

By Theorem 5.1.2.3, we have

$$Q^{\mathfrak{S}_h}_{\Pi_{k,h,h+1}}(x) = Q^{\mathfrak{S}_h}_{\mathsf{U}_{k-1,h}}(x) = \sum_{i<(k-1)/2} V_{[h-k+2,2^i,1^{k-2i-2}]}\, x^i. \tag{5.4}$$

By the same theorems, we have

$$P^{\mathfrak{S}_h}_{\Lambda_{k,h,h+1}}(x) = \mathrm{Res}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_h} P^{\mathfrak{S}_{h+1}}_{\mathsf{U}_{k,h+1}}(x) = \sum_{i<(k-1)/2} \mathrm{Res}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_h} V_{[h-2i+1,(k-2i+1)^i]/[(k-2i-1)^i]}\, x^i \tag{5.5}$$

[1]In the expression for $q^{\mathfrak{S}_h}_{k,h}(x)$, we interpret the first term to be zero if $i = 0$, and we interpret the second term to be zero if $i > 0$ and $k = h$.

and

$$Q^{\mathfrak{S}_h}_{\Lambda_{k,h,h+1}}(x) = \operatorname{Res}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_h} Q^{\mathfrak{S}_{h+1}}_{\mathsf{U}_{k,h+1}}(x) = \sum_{i<(k-1)/2} \operatorname{Res}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_h} V_{[h-k+2,2^i,1^{k-2i-1}]}\, x^i. \tag{5.6}$$

We compute the restrictions using the following lemma.

Lemma 5.1.2.8. *If λ is a partition of $h+1$, then*

$$\operatorname{Res}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_h} V_\lambda = \bigoplus_{\lambda'} V_{\lambda'},$$

where λ' ranges over partitions of h with the property that the Young diagram for λ' is obtained from the Young diagram for λ by removing a single box. If λ and μ are partitions with $|\lambda|-|\mu| = h+1$, then

$$\operatorname{Res}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_h} V_{\lambda/\mu} = \bigoplus_{\mu'} V_{\lambda/\mu'},$$

where μ' ranges over partitions with the property that the Young diagram for μ' is obtained from the Young diagram for μ by adding a single box.

Proof. The first statement is a well known special case of the Pieri rule. To prove the second statement, let ν be any partition of h. By Frobenius reciprocity, the multiplicity of V_ν in $\operatorname{Res}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_h} V_{\lambda/\mu}$ is equal to the dimension of the hom space from $\operatorname{Ind}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_h} V_\nu$ to $V_{\lambda/\mu}$, which is in turn equal to the multiplicity of V_λ in

$$\operatorname{Ind}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_{|\nu|}\times\mathfrak{S}_1\times\mathfrak{S}_{|\mu|}} V_\nu \boxtimes V_{[1]} \boxtimes V_\mu.$$

By the Pieri rule, this may be reinterpreted as the sum over all μ' of the stated form of the multiplicity of V_λ in

$$\operatorname{Ind}^{\mathfrak{S}_{h+1}}_{\mathfrak{S}_{|\nu|}\times\mathfrak{S}_{|\mu'|}} V_\nu \boxtimes V_{\mu'}.$$

In other words, it is the multiplicity of V_ν in $\bigoplus_{\mu'} V_{\lambda/\mu'}$. □

Applying the second statement of Lemma 5.1.2.8 to Equation (5.5), we find that

$$P^{\mathfrak{S}_h}_{\Lambda_{k,h,h+1}}(x) = \sum_{i<(k-1)/2} \Big(V_{[h-2i+1,(k-2i+1)^i]/[(k-2i-1)^i,1]} + V_{[h-2i+1,(k-2i+1)^i]/[k-2i,(k-2i-1)^{i-1}]}\Big)\, x^i, \tag{5.7}$$

where we interpret the second term inside the parentheses to be zero if $i=0$. Similarly, applying the first statement of Lemma 5.1.2.8 to Equation (5.6), we find that

$$Q^{\mathfrak{S}_h}_{\Lambda_{k,h,h+1}}(x) = \sum_{i<(k-1)/2} \Big(V_{[h-k+2,2^i,1^{k-2i-2}]} + V_{[h-k+2,2^{i-1},1^{k-2i}]} + V_{[h-k+1,2^i,1^{k-2i-1}]}\Big)\, x^i, \tag{5.8}$$

where we interpret the second term to be zero if $i=0$, and we interpret the third term to be zero if $i>0$ and $k=h$.

