Alma Mater Studiorum – Università di Bologna

DOTTORATO DI RICERCA IN

MATEMATICA

Ciclo XXXV

**Settore Concorsuale:** 01/A4

**Settore Scientifico Disciplinare:** MAT/07

# A COHOMOLOGICAL APPROACH TO RUELLE-POLLICOTT RESONANCES AND SPEED OF MIXING OF ANOSOV DIFFEOMORPHISMS

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Esame finale anno 2023

**Abstract**

Given a transitive Anosov diffeomorphism on a closed manifold it is known that, for smooth enough observables, the system is mixing w.r.t. the measure of maximal entropy. Therefore, it makes sense to investigate the speed of decay of correlations and to look for the so-called Ruelle-Pollicott resonances, in order to determine a complete asymptotics for the decay of correlations. In this thesis we are able to find the first terms of that asymptotics and to prove an estimate for the speed of decaying of correlations. The proof is based on a surprising connection between the action of a transfer operator on suitable anisotropic Banach spaces of currents and the action induced by the Anosov map on the de Rham cohomology.

# Contents

# Introduction

Hyperbolic dynamics has a long history in the field of dynamical systems. Both discrete-time and continuous-time systems consist of smooth maps, or smooth flows, which acts on a differentiable manifold and, at least in a region of that manifold, it defines an expanding and a contracting direction for the derivative. This double behavior produces very complicated orbits, so that the dynamics can be considered random, even if the system is entirely deterministic. Thus, these systems show a deterministic chaos. Anosov systems falls into the category of (uniformly) hyperbolic dynamical systems and they owe their name to D.V. Anosov [2], one of the great mathematicians, with S. Smale [69], R. Bowen [15], D.Ruelle [65], Ya. Sinai [66, 67], who gave the main initial contributions to the topic. The investigation of chaotic dynamical systems, which include Anosov systems, is generally complex, and the computation of a few single orbits, when achievable, turns out to be meaningless for the study of real phenomena. To give an idea of this, think for instance of a hyperbolic dynamical system which models a real phenomenon and assume that one wants to predict the evolution of some initial state $x_0$. The measurement of the initial state is subjected to errors, hence one gets an $\widetilde{x}_0$ which is close to $x_0$, but may not coincide with $x_0$. Since the system is chaotic, the computed evolution of $\widetilde{x}_0$ is usually far from the real orbit of $x_0$. In view of this fact, the usual approach to study these dynamical systems is a qualitative and quantitative statistical approach, through the use of ergodic theory [71]. The main statistical properties that one generally wants to study concern topological transitivity, topological mixing, minimality, ergodicity, unique ergodicity, mixing, exponential mixing, central limit theorems, etc. (see [47] for definitions). Some of these properties require to set a reference invariant probability measure, which is usually the SRB measure [74] or the measure of maximal entropy [54].

In this thesis, we deal with smooth Anosov diffeomorphisms $f \in C^\infty(M)$, acting on a Riemannian manifold $M$, with expansion factor $\lambda > 1$ and contraction factor $\lambda^{-1} < 1$(see Definition 1.1). We study the mixing property, and consequently the speed of mixing w.r.t. the measure of maximal entropy $\mu_{BM}$, also named Bowen-Margulis measure, after R. Bowen [13, 14] and G.Margulis [54], who gave two different, but equivalent, constructions of this important invariant measure (see Appendix D). For the sake of completeness, recall that the $f$-invariance of $\mu_{BM}$ means that, for any continuous function $\phi \in L^1(M, \mu_{BM})$, $\mu_{BM}(\phi \circ f) = \mu_{BM}(\phi)$. In addition, $\mu_{BM}$ is mixing, that is,

$$\lim_{n\to+\infty} \int_M (\phi \circ f^n)\psi \, d\mu_{BM} = \int_M \phi \, d\mu_{BM} \int_M \psi \, d\mu_{BM},$$

for any $\phi, \psi \in L^2(M, \mu_{BM})$. The mixing property is equivalent to the so-called decay of correlations, namely $\lim_{n\to+\infty} C_n^f(\phi, \psi) = 0$, for all $\phi, \psi \in L^2(M, \mu_{BM})$, where the correlation function is defined to be

$$C_n^f(\phi, \psi) = \int_M (\phi \circ f^n)\psi \, d\mu_{BM} - \int_M \phi \, d\mu_{BM} \int_M \psi \, d\mu_{BM}.$$

Once the decay of correlations is established, natural questions are: how fast is the convergence to the limit? Is there a complete asymptotics for the correlation function, for $n \to +\infty$? By classical results [15], the decay of correlations is exponential, at least for Hölder observables, in the sense that there exists a $\sigma \in (0, 1)$ such that $|C_n^f(\phi, \psi)| \leq C_{\phi,\psi}\sigma^n$, whenever $\phi$ and $\psi$ are Hölder functions on $M$. With the terminology "classical" we refer to all results obtained by coding the system, through the construction of a Markov partition, and then by studying spectral properties of transfer operator of the induced shift map (see for instance [5] for an overview of the transfer operator tool). This is the point of view adopted by the authors above cited. On the other hand, a "modern" approach to solve this kind of problems was introduced in [11], where the authors studied spectral properties of transfer operators, on appropriate anisotropic Banach spaces of distributions, without coding the system. This pioneering work opens the possibility to gain deeper results, principally with the work of C.Liverani, S.Gouëzel, V.Baladi, M.Tsujii et al. [6, 37, 38, 50, 52, 8, 9, 35, 36]. We point out that the construction of Gouëzel and Liverani is quite different from the one of Baladi and Tsujii. In effect, the first two give a geometric definition, obtained considering cones in tangent space, while the others adopt a dual point of view, considering cones in cotangent space, via Fourier transform, obtaining anisotropic Sobolev spaces. In this thesis, we follow the geometric construction of Gouëzel and Liverani. By using anisotropic Banach spaces, they proved in [38] the following result, that we rewrite with our setting in mind, and they answered our second question, i.e., the existence of the asymptotics for the correlation function, for $n \to +\infty$.

**Theorem 0.1.** *Let $f \in C^\infty$ be a topologically transitive Anosov diffeomorphism on $M$, with expansion, resp. contraction, factor $\lambda > 1$, resp. $\lambda^{-1} < 1$. Then there exists a unique measure of maximal entropy $\mu_{BM}$. In addition, for every $\epsilon > 0$ there exists $r \in \mathbb{N}$, with $\lambda^{-r} < \epsilon$, and a finite set $\Xi(r) = \{\xi_1, \dots, \xi_{n_\epsilon}\} \subset \mathbb{C}$, with $|\xi_i| < 1$, such that, for every $\xi_i$ there are a finite number $N_i \in \mathbb{Z}^+$ of nonzero bilinear forms $\{c_{\xi_i,k}(\cdot,\cdot)\}_{k=1}^{N_i}$, for which*

$$\int_M (\phi \circ f^n)\psi \, d\mu_{BM} = \int_M \phi \, d\mu_{BM} \int_M \psi \, d\mu_{BM} + \sum_{i=1}^{n_\epsilon} \sum_{k=1}^{N_i} \xi_i^n n^k c_{\xi_i,k}(\phi,\psi) + o(\epsilon^n), \quad (0.1)$$

*for any $\phi, \psi \in C^r(M)$. In particular, $\Xi(r') \supseteq \Xi(r)$, whenever $r' \geq r$.*

The equality (0.1) is generally called a Ruelle-Pollicott asymptotics, after D.Ruelle [63, 64] and M.Pollicott [60], and the complex numbers $\{1, \xi_1, \dots, \xi_{n_\epsilon}\}$ are called Ruelle-Pollicott resonances. We point out that (0.1) holds true for any $\epsilon > 0$ only for observables $\phi, \psi \in C^\infty(M)$. We also underline that above theorem only proves the existence of the asymptotics of the decay of correlations, but it does not give any information about the number and location of Ruelle-Pollicott resonances. For instance, it is well known that for hyperbolic automorphisms of tori (see Example 1.6), which are the easiest example of Anosov diffeomorphisms, $\Xi = \varnothing$ and the set of Ruelle-Pollicott resonances reduces to $\{1\}$. This fact can be easily proved by using Fourier analysis. On the other hand, there exist Anosov diffeomorphisms which admit nontrivial Ruelle-Pollicott resonances for the SRB measure [1, 68, 61], or for which there is an estimate on the number of resonances [45]. We warn the reader that some to these references use equivalent definitions of Ruelle-Pollicott resonances, such as the poles of dynamical zeta functions or the inverse of zeros of Ruelle-Fredholm determinants (see [7]).