Proof of Theorem 5.1.2.6. The $k=1$ case was treated as the base case of the induction in the proof of Theorem 5.1.2.4, so we may assume that $k>1$. We compute $p^{\mathfrak{S}_h}_{k,h}(x)$ by taking the difference between Equations (5.7) and (5.3). We observe that we have an isomorphism

$$V_{[h-2i+1,(k-2i+1)^i]/[(k-2i-1)^i,1]} \cong V_{[h-2i,(k-2i)^i]/[(k-2i-2)^i]}$$

of skew Specht modules, which follows from the fact that the skew diagrams

$$[h-2i+1,(k-2i+1)^i]/[(k-2i-1)^i,1] \text{ and } [h-2i,(k-2i)^i]/[(k-2i-2)^i]$$

are related by a horizontal translation [Kle05, Proposition 2.3.5, Lemma 2.3.12]. This leads to a cancellation which gives us the formula for $p_{k,h}^{\mathfrak{S}_h}(x)$ stated in Theorem 5.1.2.6. We compute $q_{k,h}^{\mathfrak{S}_h}(x)$ by taking the difference between Equations (5.8) and (5.4). □

An unpublished conjecture of Gedeon generalizes Conjecture 3.1.0.3 for Kazhdan–Lusztig–Stanley polynomials to the equivariant case. Here is the formal statement.

Conjecture 5.1.2.9. *Let* M *be a matroid of rank* k *on the ground set* E*, and let* W *be a finite group that acts on* E *preserving* M*. Then the coefficients of* $P^W_{\mathsf{U}_{k,E}}(x) - P^W_{\mathsf{M}}(x)$ *are honest (rather than virtual) representations of* W*.*

Remark 5.1.2.10. The fact that the constant and linear terms of $P^W_{\mathsf{U}_{k,E}}(x) - P^W_{\mathsf{M}}(x)$ are honest representations follows from [GPY17a, Corollary 2.10]. In higher degrees, the conjecture remains open.

Theorems 5.1.2.4 and 5.1.2.6 imply that Conjecture 5.1.2.9 holds for paving matroids.

Corollary 5.1.2.11. *Conjecture 5.1.2.9 holds when* M *is paving.*

Proof. If M is paving, then M may be transformed into $\mathsf{U}_{k,E}$ by relaxing finitely many W-orbits of stressed hyperplanes. Theorems 5.1.2.4 and 5.1.2.6 imply that each of these relaxations changes the equivariant Kazhdan–Lusztig polynomial by adding a correction term whose coefficients are honest representations. □

Theorems 5.1.2.4 and 5.1.2.6, along with the known formulas for uniform matroids, provide us with the tools to compute our equivariant polynomials for any paving matroid and any group of symmetries. To illustrate this, we apply our results to compute the equivariant Kazhdan–Lusztig polynomials in six specific examples.

5.1.2.3 The Vámos matroid

In this section we consider the Vámos matroid (see Definition 1.1.2.9). The ground set of V_8 is equal to [8], and it is a paving matroid of rank 4 with 5 circuit-hyperplanes corresponding to the five shaded rectangles in Figure 1.3. The automorphism group W of V_8 is generated by the following four elements:

$$r_1 = (12), \quad s_1 = (17)(28), \quad r_2 = (34), \quad \text{and} \quad s_2 = (35)(46).$$

Note that $W \cong D_4 \times D_4$, where the first factor is generated by r_1 and s_1 and the second factor is generated by r_2 and s_2.

Let $H := \{1,2,3,4\}$ and $H' := \{3,4,5,6\}$. The orbit of H under the action of W consists of the four circuit hyperplanes other than H', and the stabilizer of H is

$$W_H = \langle (12),(34),(56),(78)\rangle \cong \mathfrak{S}_2^4.$$

In contrast, H' is fixed by W. By Theorem 5.1.2.4, we have

$$P^W_{\mathsf{V}_8}(x) = \mathrm{Res}^{\mathfrak{S}_8}_W P^{\mathfrak{S}_8}_{\mathsf{U}_{4,8}}(x) - \mathrm{Ind}^W_{W_H} \mathrm{Res}^{\mathfrak{S}_4}_{W_H} p^{\mathfrak{S}_4}_{4,4}(x) - \mathrm{Res}^{\mathfrak{S}_4}_W p^{\mathfrak{S}_4}_{4,4}(x).$$

Here the first restriction is the pullback along the homomorphism from W_H to $\mathfrak{S}_4$ given by the action of W_H on $H \cong [4]$, while the second is the pullback along the homomorphism from W to $\mathfrak{S}_4$ given by the action of W on $H' \cong [4]$.

The formula for $P^{\mathfrak{S}_8}_{\mathsf{U}_{4,8}}(x)$ is given in Theorem 5.1.2.2, and the formula for $p^{\mathfrak{S}_4}_{4,4}(x)$ is given in Theorem 5.1.2.6. Note that the constant term of $P^{\mathfrak{S}_8}_{\mathsf{U}_{4,8}}(x)$ is equal to the trivial representation of dimension 1, as is the case for all loopless matroids [GPY17a, Corollary 2.10]. All three polynomials are linear, so the only non-trivial calculation is of the coefficient of x.