Concerning the measure of maximal entropy, there are few results about the existence and location of Ruelle-Pollicott resonances and a complete asymptotics (0.1) is known only in the trivial case of hyperbolic automorphisms of tori, when the Bowen-Margulis measure coincides with the SRB measure and the Lebesgue measure. Some new ideas to face this problem arose from [20, Section 5], which, in turn, is inspired by results of [31] (see also [70] for another recent application of these ideas). In fact, the authors of [31] found the complete set of Ruelle-Pollicott resonances for linear pseudo-Anosov maps on half-translation surfaces, that are a generalization to higher genus surfaces of hyperbolic automorphisms of the 2-torus [32]. Their proof is based on the investigation of the spectrum of Koopman composition operator $\mathcal{T}\phi = \phi \circ f$ on anisotropic Banach spaces of distributions, since eigenvalues of $\mathcal{T}$ coincide with Ruelle-Pollicott resonances. Moreover, since linear pseudo-Anosov maps admit, up to a finite set of points, smooth 1-dimensional stable/unstable foliations, one can take the normalized vector field $v^s$ (resp. $v^u$), tangent to the stable (resp. unstable) foliation. Then, since in their case the Lie derivatives $L_{v^s}$ and $L_{v^u}$ map eigenvectors to eigenvectors, they are able to obtain all eigenvalues of $\mathcal{T}$ and to relate them to eigenvalues of the induced action on the first de Rham cohomology group. On the other side, the authors of [20] considered transitive (not necessarily linear) Anosov diffeomorphisms of the 2-torus. They constructed anisotropic Banach spaces of currents (the 2-dimensional version of the spaces used in these thesis), obtained as the closure of differential forms w.r.t. a suitable norm, and looked for eigenvalues of the pushforward operator. Unlike [31], the role of the Lie derivative in this case is taken by the exterior derivative. By relating the spectrum of the pushforward operator on 1-currents and the induced action on the first de Rham cohomology group, they proved that there are no Ruelle-Pollicott resonances in the annulus $\{z \in \mathbb{C}| \ e^{-h_{top}} \leq z < 1\}$, for the measure of maximal entropy ($h_{top}$ represents the topological entropy of the system, see Appendix D). As a consequence, they obtained the following result for the speed of mixing relative to the measure of maximal entropy.

**Theorem 0.2.** *There exist $r \in \mathbb{N}$, $C > 0$ and $\kappa \in (0, 1)$ such that, for any couple of observables $\phi, \psi \in C^\infty(\mathbb{T}^2)$,*

$$\left| \int_{\mathbb{T}^2} (\phi \circ f^n)\psi \ d\mu_{BM} - \int_{\mathbb{T}^2} \phi \ d\mu_{BM} \int_{\mathbb{T}^2} \psi \ d\mu_{BM} \right| \leq C\|\phi\|_{C^r}\|\psi\|_{C^r}(\kappa e^{-h_{top}})^n$$

The aim of this thesis is to extend [20] to include every transitive Anosov diffeomorphism on manifolds of every dimension. Notice that every known example of Anosov diffeomorphism is topologically transitive and it is conjectured that this is always the case (see Remark 1.14). Accordingly, the following main theorem we prove in this thesis is currently the analogous, except for a small detail (see Remark 0.3), of Theorem 0.2.

**Theorem** (Main Theorem). *Let $M$ be an orientable, closed (compact and without boundary), connected Riemannian manifold and let $f\colon M \to M$ be a $C^\infty$ a topologically transitive Anosov diffeomorphism on $M$ with expanding, resp. contracting, factor $\lambda > 1$, resp. $\lambda^{-1} < 1$. Let $\theta = \max\{|\Lambda_2|, \lambda^{-1}e^{h_{top}}\}$, where $\Lambda_2$ is the second largest eigenvalue of the induced action $f_\#$ on the $d_s$-de Rham cohomology group $H^{d_s}_{dR}(M)$ ($d_s$ is the dimension of stable manifolds). Then, there exist $r \in \mathbb{N}$ and $C > 0$ such that, for every couple of observables $\phi, \psi \in C^\infty(M)$,*

$$\left| \int_M (\phi \circ f^n)\psi \; d\mu_{BM} - \int_M \phi \; d\mu_{BM} \int_M \psi \; d\mu_{BM} \right| \leq C\theta^n e^{-nh_{top}} \|\phi\|_{C^r} \|\psi\|_{C^r}$$

We actually obtain the following stronger result.

**Theorem** (Strong Theorem). *Let $M$ be an orientable, closed (compact and without boundary), connected Riemannian manifold and let $f\colon M \to M$ be a $C^\infty$ a topologically transitive Anosov diffeomorphism on $M$ with expanding, resp. contracting, factor $\lambda > 1$, resp. $\lambda^{-1} < 1$. Let $\{\Lambda_1 = e^{h_{top}}, \Lambda_2, \dots, \Lambda_m\}$ be the set of eigenvalues of the induced action $f_\#$ on $H^{d_s}_{dR}(M)$ such that $|\Lambda_i| > \lambda^{-1}e^{h_{top}}$, for any $i = 1, \dots, m$. Then, there exist $r \in \mathbb{N}$, $C > 0$ and, for any $i = 2, \dots, m$, there exist $N_i \in \mathbb{N}$ and nonzero bilinear forms $\{c_{\Lambda_i,k}(\cdot,\cdot)\}_{k=0}^{N_i-1}$, such that*

$$\begin{aligned} \Bigg| \int_M (\phi \circ f^n)\psi \; d\mu_{BM} - &\int_M \phi \; d\mu_{BM} \int_M \psi \; d\mu_{BM} - \\ &- \sum_{i=2}^{m} \sum_{k=0}^{N_i-1} (\Lambda_i e^{-h_{top}})^n n^k c_{\Lambda_i,k}(\phi,\psi) \Bigg| \leq C\lambda^{-n} \|\phi\|_{C^r} \|\psi\|_{C^r} \end{aligned}$$

*for every couple of observables $\phi, \psi \in C^\infty(M)$.*

The strong theorem shows that Ruelle-Pollicott resonances larger than $\lambda^{-1}$, as well as theirs multiplicities, are completely determined by the action induced by $f$ on de Rham cohomology. On the other hand, we are not aware of any example for which $|\Lambda_2| \geq \lambda^{-1}e^{h_{top}}$. In particular, assuming $f$ to be topologically conjugated to a hyperbolic automorphism of a torus, a property which is satisfied in many cases (see Proposition 1.18), we obtain the following corollary.

**Corollary** (Main Corollary). *Let $f\colon M \to M$ be a $C^\infty$ Anosov diffeomorphism of a torus $M$ with expanding, resp. contracting, factor $\lambda > 1$, resp. $\lambda^{-1} < 1$. Then, there exist $r \in \mathbb{N}$ and $C > 0$ such that, for every couple of observables $\phi, \psi \in C^\infty(M)$,*

$$\left| \int_M (\phi \circ f^n)\psi d\mu_{BM} - \int_M \phi d\mu_{BM} \int_M \psi d\mu_{BM} \right| \leq C\lambda^{-n} \|\phi\|_{C^r} \|\psi\|_{C^r} \tag{0.2}$$

*Moreover, when $M = \mathbb{T}^2$,*

$$\left| \int_{\mathbb{T}^2} (\phi \circ f^n)\psi d\mu_{BM} - \int_{\mathbb{T}^2} \phi d\mu_{BM} \int_{\mathbb{T}^2} \psi d\mu_{BM} \right| \leq Ce^{-nh_{top}} \|\phi\|_{C^r} \|\psi\|_{C^r} \tag{0.3}$$

**Remark 0.3.** As a consequence of this corollary, there are no Ruelle-Pollicott resonances in the annulus $\{z \in \mathbb{C} \mid \lambda^{-1} < |z| < 1\}$ ($\{z \in \mathbb{C} \mid e^{-h_top} < |z| < 1\}$ in the 2-dimensional case). Notice that the bound of (0.3) is stronger than (0.2), since $e^{h_{top}} \geq \lambda$. We point out that the authors of [20] proved a slightly stronger result for Anosov diffeomorphisms of the 2-torus. In effect, they showed that there are no Ruelle-Pollicott resonances in the annulus $\{z \in \mathbb{C} \mid e^{-h_{top}} \leq |z| < 1\}$, excluding values of modulus $|e^{-h_{top}}|$. This is a peculiarity of the 2-dimensional case and Remark 3.26 gives an idea of the proof and why it cannot be generalized to higher-dimensional cases.

The thesis is organized as follows. In Chapter 1, we firstly recall some basic concepts of Anosov diffeomorphisms, we give the setup of the problem and we restate our main results. In Chapter 2, we introduce a suitable family $\mathcal{B}^{p,q,l}$ of anisotropic Banach spaces of currents. They are obtained as the closure of spaces of differential forms on $M$ w.r.t. an appropriate anisotropic norm. Anisotropic means that this norm encodes different behaviors along stable and unstable subbundles. In particular, the elements of our Banach spaces behave as differential forms along the unstable subbundle, while they behave as currents (the dual of differential forms) along the stable subbundle. As already specified, anisotropic spaces have been largely used in the last twenty years, starting with [11], and there are nowadays many different versions. The ones that inspired our spaces are taken by [37, 38, 36], and coincide with the spaces used in [20] for the case of the 2-torus. Secondly, we prove that the pushforward operator $f_*$, i.e., the suitable transfer operator to get our results, is quasi-compact. Quasi-compact means that the spectrum of $f_*$ on $\mathcal{B}^{p,q,l}$ is made of a finite set of eigenvalues of finite multiplicity out of a small ball of radius slightly larger than the essential spectrum. Accordingly, we may consider the action of $f_*$, limited to eigenspaces corresponding to the largest eigenvalues, as the action of a matrix on a finite-dimensional vector space. We exploit Hennion's Theorem 2.9 to do it. Chapter 3 contains the key part of the thesis and it is dedicated to the investigation of the spectrum of $f_*$ on $\mathcal{B}^{p,q,l}$. For later use, we are interested in the spectrum of $f_*$ when acting on $\mathcal{B}^{p,q,d_s}$, where $d_s$ is the dimension of the stable subbundle. The first section of the chapter contains an adaptation to our setting of some results in [38]. In particular, we prove that $e^{h_{top}}$ is the unique maximal eigenvalue of $f_*|_{\mathcal{B}^{p,q,d_s}}$, it is simple and the corresponding eigenvector, joined to its dual eigenvector, defines the measure of maximal entropy. The second and third sections include the cohomology aspects of our reasoning and contain the main original contributions. We relate part of the spectrum of $f_*|_{\mathcal{B}^{p,q,d_s}}$ to the spectrum of the induced action $f_\#$ on the anisotropic de Rham cohomology $\widetilde{H}_{dR}^{p,q,d_s}$, obtained as the quotient of closed $d_s$-currents w.r.t. exact currents in $\mathcal{B}^{p,q,d_s}$. The next point consists in the proof that the anisotropic de Rham cohomology is isomorphic to the standard de Rham cohomology. It is difficult, at least to us, to prove directly that the two vector spaces are isomorphic (see Remark 3.24). Hence, in Section 3.3, we define a new family of anisotropic Banach spaces of currents $\mathcal{C}^{p,q,l}$, which are an intermediate version of the spaces $\mathcal{B}^{p,q,l}$, that is $\mathcal{B}^{p+1,q-1,l} \subseteq \mathcal{C}^{p,q,l} \subseteq \mathcal{B}^{p,q,l}$. Once we have these spaces, we are able to define the anisotropic de Rham cohomology $\bar{H}_{dR}^{p,q,l}(M)$, which turns out to be isomorphic to the standard de Rham cohomology. In conclusion, we can study the action on the standard de Rham cohomology to get information about the discrete spectrum of $f_*$ on $\mathcal{B}^{p,q,d_s}$, or equivalently on $\mathcal{C}^{p,q,d_s}$. The last section of Chapter 3 contains the proofs of the main results above stated. In Appendix A and Appendix B we recall some tools of functional analysis and Hodge theory that we use in the thesis. Appendix C contains the proof of some technical results, while Appendix D contains a basic overview of entropy theory in dynamical systems.

We conclude this introduction by saying that the thesis is almost self-contained. For the few results that are stated without proof, we indicate direct references.

# Chapter 1

# Setup and results

In this chapter we first recall some basic notions of Anosov diffeomorphisms, then we state the assumptions regarding the dynamics we are going to study and our main result.

## 1.1 A survey about Anosov diffeomorphisms

This section is a brief reminder of the main properties of Anosov diffeomorphisms. For a complete introduction to this topic we refer to the original monograph of D.V. Anosov [2], which collects most of the oldest known results about these dynamical systems. We also suggest the following more recent references [17, Chapter 5] and [47, Part 4]. We also rewrite the proof of some results, while for the others we just mention the reference.

**Definition 1.1.** *Let $M$ be a $C^1$ Riemannian manifold and let $f\colon M \to M$ be a $C^1$ diffeomorphism on $M$. The discrete dynamical system defined by repeated iterations of $f$ is said to be Anosov if $M$ is a hyperbolic set. This means that the tangent bundle $TM$ splits into two subbundles*

$$TM = E^s \oplus E^u$$

*such that:*

1. *both subbundles $E^{s/u}$ are invariant under the action of $df$, i.e.,*

$$d_x f E^{s/u}_x = E^{s/u}_{f(x)} \quad \text{for all } x \in M;$$

*2. there exist real constants $c > 0$ and $\lambda > 1$ such that, for all $n \in \mathbb{N}$,*

$$\begin{aligned} \|d_x f^{-n} v\| &> c\lambda^n \|v\| \qquad \textit{for all } v \in E^s_x, \\ \|d_x f^{n} v\| &> c\lambda^n \|v\| \qquad \textit{for all } v \in E^u_x, \end{aligned} \tag{1.1}$$

*where the norm is induced by the metric $g$ of $M$.*

*$E^s$ and $E^u$ are called the stable and the unstable subbundles, respectively.*

**Lemma 1.2.** *The subbundles $E^{s/u}$ depends continuously on $x \in M$. Accordingly, if $M$ is connected, then $\dim(E^s)$ and $\dim(E^u)$ are constant.*

PROOF. Let $\{x_k\}_{k\in\mathbb{Z}^+}$ be a sequence converging to $x_0$. Up to subsequences, we can assume that $\dim(E^s_{x_k}) = m$ is constant. Let $v_{k,1}, \ldots v_{k,m}$ be an orthonormal basis of $E^s_{x_k}$. Since the unit tangent bundle $T^1M$ is compact, up to considering subsequences, we obtain an orthonormal basis $v_{0,1}, \ldots, v_{0,m}$ of $E^s_{x_0}$, which satisfies the first condition of (1.1), since $df$ is continuous. In addition, $\dim(E^s_{x_0}) \geq m = \dim(E^s_{x_k})$. Repeating the argument for $E^u$ on a subsequence of $\{x_k\}$, one obtains that $\dim(E^u_{x_0}) \geq \dim(E^u_{x_k})$. But, $\dim(E^s_{x_0}) + \dim(E^u_{x_0}) = \dim(M)$, hence $E^s$ and $E^u$ depends continuously on $x$ and, if $M$ is connected, their dimension is constant.

Q.E.D.

The following lemma shows that there exists an equivalent adapted metric $\bar{g}$, such that (1.1) holds for $c = 1$.

**Lemma 1.3** ([55, Mather]). *For every $\widetilde{\lambda} \in (1, \lambda)$ and $\epsilon > 0$, there exists a metric $\bar{g}$, equivalent to $g$, such that (1.1) holds true for $c = 1$ and for $\lambda$ replaced by $\widetilde{\lambda}$. Moreover,*

$$\bar{g}(v_s, v_u) < \epsilon$$

*for all $v_s \in E^s$ and $v^u \in E^u$.*

PROOF. Fix a constant $\widetilde{\lambda} \in (1, \lambda)$ and consider $K \in \mathbb{N}$ such that $c(\lambda/\widetilde{\lambda})^K > 1$. Let $v, w \in T_xM$ and assume $v = v_s + v_u, w = w_s + w_u \in E^s_x \oplus E^u_x$ is the unique splitting into stable and unstable components. We define

$$\widetilde{g}_x(v, w) = \widetilde{g}_x(v_s, w_s) + \widetilde{g}_x(v_u, w_u),$$

where

$$\widetilde{g}_x(v_s, w_s) = \sum_{k=0}^{K-1} \widetilde{\lambda}^{-k} g_{f^{-k}(x)}(d_x f^{-k} v_s, d_x f^{-k} w_s),$$

$$\widetilde{g}_x(v_u, w_u) = \sum_{k=0}^{K-1} \widetilde{\lambda}^{-k} g_{f^k(x)}(d_x f^k v_u, d_x f^k w_u).$$

In order to simplify the notation, from now on we drop the dependence on $x \in M$. Given $v_s \in E^s$ and $v_u \in E^u$ their induced norms are

$$|v_s|^2 = \sum_{k=0}^{K-1} \|\widetilde{\lambda}^{-k} df^{-k} v_s\|^2, \quad |v_u|^2 = \sum_{k=0}^{K-1} \|\widetilde{\lambda}^{-k} df^{k} v_u\|^2.$$

Thus, for $v_s \in E^s$,

$$\begin{aligned}
&|df^{-1} v_s|^2 = \sum_{k=0}^{K-1} \|\widetilde{\lambda}^{-k} df^{-k-1} v_s\|^2 = \widetilde{\lambda}^2 \sum_{j=1}^{K} \|\widetilde{\lambda}^{-j} df^{-j} v_s\|^2 = \\
&= \widetilde{\lambda}^2 (|v_s|^2 - \|v_s\|^2 + \|\widetilde{\lambda}^{-K} df^{-K} v_s\|^2) > \widetilde{\lambda}^2 (|v_s|^2 - \|v_s\|^2 + c(\lambda \widetilde{\lambda}^{-1})^K \|v_s\|) > \\
&> \widetilde{\lambda}^2 |v_s|^2
\end{aligned}$$

Similarly, one can prove that $|df v_u|^2 > \widetilde{\lambda}^2 |v_u|^2$ for $v_u \in E^u$. Notice that $\widetilde{g}$ is a continuous but generally not smooth metric, because it depends on the splitting of the tangent bundle into stable/unstable subbundles which is generally not smooth. Moreover, we remark that $E^s$ and $E^u$ are orthogonal with respect to $\widetilde{g}$. Finally, using classical results of differential geometry [43], given a small $\epsilon > 0$, one can approximate $\widetilde{g}$ with a smooth adapted metric $\bar{g}$, which is hyperbolic with constant $\widetilde{\lambda} - \epsilon$ and with the angle between stable and unstable subbundles uniformly bounded by $\epsilon$. Q.E.D.