The calculation can be done explicitly using character tables. We use the following standard representation of the character table for D_4:

	e	s	r^2	sr	r
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	-1	-1	1	1
χ_4	1	-1	1	1	-1
χ_5	2	0	0	-2	0

The irreducible characters of $W \cong D_4 \times D_4$ are of the form $\chi_i \boxtimes \chi_j$ for $i, j \in \{1, \ldots, 5\}$. After performing all of the restrictions and inductions, we find that the character of the linear term of $P^W_{\mathsf{V}_8}(x)$ is equal to

$$\begin{aligned}3\chi_1 \boxtimes \chi_1 + \chi_1 \boxtimes \chi_2 + \chi_1 \boxtimes \chi_4 + 2\chi_2 \boxtimes \chi_1 + \chi_2 \boxtimes \chi_2 + \chi_2 \boxtimes \chi_4 + \chi_4 \boxtimes \chi_1 + \chi_4 \boxtimes \chi_2 \\ +\chi_1 \boxtimes \chi_5 + \chi_2 \boxtimes \chi_5 + \chi_4 \boxtimes \chi_5 + 2\chi_5 \boxtimes \chi_1 + \chi_5 \boxtimes \chi_2 + \chi_5 \boxtimes \chi_5 + 2\chi_5 \boxtimes \chi_5.\end{aligned}$$

We observe that the value of this character on the identity is 33, so the non-equivariant Kazhdan–Lusztig polynomial of V_8 is $P_{\mathsf{V}_8}(x) = 1 + 33x$.

5.1.2.4 Steiner systems

We introduced Steiner systems in Definition 1.1.2.12.

Remark 5.1.2.12. The Mathieu groups M_{11}, M_{12}, M_{23}, and M_{24} are each equal to the automorphism groups of their corresponding Steiner systems. In contrast, M_{22} is the unique index 2 subgroup of the automorphism group of $S(3, 6, 22)$.

We will use the same notation to refer to a Steiner system and its associated matroid. For example, we will denote by $P^{M_{24}}_{S(5,8,24)}(x)$ the M_{24}-equivariant Kazhdan–Lusztig polynomial of the matroid associated with the Steiner system $S(5, 8, 24)$. We will refer to irreducible characters of the Mathieu groups by the same indices used in the $\mathbb{ATLAS}$ of Finite Groups [CCN+85].

Proposition 5.1.2.13. *The equivariant Kazhdan–Lusztig polynomials of the matroids associated with the aforementioned Steiner systems are characterized as follows:*

$$\begin{aligned}
\operatorname{char} P^{M_{11}}_{S(4,5,11)}(x) &= \chi_1 + (\chi_5 + \chi_8)\, x + (\chi_5 + \chi_8)\, x^2 \\
\operatorname{char} P^{M_{12}}_{S(5,6,12)}(x) &= \chi_1 + (\chi_3 + \chi_7 + \chi_8)\, x + (\chi_3 + \chi_7 + \chi_8 + \chi_{11} + \chi_{12} + \chi_{14})\, x^2 \\
\operatorname{char} P^{M_{22}}_{S(3,6,22)}(x) &= \chi_1 + \chi_5\, x \\
\operatorname{char} P^{M_{23}}_{S(4,7,23)}(x) &= \chi_1 + \chi_5\, x + \chi_9\, x^2 \\
\operatorname{char} P^{M_{24}}_{S(5,8,24)}(x) &= \chi_1 + (\chi_8 + \chi_9)\, x + (\chi_9 + \chi_{14} + \chi_{21})\, x^2.
\end{aligned}$$

Non-equivariantly, we have

$$\begin{aligned}
P_{S(4,5,11)}(x) &= 1 + 55x + 55x^2 \\
P_{S(5,6,12)}(x) &= 1 + 120x + 429x^2 \\
P_{S(3,6,22)}(x) &= 1 + 55x \\
P_{S(4,7,23)}(x) &= 1 + 230x + 253x^2 \\
P_{S(5,8,24)}(x) &= 1 + 735x + 4830x^2.
\end{aligned}$$