An equivalent definition of Anosov diffeomorphisms involves invariant cones. Recall that a subset $K$ of a vector space is a cone if $cv \in K$, for any vector $v \in K$, and for any constant $c$. When $f$ is an Anosov diffeomorphism, we can write any vector $v_x \in T_x M$, in a unique way, as a sum $v_x = v_x^s + v_x^u$, with $v_x^s \in E_x^s$ and $v_x^u \in E_x^u$. Accordingly, for any $\alpha \in (0,1)$, we define the following families of stable/unstable cones:

$$\begin{aligned}
\mathcal{C}_x^{s,\alpha} &= \{v \in T_x M |\ v = v^s + v^u,\ \|v^u\| \leq \alpha \|v^s\|\}, \\
\mathcal{C}_x^{u,\alpha} &= \{v \in T_x M |\ v = v^s + v^u,\ \|v^s\| \leq \alpha \|v^u\|\}
\end{aligned} \tag{1.2}$$

**Proposition 1.4.** *For any $\alpha \in (0,1)$ and for any $x \in M$*

$$d_x f^{-1} \mathcal{C}_x^{s,\alpha} \subseteq int(\mathcal{C}_{f^{-1}(x)}^{s,\alpha}) \cup \{0\}, \ d_x f \mathcal{C}_x^{u,\alpha} \subseteq int(\mathcal{C}_{f(x)}^{u,\alpha}) \cup \{0\}.$$

*Moreover, for any $\delta \in (0, \lambda)$, there exists $\alpha \in (0, 1)$ such that*

$$\|d_x f^{-n} v\| > (\lambda - \delta)^n \|v\| \qquad \text{if } v \in \mathcal{C}_x^{s,\alpha},$$
$$\|d_x f^{n} v\| > (\lambda - \delta)^n \|v\| \qquad \text{if } v \in \mathcal{C}_x^{u,\alpha}.$$

PROOF. Let $v \in \mathcal{C}_x^{s,\alpha}$ be a nonzero vector in the stable cone and let $v = v^u + v^s$ be the decomposition of $v$ along the unstable and stable directions with $\|v^u\| \leq \alpha \|v^s\|$. Accordingly, $d_x f^{-1} v = d_x f^{-1} v^u + d_x f^{-1} v^s$ and, by using (1.1), we obtain that

$$\|d_x f^{-1} v^u\| < \lambda^{-1} \|v^u\| \leq \lambda^{-1} \alpha \|v^s\| < \lambda^{-2} \alpha \|d_x f^{-1} v^s\|.$$

We conclude that $d_x f^{-1} v \in int(\mathcal{C}^{s,\alpha}_{f^{-1}(x)})$. A similar computation shows the second inclusion.

Let us prove the second part of the statement. By using the inclusions we have just proved, we can limit to consider the case $n = 1$. Let $v = v^u + v^s \in \mathcal{C}_x^{s,\alpha}$ be as above. Then

$$\|d_x f^{-1} v\| > \lambda \|v^s\| - \lambda^{-1} \|v^u\| \geq (\lambda - \lambda^{-1}\alpha) \|v^s\| \geq \left( \frac{\lambda - \lambda^{-1}\alpha}{1 + \alpha} \right) \|v\|,$$

where we used that $\|v^u\| \leq \alpha \|v^s\|$ and $\|v\| \leq (1 + \alpha)\|v^s\|$. By setting

$$\alpha \leq \frac{\delta}{\lambda + \lambda^{-1} - \delta},$$

we obtain that

$$\frac{\lambda - \lambda^{-1}\alpha}{1 + \alpha} \geq \lambda - \delta,$$

which concludes the proof of the first inequality. The second one can be proved with a similar argument. Q.E.D.

Next proposition shows that one can actually define Anosov diffeomorphisms using families of stable/unstable cones.

**Proposition 1.5.** *Let $f$ be a $C^1$ diffeomorphism on the Riemannian manifold $M$. Let us suppose that there exists $\alpha > 0$ for which there are two continuous bundles $\bar{E}^{s/u}$ that define two families of cones $\mathcal{C}_x^{s/u,\alpha}$, as in (1.2). Assume that*

1. $d_x f^{-1} \mathcal{C}_x^{s,\alpha} \subseteq int(\mathcal{C}^{s,\alpha}_{f^{-1}(x)}) \cup \{0\}$ *and* $d_x f \mathcal{C}_x^{u,\alpha} \subseteq int(\mathcal{C}^{u,\alpha}_{f(x)}) \cup \{0\}$;
2. $\|d_x f v\| < \|v\|$ *if* $v \in \mathcal{C}_x^{s,\alpha} \setminus \{0\}$ *and* $\|d_x f^{-1} v\| < \|v\|$ *if* $v \in \mathcal{C}_x^{u,\alpha} \setminus \{0\}$.

*Then $f$ is an Anosov diffeomorphism.*

PROOF. Since the unit tangent bundle $T^1M$ is compact, there exists $\lambda > 1$ such that

$$\begin{aligned} \|d_x f^{-1} v\| &> \lambda \|v\| \qquad \text{if } v \in \mathcal{C}_x^{s,\alpha} \\ \|d_x f v\| &> \lambda \|v\| \qquad \text{if } v \in \mathcal{C}_x^{u,\alpha} \end{aligned}$$

Next, define

$$E_x^s = \bigcap_{k\in\mathbb{N}} d_{f^k(x)} f^{-k} \mathcal{C}^{s,\alpha}_{f^k(x)} \text{ and } E_x^u = \bigcap_{k\in\mathbb{N}} d_{f^{-k}(x)} f^{k} \mathcal{C}^{u,\alpha}_{f^{-k}(x)}.$$

Since $E_x^s$, resp. $E_x^u$, belong to the stable, resp. unstable, cone $\mathcal{C}_x^{s,\alpha}$, resp. $C_x^{u,\alpha}$, the condition (1.1) is verified. In addition, if $v \in E_x^s \cap E_x^u$, then $v \in \mathcal{C}_x^{s,\alpha} \cap \mathcal{C}_x^{u,\alpha}$, which implies $v = 0$. Since the dimensions of $E^s$ and $E^u$ coincide with the dimensions of $\bar{E}^s$ and $\bar{E}^u$, respectively, we conclude that $TM = E^s \oplus E^u$.

Q.E.D.

**Example 1.6** (hyperbolic automorphisms of tori). *Let $M = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the $d$-dimensional torus, with $d \geq 2$. Given a matrix $A \in SL(d,\mathbb{Z})$ with no eigenvalues of modulus 1, the action of $A$ on $\mathbb{R}^d$ induces a hyperbolic automorphism of $\mathbb{T}^d$*

$$\begin{aligned} f \colon \mathbb{T}^d &\to \mathbb{T}^d \\ x &\mapsto Ax \mod \mathbb{Z} \end{aligned}$$

*This is the easiest example of Anosov diffeomorphism, whose stable (resp. unstable) subspace is the direct sum of generalized eigenspaces corresponding to the eigenvalues of modulus smaller (resp. greater) than 1.*

Hyperbolic toral automorphisms do not exhaust the set of Anosov diffeomorphisms. In effect, we have the following

**Proposition 1.7.** *[17, Corollary 5.5.2] The Anosov property is an open condition in the space of smooth diffeomorphisms with respect to the $C^1$ topology.*

**Example 1.8.** *Consider the linear automorphism of $\mathbb{T}^2$*

$$\begin{aligned} f_0 \colon \mathbb{T}^2 &\to \mathbb{T}^2 \\ \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto \begin{bmatrix} 2x+y \\ x+y \end{bmatrix}, \end{aligned}$$

*Given $\delta \in (0,1)$, let $f_\delta$ be the family of $C^\infty$ diffeomorphisms*

$$\begin{aligned} f_\delta \colon \mathbb{T}^2 &\to \mathbb{T}^2 \\ \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto \begin{bmatrix} 2x + y - \frac{\delta}{2\pi}\sin(2\pi x) \\ x + y - \frac{\delta}{2\pi}\sin(2\pi x) \end{bmatrix}, \end{aligned}$$

*One can easily check that $\|f_\delta - f_0\|_{C^1} \leq C\delta$, hence, by Proposition 1.7, there exists $\bar{\delta} > 0$ such that, for every $\delta \in (0, \bar{\delta})$, $f_\delta$ is a nonlinear Anosov diffeomorphism. To learn more about this nonlinear examples see for instance [35, 49, 51].*

In addition, the following result also holds

**Proposition 1.9.** *[17, Corollary 5.5.4] Anosov diffeomorphisms are structurally stable (a diffeomorphism $f$ of $M$ is structurally stable if, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for any other diffeomorphism $g$, with $\|f - g\|_{C^1} < \delta$, there exists a homeomorphism $h$ of $M$ for which $h \circ f = g \circ h$ and $\|h - \mathrm{id}\|_{C^0} < \epsilon$).*

Proposition 1.7 and Proposition 1.9 imply that, if $f$ is an Anosov diffeomorphism and $g$ is $C^1$ close to $f$, then $g$ is Anosov and $f$ is topologically conjugated to $g$ (see Definition 1.17), with a conjugacy homeomorphism $h$ close to the identity.

An Anosov diffeomorphism defines a geometric structure on $M$ which is summarized by the following stable/unstable manifold theorem.

**Theorem 1.10.** *[17, Theorem 5.6.4, Proposition 5.6.5, Corollary 5.6.6, Proposition 5.9.1] Let $f$ be a $C^r$ Anosov diffeomorphism, with $r \geq 1$ or $r = \infty$. Assume that $\lambda > 1$ is the expansion factor and that the metric $g$ is adapted. Then there exist $\epsilon > 0$ and $\delta > 0$ such that, for any $x \in M$,*

1. *denoting by $d$ the distance induced by the metric $g$, the local stable manifolds*

   $$W^s_\epsilon(x) = \{y \in M |\ d(f^n(x), f^n(y)) < \epsilon \text{ for any } n \geq 1\}$$

   *and the local unstable manifolds*

   $$W^u_\epsilon(x) = \{y \in M |\ d(f^{-n}(x), f^{-n}(y)) < \epsilon \text{ for any } n \geq 1\}$$

   *are $C^r$ embedded disks.*

2. *$T_y W^s_\epsilon(x) = E^s(y)$, if $y \in W^s_\epsilon(x)$, and $T_y W^u_\epsilon(x) = E^u(y)$, if $y \in W^u_\epsilon(x)$.*

3. *$f(W^s_\epsilon(x)) \subseteq W^s_\epsilon(f(x))$ and $f^{-1}(W^u_\epsilon(x)) \subseteq W^u_\epsilon(f^{-1}(x))$.*

4. *Let $d^s$, resp. $d^u$, be the distance induced by $d$ on $W^s_\epsilon(x)$, resp. $W^u_\epsilon(x)$. Then*
$$d^s(f^{-1}(y), f^{-1}(z)) > \lambda d^s(y, z),$$
*when $y, z \in W^s_\epsilon(x)$, and*
$$d^u(f(y), f(z)) > \lambda d^u(y, z),$$
*when $y, z \in W^u_\epsilon(x)$.*

5. *Denote by $\exp_p\colon T_pM \to M$ the exponential map at $p$. If $0 < d(x, y) < \epsilon$ and $\exp_x^{-1}(y)$ belongs to the stable cone $\mathcal{C}^{s,\alpha}_x$, then*
$$d(f^{-1}(x), f^{-1}(y)) > \lambda d(x, y)$$
*If $0 < d(x, y) < \epsilon$ and $\exp_x^{-1}(y)$ belongs to the unstable cone $\mathcal{C}^{u,\alpha}_x$, then*
$$d(f(x), f(y)) > \lambda d(x, y)$$

6. *If $y \in W^s_\epsilon(x)$, there exists $\epsilon'$ such that $W^s_{\epsilon'}(y) \subseteq W^s_\epsilon(x)$. If $y \in W^u_\epsilon(x)$, there exists $\epsilon'$ such that $W^u_{\epsilon'}(y) \subseteq W^u_\epsilon(x)$.*

7. *For every $\epsilon > 0$ there exists $\delta > 0$ such that, if $d(x, y) < \delta$, then $W^s_\epsilon(x)$ and $W^u_\epsilon(y)$ are uniformly transversal and $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of a single point, denoted by $[x, y]$.*

8. *The global stable manifolds*
$$W^s(x) = \{y \in M |\ \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0\}$$
*and the global unstable manifolds*
$$W^u(x) = \{y \in M |\ \lim_{n\to+\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}$$
*are $C^r$ immersed manifolds. In particular, there exists $\bar\epsilon > 0$ such that, for any $\epsilon \in (0, \bar\epsilon)$*
$$W^s(x) = \bigcup_{n=0}^{+\infty} f^{-n}(W^s_\epsilon(f^n(x))), \quad W^u(x) = \bigcup_{n=0}^{+\infty} f^{n}(W^u_\epsilon(f^{-n}(x)))$$
*As a consequence, $d^s(f^{-1}(x), f^{-1}(y)) > \lambda d^s(x, y)$, whenever $y \in W^s(x)$, and $d^u(f(x), f(y)) > \lambda d^u(x, y)$, whenever $y \in W^u(x)$. $f(W^s(x)) = W^s(f(x))$ and $f(W^u(x)) = W^u(f(x))$.*

**Remark 1.11.** Notice that global stable/unstable manifolds define two transversal foliations, called the stable/unstable foliation. We point out that, despite each leaf of the stable/unstable foliation has the same degree of smoothness as $f$, the foliation itself is generally only Hölder [47, Theorem 19.1.6].

We recall some basic concepts of topological dynamics.

**Definition 1.12.** *A topological dynamical system $f\colon X \to X$, where $f$ is a homeomorphism of a topological space $X$, is topologically transitive if, for any $U$, $V$ nonempty open sets, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$. When $X$ is a compact metric space, topological transitivity is equivalent to the existence of a dense orbit [71, Theorem 5.8], i.e., there exists $x \in X$ such that $\overline{\{f^n(x) |\ n \in \mathbb{Z}\}} = M$. $f$ is topologically mixing if, for any $U$, $V$ nonempty open sets, there exists $n_0 > 0$ such that $f^n(U) \cap V \neq \varnothing$, for any $n > n_0$.*

For Anosov diffeomorphisms the following equivalence holds.

**Proposition 1.13.** *[17, Theorem 5.10.3] Let $f$ be an Anosov diffeomorphism on a connected manifold $M$. Then the following properties are equivalent.*

- *Every unstable manifold is dense in $M$;*
- *Every stable manifold is dense in $M$;*
- *$f$ is topologically transitive;*
- *$f$ is topologically mixing.*

**Remark 1.14.** Every known example of Anosov diffeomorphism is topologically transitive, hence Proposition 1.13 applies. In effect, it is conjectured that every Anosov diffeomorphism is topologically transitive.

Up to now, we only gave examples of Anosov diffeomorphisms on tori (Example 1.6, Example 1.8). On the other hand, there are also other manifolds (i.e., manifolds not homeomorphic to tori) which admit Anosov diffeomorphism. The following example by Smale should be the first appearance of nontoral Anosov diffeomorphism.

**Example 1.15.** *[69, Section I.3.] Let $G_1$, $G_2$ two copies of the Heisenberg Lie group. Hence, the Lie Algebra $\mathfrak{g}_i$, $i = 1, 2$, is generated by $\{X_i, Y_i, Z_i\}$ with $[X_i, Y_i] = Z_i$ and*

$[X_i, Z_i] = [Y_i, Z_i] = 0$. *Let us consider* $G = G_1 \times G_2$ *and let us define a hyperbolic automorphism of* $f$ *on* $G$ *by defining the action of* $df$ *on* $\mathfrak{g}_i$, $i = 1, 2$. *Set* $\lambda = 2 + \sqrt{3}$ *and assume that*

$$\begin{aligned} df(X_1) &= \lambda X_1 & \qquad df(X_2) &= \lambda^{-1} X_2 \\ df(Y_1) &= \lambda^2 Y_1 & \qquad df(Y_2) &= \lambda^{-2} Y_2 \\ df(Z_1) &= \lambda^3 Z_1 & \qquad df(Z_2) &= \lambda^{-3} Z_2 \end{aligned}$$

*Recall that every element of* $G_i$, *resp.* $\mathfrak{g}_i$, *can be represented as a* $3 \times 3$ *matrix*

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \ \textit{resp.} \ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix},$$

*where* $x, y, z \in \mathbb{R}$. *Let* $\mathbb{Q}(\sqrt{3})$ *be the field of rational numbers extended with* $\sqrt{3}$ *and let* $\sigma\colon \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt{3})$ *be the nontrivial Galois automorphism. Let* $\mathfrak{h}$ *be the subgroup of* $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ *containing the* $6 \times 6$ *matrices*

$$\begin{pmatrix} 0 & x & z & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(x) & \sigma(z) \\ 0 & 0 & 0 & 0 & 0 & \sigma(y) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathfrak{h}$ *is a lattice of* $\mathfrak{g}$, *while* $H = \exp(\mathfrak{h})$, *which is defined through the exponential map, is a uniform discrete subgroup of* $G$. *The quotient* $M = G/H$ *is a manifold, actually a nilmanifold, i.e, the quotient of a nilpotent Lie group w.r.t. a uniform lattice. Since* $f$ *preserves* $H$, *it induces an Anosov action on* $M$. *Finally, by topological reasons,* $M$ *cannot be homeomorphic to a torus.*

**Remark 1.16.** Notice that tori are particular examples of nilmanifolds, i.e., they are the quotient of the nilpotent Lie group $\mathbb{R}^n$ w.r.t. the lattice $\mathbb{Z}^n$. Accordingly, nilmanifolds Anosov diffeomorphisms include toral Anosov diffeomorphisms. There are also other manifolds, called infranilmanifolds, that are not nilmanifolds, but are finitely covered by a nilmanifold, and that could support this dynamical systems. We do not get into details and refer to [42, 24, 25]. We only mention that these algebraic manifolds cover all known examples of Anosov diffeomorphisms. In fact, it is conjectured [69] that there are no other manifolds, in addition to infranilmanifolds, which admit Anosov diffeomorphisms. To conclude, we specify that there are infranilmanifolds which do not admit Anosov diffeomorphisms and there is a lot of recent literature about the problem of classifying infranilmanifolds which do (see for instance [26, 48, 28]).