Proof. All of these calculations are done using only Theorems 5.1.2.4 and 5.1.2.6 along with the character tables found in the $\mathbb{ATLAS}$. We provide a brief outline of the calculation only for the most interesting case, namely $S(5,8,24)$. The ground set of the matroid $S(5,8,24)$ is $\{1,\ldots,24\}$. The group M_{24} acts transitively on the set of blocks. We have a distinguished block $H = \{1,\ldots,8\}$, whose stabilizer group is isomorphic to $\mathfrak{A}_8 \ltimes \mathbb{F}_2^4$, where the alternating group $\mathfrak{A}_8 \cong \mathrm{GL}_4(\mathbb{F}_2)$ acts linearly on the vector space $\mathbb{F}_2^4$. The homomorphism from the stabilizer group to $\mathfrak{S}_H \cong \mathfrak{S}_8$ is given by the projection onto $\mathfrak{A}_8$ followed by the inclusion of $\mathfrak{A}_8$ into $\mathfrak{S}_8$. Theorem 5.1.2.4 tells us that

$$P^{M_{24}}_{S(5,8,24)}(x) = \mathrm{Res}^{\mathfrak{S}_{24}}_{M_{24}} P^{\mathfrak{S}_{24}}_{\mathsf{U}_{6,24}}(x) - \mathrm{Ind}^{M_{24}}_{\mathfrak{A}_8 \ltimes \mathbb{F}_2^4} \mathrm{Res}^{\mathfrak{S}_8}_{\mathfrak{A}_8 \ltimes \mathbb{F}_2^4} p^{\mathfrak{S}_8}_{6,8}(x).$$

Using the formula for $P^{\mathfrak{S}_{24}}_{\mathsf{U}_{6,24}}(x)$ in Theorem 5.1.2.2 and the formula for $p^{\mathfrak{S}_4}_{4,4}(x)$ given in Theorem 5.1.2.6, this becomes a straightforward (if cumbersome) computer computation. □

5.2 Categorical valuative invariants

In Corollary 1.3.3.5, Theorem 3.1.4.3, Theorem 3.1.4.1 and Theorem 3.1.4.2 we showed that our Hilbert–Poincaré series are valuative and we heavily exploited it in Section 4 to produce fast formulas for paving matroids. However, the valuativity of these invariants still appears mysterious. One can prove that these various polynomials are valuative, but we lack a clear understanding of why they should be valuative. This last section is dedicated to giving an explanation for this phenomenon by producing canonical exact sequences of graded vector spaces. We include a full discussion for the Orlik–Solomon algebra $\mathrm{OS}(\mathsf{M})$.

5.2.1 Categories of matroids

Our goal in this section is to give precise definitions of a category of matroids, and what it means for a functor from such a category to be valuative. Let $\mathcal{M}$ be the category in which an object consists of a pair (E, M), where E is a finite set and M is a matroid on E, and a morphism from (E, M) to (E', M') is a (rank preserving) weak map, as defined in Definition 1.1.1.19. For any finite set E, we define $\mathcal{M}(E)$ to be the full subcategory of $\mathcal{M}$ consisting of matroids on E, and we define $\mathcal{M}_{\mathrm{id}}(E)$ to be the subcategory of $\mathcal{M}(E)$ consisting of only morphisms given by the identity map id_E. We then write $\mathcal{M}^+_{\mathrm{id}}(E)$ for the additive closure of this category, where now objects are formal direct sums of matroids and morphisms are matrices of morphisms in $\mathcal{M}^+_{\mathrm{id}}(E)$. Lastly, let $\mathrm{Ch}(\mathcal{M}(E))$ denote the (additive) category of chain complexes in $\mathcal{M}^+_{\mathrm{id}}(E)$, with the homological convention that differentials decrease homological degree by one. Morphisms are chain maps between complexes. Let $\mathcal{K}(\mathcal{M}(E))$ denote the *homotopy category* of $\mathcal{M}^+_{\mathrm{id}}(E)$, the quotient of $\mathrm{Ch}(\mathcal{M}(E))$ by the ideal of nulhomotopic chain maps. This is a triangulated category. As usual, we use a superscript $\mathrm{Ch}^b(\mathcal{M}(E))$ and $\mathcal{K}^b(\mathcal{M}(E))$ to indicate the full subcategory whose objects are bounded complexes.

We let $[1]$ denote the usual homological shift on complexes, so that $C[1]$ in degree i agrees with C in degree $i+1$.

Let $\mathcal{A}$ be an additive category, and let A be its split Grothendieck group. For an object X of $\mathcal{A}$ we let $[X]$ denote its symbol in A. We will be interested in functors Φ to $\mathcal{A}$ from $\mathcal{M}_{\mathrm{id}}(E)$. Then Φ induces a homomorphism from $\mathrm{Mat}(E)$ to A, sending a matroid M to $[\Phi(\mathsf{M})]$. We say that the functor *categorifies* the homomorphism.