In view of Remark 1.16, every known manifold admitting Anosov diffeomorphism is of algebraic nature. Accordingly, one can define hyperbolic automorphisms of infranilmanifolds, which generalize the concept of hyperbolic automorphisms on tori. Thus, given an infranilmanifold $M$ admitting Anosov diffeomorphisms, there are examples of linear hyperbolic invertible maps on $M$, which can be easily studied, because of linearity (Example 1.15 is a hyperbolic automorphism of a 6-dimensional nilmanifold). Nonlinear Anosov diffeomorphisms are, of course, much more complicated, but, in many cases, one can relate it to linear cases. In effect, let us recall the following definition.

**Definition 1.17.** *Let $f_i\colon M_i \to M_i$, $i = 1, 2$, be two (at least continuous) invertible dynamical systems. $f_1$ is topologically conjugated to $f_2$ if there exists a homeomorphism $h\colon M_1 \to M_2$, which makes the following*

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_1 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ M_2 & \xrightarrow{f_2} & M_2 \end{array}$$

*a commutative diagram. Consequently, $M_1$ and $M_2$ are homeomorphic.*

By classical results of Franks and Newhouse we have the following proposition.

**Proposition 1.18.** *The following statements hold true.*

1. *Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be an Anosov diffeomorphism of the torus $\mathbb{T}^d$. Then $f$ is topologically conjugated to a hyperbolic automorphism of $\mathbb{T}^d$ [34].*

2. *Let $f\colon M \to M$ be a codimension 1 Anosov diffeomorphism of $M$, i.e., assume that the dimension of the stable or unstable subbundle is 1. Then $f$ is topologically conjugated to a hyperbolic automorphism of the torus $\mathbb{T}^{\dim(M)}$. Accordingly, $M$ is homeomorphic to a torus [33, 58].*

Franks' theorem (statement *1.* above) was extended by Manning [53] to include Anosov diffeomorphisms on infranilmanifold. Only recently [25], it was discovered that there is a mistake in Manning's proof, because he makes use of a lemma by Auslander [3], which in turn is false. What remains true of Manning's paper is the following proposition.

**Proposition 1.19.** *Let $f\colon M \to M$ be an Anosov diffeomorphism of the nilmanifold $M$. Then $f$ is topologically conjugated to a hyperbolic automorphism of $M$.*

The author of [25] gives an explicit example of an Anosov diffeomorphism of an infranilmanifold $M$ which is not topologically conjugated to a hyperbolic automorphism of $M$. Defining the broader concept of hyperbolic affine automorphism of an infranilmanifold he also conjectured the following.

**Conjecture 1** ([25])**.** *Let $f\colon M \to M$ be an Anosov diffeomorphism of the infranilmanifold $M$. Then $f$ is topologically conjugated to a hyperbolic affine automorphism of $M$.*

**Remark 1.20.** The veracity of this conjecture is reasonable. In effect, Gromov [39] firstly proved a parallel, actually stronger, results for expanding maps. Then the author of [25] fixed the same mistake as above. In particular, they proved that every expanding map of a compact manifold $M$ is topologically conjugated to an affine expanding infranilmanifold endomorphism. As a consequence, it is reasonable to state the following stronger conjecture.

**Conjecture 2** ([25])**.** *Let $f\colon M \to M$ be an Anosov diffeomorphism of the compact manifold $M$. Then $f$ is topologically conjugated to a hyperbolic affine automorphism of the infranilmanifold $M$.*

We conclude this section devoted to the basic prerequisites with a warning about notation.

**Notation.** *Throughout this thesis we denote by $C$ a generic constant that could depend on the manifold, the dynamics or the atlas of $M$. We underline that $C$ may change also inside the same equation. If we want to point out the dependence of $C$ from a parameter $a$, we write $C_a$. Also this constant could change at any occurrence inside a single equation.*

## 1.2 The dynamical system

In this section we set assumptions about the dynamical system we are going to study in this thesis.

Let $M$ be an orientable, closed (compact and without boundary), connected Riemannian manifold, endowed with the metric $g$. We consider a smooth diffeomorphism $f \in C^{\infty}(M)$ on the manifold which satisfies the Anosov property. In view of Proposition 1.4 and Proposition 1.5, we can assume that $f$ satisfies the hypotheses of Definition 1.1, or, equivalently, the cones condition (1.2), with expansion factor $\lambda > 1$. We always assume to work with an adapted metric $g$, in the sense of Lemma 1.3 and we denote by $d_s$, resp. $d_u$, the dimension of the stable, resp. unstable, subbundle $E^s$, resp. $E^u$.

Next, we need to introduce cohomology. By classical algebraic topology, every homeomorphism $f$ on a topological manifold $M$ induces an action on homology and cohomology. For a complete overview of this subject we refer the reader to [40]. Hence, we simply recall the basics of de Rham cohomology. In fact, since $M$ is a differentiable manifold and $f$ is a diffeomorphism, it makes sense to consider the space of $C^{\infty}$ differential forms $\Omega^l(M)$ endowed with the coboundary operator $d\colon \Omega^l(M) \to \Omega^{l+1}(M)$ given by the exterior derivative. Since $d \circ d = 0$, this defines a cochain complex. A differential form $\omega \in \Omega^l(M)$ is closed if $d\omega = 0$, while it is exact if there exists $u \in \Omega^{l-1}(M)$ such that $du = \omega$. From $d \circ d = 0$, one obtains that exact forms are a vector subspace of closed forms. Finally, one defines the de Rham cohomology group $H^k_{dR}(M)$ as the quotient of closed $k$-forms w.r.t. exact $k$-forms. The pushforward $f_*$ of a $C^{\infty}$-diffeomorphism $f$ on $M$ preserves closed and exact forms, thus it induces a linear map from the cohomology group $H^k_{dR}(M)$ to itself defined by $f_{\#}[\omega] = [f_*\omega]$. In particular, let us consider the action of $f_{\#}$ on $H^{d_s}_{dR}(M)$, i.e., the de Rham cohomology group of degree corresponding to the dimension of the stable subbundle. Since $M$ is a compact manifold, the action of $f_{\#}$ is a linear automorphism of a finite dimensional vector spaces. Therefore, it admits a finite (complex) spectrum $\{\Lambda_1, \dots, \Lambda_N\}$ with

$$|\Lambda_1| \geq |\Lambda_2| \geq \cdots \geq |\Lambda_{N-1}| \geq |\Lambda_N| > 0$$

In Chapter 3, we prove that $\Lambda_1 = e^{h_{top}} \in \mathbb{R}$, where $h_{top}$ is the topological entropy of the dynamical system (see Appendix D for details about topological entropy). In addition, let us consider the second highest eigenvalue $\Lambda_2$ and let us set

$$\theta = \max\{|\Lambda_2|, \lambda^{-1}e^{h_{top}}\},$$

where again $\lambda$ represents the expansion factor of the Anosov splitting (1.1).

To conclude this section, we also need to assume the following.

**Assumption 1.** *The Anosov diffeomorphism $f$ is topologically transitive.*

As already anticipated in Remark 1.14, this is not a strict condition and it is satisfied by every known Anosov diffeomorphism.

### 1.2.1 Statement of the main theorem

We can now state our main result. As already specified in the introduction, it generalizes [20, Theorem 5.10] to Anosov diffeomorphisms on generic manifold (not necessarily tori) with $\dim(M) \geq 2$.

**Theorem 1.21.** *Let $M$ be an orientable, closed (compact and without boundary), connected Riemannian manifold. Let $f\colon M \to M$ be a topologically transitive $C^\infty$ Anosov diffeomorphism. Then, there exist $r \in \mathbb{N}$ and $C > 0$ such that*

$$\left| \int_M \phi\psi \circ f^n d\mu_{BM} - \int_M \phi d\mu_{BM} \int_M \psi d\mu_{BM} \right| \leq C\theta^n e^{-nh_{top}} \|\phi\|_{C^r} \|\psi\|_{C^r}$$

*for any $\phi, \psi \in C^\infty(M)$.*

We actually obtain Theorem 1.21 as consequence of the following stronger result, which gives partial Ruelle-Pollicott asymptotics for these systems.