Example 5.2.1.1. Let $\mathcal{A}$ be the category of finite dimensional graded vector spaces over $\mathbb{Q}$. The *Orlik–Solomon functor* $\mathrm{OS} : \mathcal{M} \to \mathcal{A}$ takes a matroid M to its Orlik–Solomon algebra $\mathrm{OS}(\mathsf{M})$, and sends a weak map $\varphi : (E, \mathsf{M}) \to (E', \mathsf{M}')$ to the algebra homomorphism $\mathrm{OS}(\varphi) : \mathrm{OS}(\mathsf{M}) \to \mathrm{OS}(\mathsf{M}')$ given by

$$\mathrm{OS}(\varphi) : u_e \mapsto u_{\varphi(e)}, \qquad \text{for all } e \in E.$$

The Grothendieck group of $\mathcal{A}$ is isomorphic to the polynomial ring $\mathbb{Z}[x]$, and the functor OS categorifies the Poincaré polynomial $\pi_{\mathsf{M}}(x)$.

5.2.2 The complex associated to a decomposition

Let $\mathcal{N}$ be a decomposition of a matroid polytope $\mathcal{P}(\mathsf{M})$ of dimension d. We define an *orientation* Ω of $\mathcal{N}$ to be an arbitrary choice of orientation of each polyhedron in $\mathcal{N}$, along with a choice of orientation of $\mathcal{P}(\mathsf{M})$ itself. Given the pair $(\mathcal{N}, \Omega)$, we define an object $C^{\Omega}_{\bullet}(\mathcal{N})$ as follows:

- If $k < 0$ or $k > d+1$, $C^{\Omega}_k(\mathcal{N}) = 0$.

- $C^{\Omega}_{d+1}(\mathcal{N}) = \mathsf{M}$.
- For all $0 \leq k \leq d$, $C^{\Omega}_{k}(\mathcal{N}) := \bigoplus_{\mathsf{N}\in\mathcal{N}_k} \mathsf{N}$.

If $\mathsf{N}' \in \mathcal{N}_{k-1}$ is a facet of $\mathsf{N} \in \mathcal{N}_k$, then the (N,N') component of the differential $C^{\Omega}_{k}(\mathcal{N}) \to C^{\Omega}_{k-1}(\mathcal{N})$ is given by ± 1 times $\iota_{\mathsf{N},\mathsf{N}'}$, depending on whether or not the orientation of $\mathcal{P}(\mathsf{N}')$ matches the one induced by the orientation of $\mathcal{P}(\mathsf{N})$. If N' is not a facet of N, then that component of the differential is zero. For each $\mathsf{N} \in \mathcal{N}_d$, the (M,N) component of the differential $C^{\Omega}_{d+1}(\mathcal{N}) \to C^{\Omega}_{d}(\mathcal{N})$ is given by ± 1 times $\iota_{\mathsf{M},\mathsf{N}}$, depending on whether or not the orientation of $\mathcal{P}(\mathsf{N})$ matches the one induced by the orientation of $\mathcal{P}(\mathsf{M})$. The statement that the differential squares to zero is straightforward.

Example 5.2.2.1. Consider the decomposition $\mathcal{N}$ from Examples 1.3.2.1 and 1.3.2.4. The complex $C^{\Omega}_{\bullet}(\mathcal{N})$ takes the form depicted in Figure 5.1. Choose an orientation of the 3-dimensional vector space $\{x \mid x_1 + x_2 + x_3 + x_4 = 2\} \subset \mathbb{R}^E$, and use this to induce orientations $\Omega(\mathsf{M})$, $\Omega(\mathsf{N})$, and $\Omega(\mathsf{N}')$. Choose $\Omega(\mathsf{N}'')$ to be the orientation induced by realizing $\mathcal{P}(\mathsf{N}'')$ as a facet of $\mathcal{P}(\mathsf{N})$, which is the opposite of the orientation induced by realizing $\mathcal{P}(\mathsf{N}'')$ as a facet of $\mathcal{P}(\mathsf{N}')$. We have

$$\mathrm{Hom}_{\mathcal{M}^{+}_{\mathrm{id}}(E)}(\mathsf{M}, \mathsf{N} \oplus \mathsf{N}') \cong \mathbb{Q}^2,$$

and our first differential corresponds to the element $(1,1)$. We also have

$$\mathrm{Hom}_{\mathcal{M}^{+}_{\mathrm{id}}(E)}(\mathsf{N} \oplus \mathsf{N}', \mathsf{N}'') \cong \mathbb{Q}^2,$$

and our second differential corresponds to the element $(1,-1)$. The composition is given by dot product, and our differential squares to zero because $(1,1)$ is orthogonal to $(1,-1)$.