**Theorem 1.22.** *Let $M$ be an orientable, closed (compact and without boundary), connected Riemannian manifold. Let $f\colon M \to M$ be a topologically transitive $C^\infty$ Anosov diffeomorphism. Assume that, for any $i = 1, \ldots, m$, the complex eigenvalue $\Lambda_i$ satisfies $|\Lambda_i| > \lambda^{-1}e^{h_{top}}$. Then, there exist $r \in \mathbb{N}$, $C > 0$ and, for any $i = 2, \ldots, m$, $N_i \in \mathbb{N}$ and nonzero bilinear forms $\{c_{\Lambda_i,k}(\cdot,\cdot)\}_{k=1}^{N_i}$, such that*

$$\left| \int_M \phi\psi \circ f^n d\mu_{BM} - \int_M \phi d\mu_{BM} \int_M \psi d\mu_{BM} - \right.$$
$$\left. - \sum_{i=1}^{m} \sum_{k=2}^{N_i} (\Lambda_i e^{-h_{top}})^n n^k c_{\Lambda_i,k}(\phi,\psi) \right| \leq C\lambda^{-n} \|\phi\|_{C^r} \|\psi\|_{C^r}$$

*for every $\phi, \psi \in C^\infty(M)$.*

Notice that we refer to Theorem 1.21, and not to the stronger Theorem 1.22, as our main result. The reason is that we do not have any example of Anosov diffeomorphism for which $|\Lambda_2| > \lambda^{-1}$. In particular, let us assume that $f\colon M \to M$ is a $C^\infty$ Anosov diffeomorphism of a torus $M$. Then, by Proposition 1.18, $f$ is topologically conjugated to a hyperbolic automorphism of $M$. This assumption actually reduces the set of Anosov diffeomorphisms to which the following Corollary 1.23 can be applied. For instance, Example 1.15 gives an Anosov diffeomorphism on a nilmanifold which is not a torus. On the other hand, since every codimension 1 Anosov diffeomorphism acts on a torus (Proposition 1.18 - *2.*), the following Corollary 1.23, applies, at least, to every Anosov diffeomorphism on a manifold $M$ with $\dim(M) \leq 3$.

**Corollary 1.23.** *Let $f\colon M \to M$ be a $C^\infty$ Anosov diffeomorphism of a torus $M$. Then, there exist $r \in \mathbb{N}$ and $C > 0$ such that*

$$\left| \int_M \phi\psi \circ f^n d\mu_{BM} - \int_M \phi d\mu_{BM} \int_M \psi d\mu_{BM} \right| \leq C\lambda^{-n} \|\phi\|_{C^r} \|\psi\|_{C^r}$$

*for any $\phi, \psi \in C^\infty(M)$. Moreover, when $M = \mathbb{T}^2$,*

$$\left| \int_{\mathbb{T}^2} (\phi \circ f^n)\psi d\mu_{BM} - \int_{\mathbb{T}^2} \phi d\mu_{BM} \int_{\mathbb{T}^2} \psi d\mu_{BM} \right| \leq Ce^{-nh_{top}} \|\phi\|_{C^r} \|\psi\|_{C^r}$$

Notice that we have also dropped the assumption on topological transitivity. In effect, every Anosov diffeomorphism of a torus is topologically conjugate to a hyperbolic automorphism, the topological transitivity is invariant under topological conjugacy and it is well-known (see for instance [56]) that every hyperbolic automorphism of a torus is topologically transitive.

**Remark 1.24.** Corollary 1.23 follows from Theorem 1.21 by proving that $\theta = \max\{|\Lambda_2|, \lambda^{-1}e^{h_{top}}\} = \lambda^{-1}e^{h_{top}}$ We obtain this estimate for tori, because, in these cases, we can easily compute the eigenvalues of the induced action on cohomology and we are able to prove that $|\Lambda_2| \leq \lambda^{-1}e^{h_{top}}$. If Conjecture 2 were true, it would be possible to extend Theorem 1.22 or Corollary 1.23 to every Anosov diffeomorphism, up to computing the spectrum of the action on de Rham cohomology for the corresponding infranilmanifold.

# Chapter 2

# Anisotropic Banach spaces of currents

This chapter is devoted to the construction of a family of anisotropic Banach spaces which makes the transfer operator we are interested in quasi-compact (see Section 2.4 for details and Appendix A for a short introduction to quasi-compact operators). Anisotropic means that the elements belonging to our spaces have different behaviors along stable and unstable manifolds. In particular, our objects behave as smooth differential forms in the unstable direction, while they behave as distributional differential forms, actually currents, in the stable direction (see Definition 2.5 and Lemma 2.8). We cannot avoid mentioning [11] where similar spaces were firstly defined to investigate Anosov diffeomorphisms, without coding the system. The spaces we use in this chapter recall the ones introduced in [37, 36](see also [4, 20, 21, 35, 38]). In particular, Section 2.2 is focused on a suitable set of stable leaves and it refers to [37, Section 3]. Then, Section 2.3 is a simplified version of [36, Section 3.2], because we work with Anosov diffeomorphisms, while the cited paper treats Anosov flows. For a recent survey of this construction we also refer to [27].

## 2.1 Local charts and norms

We now introduce local coordinates on the manifold, conformed to the hyperbolic dynamics. Given a small constant $\rho > 0$, there exists a smooth atlas $(U_i, \psi_i)_{i=1}^{m}$, such that[1]

$$\overline{\mathcal{B}_{d_s}(0,3\rho)\times\mathcal{B}_{d_u}(0,3\rho)}\subseteq U_i$$

and the maps $\psi_i\colon U_i\to M$ satisfy the following properties:[2]

1. $\bigcup_{i=1}^m \psi_i(\mathcal{B}(0,\rho))=M$;
2. $d_0\psi_i(\mathbb{R}^{d_s}\times\{0\})=E^s_{p_i}$ and $d_0\psi_i(\{0\}\times\mathbb{R}^{d_u})=E^u_{p_i}$, where $p_i=\psi_i(0)$;
3. For every $x\in U_i$, let

$$\zeta^s_{x,i}=\{v+w\in T_xU_i : v\in\mathbb{R}^{d_s}\times\{0\},\ w\in\{0\}\times\mathbb{R}^{d_u},\ \|w\|\le\|v\|\},$$

and let

$$\zeta^u_{x,i}=\{v+w\in T_xU_i : v\in\mathbb{R}^{d_s}\times\{0\},\ w\in\{0\}\times\mathbb{R}^{d_u},\ \|v\|\le\|w\|\}.$$

Choosing $\rho>0$ small enough, we require that the Euclidean stable/unstable cones $\zeta^{s/u}_{x,i}$ satisfy

$$\mathcal{C}^s_{\psi_i(x)}\subseteq d_x\psi_i\zeta^s_{x,i},\quad \mathcal{C}^u_{\psi_i(x)}\subseteq d_x\psi_i\zeta^u_{x,i}$$

$$d_{\psi_i(x)}f^{-1}(\mathcal{C}^s_{\psi_i(x)}\setminus\{0\})\subseteq d_{\psi_i(x)}f^{-1}d_x\psi_i(\zeta^s_{x,i}\setminus\{0\})\subseteq\operatorname{int}(\mathcal{C}^s_{f^{-1}\circ\psi_i(x)}),$$
$$d_{\psi_i(x)}f(\mathcal{C}^u_{\psi_i(x)}\setminus\{0\})\subseteq d_{\psi_i(x)}f d_x\psi_i(\zeta^u_{x,i}\setminus\{0\})\subseteq\operatorname{int}(\mathcal{C}^u_{f\circ\psi_i(x)}),$$

where $\mathcal{C}^{s/u}=\mathcal{C}^{s/u,\alpha}$ are the stable/unstable bundles of (1.2) for some $\alpha\in(0,1)$. As a consequence, $\zeta^{s/u}_{x,i}$ fulfill Property 1. of Proposition 1.5 with respect to the Euclidean metric and also the cone hyperbolicity property, i.e.,

$$d_x(\psi_j^{-1}\circ f^{-1}\circ\psi_i)\zeta^s_{x,i}\subset\operatorname{int}(\zeta^s_{\psi_j^{-1}\circ f^{-1}\circ\psi_i(x),j})\cup\{0\},$$

$$d_x(\psi_j^{-1}\circ f\circ\psi_i)\zeta^u_{x,i}\subset\operatorname{int}(\zeta^u_{\psi_j^{-1}\circ f\circ\psi_i(x),j})\cup\{0\}.$$

Once we have fixed local charts, we introduce appropriate norms on some spaces of functions, whose properties will be used over and over again in this chapter. We first recall that a Banach space $(\mathcal{B},\|\cdot\|)$ is a Banach algebra if it is a linear algebra

[1] We denote by $\mathcal{B}_t(p,r)$ the ball centered in $p$ of radius $r$ in $\mathbb{R}^t$. When the subscript $t$ is not specified we mean $t=\dim(M)$.