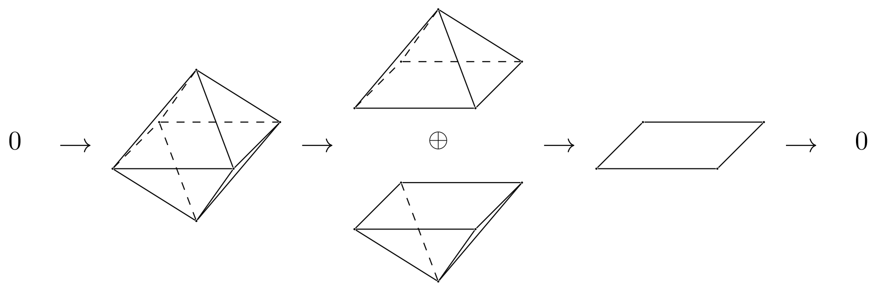

Figure 5.1: The complex $C^{\Omega}_{\bullet}(\mathcal{N})$ arising from the decomposition of $\mathsf{M} = \mathsf{U}_{2,4}$.

5.2.3 Valuativity

Let $\mathcal{A}$ be an additive $\mathbb{K}$-linear category. Any functor $\Phi : \mathcal{M}(E) \to \mathcal{A}$ induces a functor $\mathcal{M}^{+}(E) \to \mathcal{A}$. This in turn induces a functor $\mathrm{Ch}^b(\mathcal{M}(E)) \to \mathrm{Ch}^b(\mathcal{A})$ and a functor $\mathcal{K}^b(\mathcal{M}(E)) \to \mathcal{K}^b(\mathcal{A})$. We abusively denote all these functors by the letter Φ.

We say that a functor $\Phi : \mathcal{M}(E) \to \mathcal{A}$ is *valuative* if, for any pair $(\mathcal{N}, \Omega)$, the complex $\Phi(C^{\Omega}_{\bullet}(\mathcal{N}))$ is split exact. Similarly, we say that $\Phi : \mathcal{M}_{\mathrm{id}}(E) \to \mathcal{A}$ is *valuative* if, for any pair $(\mathcal{N}, \Omega)$, $\Phi(C^{\Omega}_{\bullet}(\mathcal{N}))$ is split exact.

By Theorem 1.3.2.2, any valuative functor categorifies a valuative homomorphism.

As a basic example, consider the trivial functor $\tau : \mathcal{M}(E) \to \mathrm{Vec}_{\mathbb{Q}}$ that takes all matroids to $\mathbb{Q}$ and all morphisms to the identity map. This categorifies the homomorphism that evaluates to 1 on every matroid.

Proposition 5.2.3.1. *The trivial functor τ is valuative.*

Proof. Let $d = \dim \mathcal{P}(\mathsf{M})$, and consider the complex $\tau(C^{\Omega}_{\leq d}(\mathcal{N}))$ obtained by removing the term in degree $d+1$. This complex coincides with the cellular chain complex that computes the homology of the one point compactification of $\mathcal{P}(\mathsf{M})$ relative to the one point compactification of $\partial\mathcal{P}(\mathsf{M})$, which is 1-dimensional and concentrated in degree d. We have an exact sequence of chain complexes

$$0 \to \tau(C^{\Omega}_{\leq d}(\mathcal{N})) \to \tau(C^{\Omega}_{\bullet}(\mathcal{N})) \to \mathbb{Q}[-d-1] \to 0.$$

We have observed that $\tau(C^{\Omega}_{\leq d}(\mathcal{N}))$ has 1-dimensional homology concentrated in degree d, while $\mathbb{Q}[-d-1]$ has 1-dimensional homology concentrated in degree $d+1$. The boundary map in the long exact sequence in homology is an isomorphism, which implies that the homology of $\tau(C^{\Omega}_{\bullet}(\mathcal{N}))$ vanishes. □

5.2.4 The Orlik–Solomon functor

The purpose of this section is to prove that the Orlik–Solomon functor of Example 5.2.1.1 is valuative. Observe that, if $\mathrm{id}_E : \mathsf{M} \to \mathsf{M}'$ is a morphism in $\mathcal{M}_{\mathrm{id}}(E)$, then the associated map from $\mathrm{OS}(\mathsf{M})$ to $\mathrm{OS}(\mathsf{M}')$ is filtered, and therefore induces a map from $\mathrm{gr}\,\mathrm{OS}(\mathsf{M})$ to $\mathrm{gr}\,\mathrm{OS}(\mathsf{M}')$. Let us describe this map explicitly.
The set $\{u_S \mid S \in \mathrm{nbc}(\mathsf{M})\}$ is a basis for $\mathrm{gr}\,\mathrm{OS}(\mathsf{M})$. The map $\mathrm{gr}\,\mathrm{OS}(\mathsf{M}) \to \mathrm{gr}\,\mathrm{OS}(\mathsf{M}')$ takes u_S to u_S if $S \in \mathrm{nbc}(\mathsf{M}')$ and to 0 otherwise. Let $\mathcal{N}$ be a decomposition of a matroid M on the ground set E, and let $d = d(\mathsf{M})$. For any $S \in \mathrm{nbc}(\mathsf{M})$, consider the quotient complex $V^{\Omega}_{\bullet}(\mathcal{N}, S)$ of $\tau(C^{\Omega}_{\bullet}(\mathcal{N}))$ given by putting $V^{\Omega}_{d+1}(\mathcal{N}, S) = \mathbb{Q}$ and

$$V^{\Omega}_{k}(\mathcal{N}, S) := \bigoplus_{\substack{N \in \mathcal{N}_k \\ S \in \mathrm{nbc}(\mathsf{N})}} \mathbb{Q}$$

for all $0 \leq k \leq d$. More informally, $V^{\Omega}_{\bullet}(\mathcal{N}, S)$ is obtained from $\tau(C^{\Omega}_{\bullet}(\mathcal{N}))$ by killing all terms corresponding to internal faces $\mathsf{N} \in \mathcal{N}$ for which $S \notin \mathrm{nbc}(\mathsf{N})$. The previous paragraph implies that we have an isomorphism of complexes

$$\mathrm{gr}\,\mathrm{OS}(C^{\Omega}_{\bullet}(\mathcal{N})) \;\cong \bigoplus_{S \in \mathrm{nbc}(\mathsf{M})} V^{\Omega}_{\bullet}(\mathcal{N}, S)\big[-\#S\big]. \tag{5.9}$$

Our strategy will be to prove that $V^{\Omega}_{\bullet}(\mathcal{N}, S)$ is exact, and use this to prove Theorem 5.2.4.3.

5.2.4.1 Characterizing the nbc condition

Fix a subset $S \subset E = \{1, \dots, n\}$. For each $e \in E$, let $S_e := \{s \in S \mid s > e\}$, and consider the open half-space

$$H^{+}_{e,S} := \left\{ x \in \mathbb{R}^E \mid \sum_{f \in S_e \cup \{e\}} x_f > \#S_e \right\}.$$

Lemma 5.2.4.1. *If M is a matroid on E, the following statements are equivalent:*

(i) $S \in \mathrm{nbc}(\mathsf{M})$,

(ii) $S_e \cup \{e\}$*, is independent for all* $e \in E$,

(iii) $\mathcal{P}(\mathsf{M}) \cap H^{+}_{e,S} \neq \emptyset$ *for all* $e \in E$,

(iv) $\mathcal{P}(\mathsf{M}) \cap \bigcap_{e\in E} H^+_{e,S} \neq \emptyset$.

Proof. The equivalence of (i) and (ii) is immediate from the definition of a broken circuit. We next prove the equivalence of (ii) and (iii). If $S_e \cup \{e\}$ is independent, then it is contained in some basis B, and $x_B \in \mathcal{P}(\mathsf{M}) \cap H^+_{e,S}$. Conversely, suppose that $x \in \mathcal{P}(\mathsf{M}) \cap H^+_{e,S}$. Then we have

$$\#S_e < \sum_{f\in S_e\cup\{e\}} x_f < \operatorname{rk}\big(S_e \cup \{e\}\big),$$

where the first inequality comes from the fact that $x \in H^+_{e,S}$ and the second comes from the fact that $x \in \mathcal{P}(\mathsf{M})$. This implies that the cardinality of $S_e \cup \{e\}$ is equal to its rank, which means that it is independent.

We have now established the equivalence of (i), (ii), and (iii). The fact that (iv) implies (iii) is obvious, thus we can finish the proof by showing that (ii) implies (iv). Assume that (ii) holds, and for each $e \in E$, choose a basis B_e containing $S_e \cup \{e\}$. In addition, choose real numbers $\varepsilon_0, \dots, \varepsilon_n$ with $\varepsilon_0 = 1$, $\varepsilon_n = 0$, and $\varepsilon_e < \varepsilon_{e-1}/\big(\#S_e + 1\big)$ for all $e \in E$. Let

$$x := \sum_{e\in E} (\varepsilon_{e-1} - \varepsilon_e)\, x_{B_e} \in \mathbb{R}^E.$$

The sum of the coefficients appearing in the definition of x is equal to $\varepsilon_0 - \varepsilon_n = 1$, thus x is in the convex hull of $\{x_{B_e} \mid e \in E\}$, which is contained in $\mathcal{P}(\mathsf{M})$. It thus remains only to prove that $x \in H^+_{e,S}$ for all $e \in E$. We have

$$\sum_{f\in S_e\cup\{e\}} x_f = \sum_{f\in S_e\cup\{e\}} \sum_{B_g\ni f} (\varepsilon_{g-1} - \varepsilon_g) = \sum_{f\in S_e} \sum_{B_g\ni f} (\varepsilon_{g-1} - \varepsilon_g) + \sum_{B_g\ni e} (\varepsilon_{g-1} - \varepsilon_g). \tag{5.10}$$

Note that, if $g \le f$ and $f \in S$, then $f \in S_g \cup \{g\} \subset B_g$. This implies that

$$\sum_{f\in S_e} \sum_{B_g\ni f} (\varepsilon_{g-1} - \varepsilon_g) \ge \sum_{f\in S_e} \sum_{g\le f} (\varepsilon_{g-1} - \varepsilon_g) = \sum_{f\in S_e} (\varepsilon_0 - \varepsilon_f) \ge \sum_{f\in S_e} (\varepsilon_0 - \varepsilon_e) = \#S_e(1-\varepsilon_e). \tag{5.11}$$

In addition, we have $e \in B_e$, and therefore

$$\sum_{B_g\ni e} (\varepsilon_{g-1} - \varepsilon_g) \ge \varepsilon_{e-1} - \varepsilon_e > \#S_e \varepsilon_e. \tag{5.12}$$

Combining Equations (5.10), (5.11), and (5.12), we find that

$$\sum_{f\in S_e\cup\{e\}} x_f > \#S_e(1-\varepsilon_e) + \#S_e\varepsilon_e = \#S_e,$$

and therefore $x \in H^+_{e,S}$. □

5.2.4.2 A proof of the exactness

Fix a matroid M on the ground set E, a decomposition $\mathcal{N}$ of M with orientation Ω, and a set $S \in \operatorname{nbc}(\mathsf{M})$. Our goal in this section is to use Lemma 5.2.4.1 to prove the following proposition.

Proposition 5.2.4.2. *The complex $V^\Omega_\bullet(\mathcal{N}, S)$ is exact.*

Proof. We will proceed in the same manner as the proof of Proposition 5.2.3.1.[2] As in that argument, let $d = d(\mathsf{M})$, and let $V^\Omega_{\leq d}(\mathcal{N}, S)$ be the complex obtained from $V^\Omega_\bullet(\mathcal{N}, S)$ by removing the term in degree $d+1$. Our plan is to give a topological interpretation of this complex.

For any polytope $P \subset \mathbb{R}^E$, let $\mathring{P}$ denote its relative interior. Let

$$U := \mathring{\mathcal{P}}(\mathsf{M}) \cap \bigcap_{e \in E} H^+_{e,S}.$$

Since U is an intersection of convex open subsets of $\mathcal{P}(\mathsf{M})$, it is itself a convex open subset of $\mathcal{P}(\mathsf{M})$. By Lemma 5.2.4.1, U is nonempty, therefore $(\bar{U}, \partial U) \cong (B^d, S^{d-1})$.

For all $\mathsf{N} \in \mathcal{N}$, let $U_\mathsf{N} := U \cap \mathring{\mathcal{P}}(\mathsf{N})$. Lemma 5.2.4.1 implies that $U_N \neq \emptyset$ if and only if N is an interior face and $S \in \mathrm{nbc}(\mathsf{N})$. The set U is the disjoint union of the convex open sets U_N, and adding a single 0-cell gives us a cell decomposition of the quotient $\bar{U}/\partial U$. The complex $V^\Omega_{\leq d}(\mathcal{N}, S)$ is precisely the cell complex that computes the reduced homology $\tilde{H}_\bullet(\bar{U}, \partial U) \cong \mathbb{Q}[-d]$.

The remainder of the proof is identical to the proof of Proposition 5.2.3.1. We have an exact sequence of chain complexes

$$0 \to V^\Omega_{\leq d}(\mathcal{N}, S) \to V^\Omega_\bullet(\mathcal{N}, S) \to \mathbb{Q}[-d-1] \to 0.$$

We have observed that $V^\Omega_{\leq d}(\mathcal{N}, S)$ has 1-dimensional homology concentrated in degree d, while $\mathbb{Q}[-d-1]$ has 1-dimensional homology concentrated in degree $d+1$. The boundary map in the long exact sequence in homology is an isomorphism, which implies that the homology of $V^\Omega_\bullet(\mathcal{N}, S)$ vanishes. □

Theorem 5.2.4.3. *The categorical invariant* OS *is valuative.*

Proof. We need to show that, for any matroid M on E and any decomposition $\mathcal{N}$ of M with orientation Ω, $C^\Omega_\bullet(\mathcal{N})$ is exact. By Equation (5.9) and Proposition 5.2.4.2, $C^\Omega_\bullet(\mathcal{N})$ admits a filtration whose associated graded is exact. The spectral sequence of the filtered complex has E_1 page equal to the homology of the associated graded and converges to the homology of the original complex. In this case, the E_1 page is zero, so the original complex must be exact, as well. □

[2]In the special case where M is loopless and $S = \emptyset$, Proposition 5.2.4.2 follows from Proposition 5.2.3.1.

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