[2] One may use the exponential map to construct such charts around every point of the manifold. A finite covering of $M$ gives the atlas we need.

and $\|xy\| \le \|x\|\|y\|$, whenever $x, y \in \mathcal{B}$. We define particular norms on $C^r(U, \mathcal{B})$, for $r \in \mathbb{N}$, $U \subseteq \mathbb{R}^d$ and $\mathcal{B}$ Banach algebra, so that this becomes a Banach algebra. We point out that, in our context, $r$ will be chosen as big as we want, because our Anosov map is $C^\infty$. Given $a \ge 1$, to be fixed at the end of the proof of Lemma 2.1, we define by induction

$$\|\phi\|_{C^0(U,\mathcal{B})} = \sup_{v \in U} \|\phi(v)\|,$$

$$\|\phi\|_{C^{k+1}(U,\mathcal{B})} = \sup_{i=1,\dots,d} \| \partial_{x_i} \phi\|_{C^k} + a\|\phi\|_{C^k}, \text{ for } k \ge 0.$$

A short computation shows that $C^r(U, \mathcal{B})$ is a Banach algebra and

$$\|\phi\|_{C^k} = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} \sup_{|\beta|=i} \| \partial^\beta \phi\|_{C^0}. \tag{2.1}$$

Finally, given a linear map $L\colon C^r(U, \mathcal{B}) \to C^r(U, \mathcal{B})$, we define the usual operator norm

$$\|L\|_{(C^r)^\star} = \sup_{\substack{\phi \in C^r \\ \|\phi\|_{C^r} \le 1}} \|L\phi\|_{C^r}.$$

## 2.2 Set of stable leaves

First of all, we describe the set of stable leaves that we are going to use to define our anisotropic Banach spaces. We remark that by stable leaf we do not mean a piece of stable manifold, but instead a small piece of manifold whose tangent space belongs to the stable cone bundle $\mathcal{C}^s$.

We consider the following set of stable graphs in $\mathbb{R}^{\dim(M)}$

$$\begin{aligned}\mathcal{F} = \{F \in C^r(\mathcal{B}_{d_s}(0, 2\rho); \mathbb{R}^{d_u})|\ F(0) = 0,\\ \|F\|_{C^0(\mathcal{B}_{d_s}(0,2\rho))} \le 2\rho,\ \|dF\|_{(C^r(\mathcal{B}_{d_s}(0,2\rho)))^\star} \le 1\}.\end{aligned} \tag{2.2}$$

Given $F \in \mathcal{F}$ and a point $x \in \mathcal{B}(0, \rho)$ let $G_{x,F}$ be the graph of $F$ in $\mathbb{R}^{\dim(M)}$ centered at $x$, namely $G_{x,F}(\mathcal{B}_{d_s}(0, 2\rho)) = x+(y, F(y))_{y\in\mathcal{B}_{d_s}(0,2\rho)}$. Notice that the tangent space to the graph $G_{x,F}$ belongs to the Euclidean stable cone $\zeta^s_{x,i}$. Finally, we define the set of full admissible leaves

$$\widetilde{\Sigma} = \{\psi_i \circ G_{x,F}(\mathcal{B}_{d_s}(0, 2\rho))|\ x \in \mathcal{B}(0, \rho),\ F \in \mathcal{F},\ i = 1, \dots, m\}$$

and the set of admissible leaves

$$\Sigma = \{\psi_i \circ G_{x,F}(\mathcal{B}_{d_s}(0,\rho))|\ x \in \mathcal{B}(0,\rho),\ F \in \mathcal{F},\ i = 1,\dots,m\}.$$

Observe that, for any admissible leaf $W \in \Sigma$, there is a full admissible leaf $\widetilde{W} \in \widetilde{\Sigma}$ containing $W$. Moreover, notice that the sets of leaves $\Sigma$ and $\widetilde{\Sigma}$ are well-defined. In effect, the graph of $F \in \mathcal{F}$ is included in $\mathcal{B}_{d_s}(0,2\rho)\times\mathcal{B}_{d_u}(0,2\rho)$ and, since $x \in \mathcal{B}(0,\rho)$, $G_{x,F}(\mathcal{B}_{d_s}(0,2\rho)) \subset \mathcal{B}_{d_s}(0,3\rho) \times \mathcal{B}_{d_u}(0,3\rho) \subseteq U_i$, for all $i = 1,\dots,m$.

The importance of the set $\Sigma$ is given by the following lemma. This is a simplified version of [37, Lemma 3.3], where the authors proved a similar result for what they call $\gamma$-admissible leaves, which are useful to study perturbations of the Anosov system.

**Lemma 2.1.** *There exists $n_0 \in \mathbb{N}$ and $\rho > 0$ small enough such that for each full admissible leaf $\widetilde{W}$, with corresponding admissible leaf $W$, and for each $n \geq n_0$, there exists a finite number (depending only on $n$) of admissible leaves $W_1,\dots,W_l \in \Sigma$ such that*

1. $f^{-n}(W) \subseteq \cup_{i=1}^{l} W_i \subseteq f^{-n}(\widetilde{W})$
2. *The leaves $W_1,\dots,W_l$ admit a uniformly finite (in $W$ and $n$) number of overlaps.*
3. *There exists a constant $C_\rho$ and a $C^r$ partition of unity $\eta_1,\dots,\eta_l$ subordinated to $\{W_1,\dots,W_l\}$ on $f^{-n}(W)$, such that $\|\eta_i\|_{C^r} \leq C_\rho$, for any $i = 1,\dots,l$.*

Proof. Let $W = \psi_i \circ G_{x,F}(\mathcal{B}_{d_s}(0,\rho))$ be an admissible leaf and let $\widetilde{W}$ be the corresponding full admissible leaf. Since the tangent space to $\widetilde{W}$ belongs to the stable cone, there exists $n_0 \in \mathbb{N}$ such that the distance between the boundary of $f^{-n}(W)$ and the boundary of $f^{-n}(\widetilde{W})$ is greater than $2r$, for every $n \geq n_0$. Let $n \geq n_0$ and $p \in f^{-n}(W)$. By definition there exists $t \in \mathcal{B}_{d_s}(0,\rho)$ such that $p = f^{-n}\circ\psi_i\circ G_{x,F}(t)$. Moreover, by property 1. of the charts, there exists $j \in \{1,\dots,m\}$ such that $p \in \psi_j(\mathcal{B}(0,\rho))$; thus we denote by $y = \psi_j^{-1}(p) \in \mathcal{B}(0,\rho)$. The uniform hyperbolicity of the Anosov map implies that there exists $F_p \in C^r(\mathcal{B}_{d_s}(0,2\rho),\mathbb{R}^{d_u})$ such that $F_p(0) = 0$, $\|F_p\|_{C^0} \leq 2\rho$ and $\widetilde{W}_p = \psi_j \circ G_{y,F_p}(\mathcal{B}_{d_s}(0,2\rho)) \subseteq f^{-n}(\widetilde{W})$. We need to prove that $\|dF_p\|_{C^{r-1}}$ is bounded by 1, so that $F_p \in \mathcal{F}$. Let $f_{i,j}^{-n} =$

$\psi_j^{-1} \circ f^{-n} \circ \psi_i \colon U_i \to U_j$ be the dynamics induced by $f^{-n}$ on the charts. There exists a subset $I \subseteq \mathcal{B}_{d_s}(0, 2\rho)$ and a bijection $\alpha \colon I \to \mathcal{B}_{d_s}(0, 2\rho)$ such that

$$f_{i,j}^{-n}(G_{x,F}(s)) = f_{i,j}^{-n}(x + (s, F(s))) = G_{y,F_p}(\alpha(s)) = y + (\alpha(s), F_p(\alpha(s))).$$

Computing the differential at $s \in I$ we obtain

$$d_{G_{x,F}(s)} f_{i,j}^{-n} \begin{pmatrix} id \\ d_s F \end{pmatrix} = \begin{pmatrix} d_s \alpha \\ d_{\alpha(s)} F_p d_s \alpha \end{pmatrix} \tag{2.3}$$

Notice that, by assumption 2. on the charts, if $x = p_i = \psi_i(0)$ and $y = p_j = \psi_j(0)$, then the differential

$$d_{p_i} f_{i,j}^{-n} = \begin{pmatrix} A_{i,j}(p_i) & 0 \\ 0 & D_{i,j}(p_i) \end{pmatrix},$$

where $A_{i,j}(p_i)$ is a $d_s \times d_s$ matrix such that $\|A_{i,j}(p_i)^{-1}\|^\star \leq \lambda^{-n}$, while $D_{i,j}(p_i)$ is a $d_u \times d_u$ matrix such that $\|D_{i,j}(p_i)\|^\star \leq \lambda^{-n}$. By continuity of the differential, given every $\delta > 0$ there exists $\rho > 0$ small enough such that, for every $i, j \in \{1, \ldots, m\}$ and for all $z \in U_i$,

$$d_z f_{i,j}^{-n} = d_{p_i} f_{i,j}^{-n} + \Delta_{i,j}(z) = \begin{pmatrix} A_{i,j}(z) & B_{i,j}(z) \\ C_{i,j}(z) & D_{i,j}(z) \end{pmatrix},$$

where $\Delta_{i,j} = \begin{pmatrix} \Delta_{i,j}^{d_s,d_s} & \Delta_{i,j}^{d_u,d_s} \\ \Delta_{i,j}^{d_s,d_u} & \Delta_{i,j}^{d_u,d_u} \end{pmatrix}$ is a $\dim(M) \times \dim(M)$ matrix such that $\|\Delta_{i,j}\| < \delta$, $A_{i,j}(z) = A_{i,j}(p_i) + \Delta_{i,j}^{d_s,d_s}(z)$, $B_{i,j}(z) = \Delta_{i,j}^{d_u,d_s}(z)$, $C_{i,j}(z) = \Delta_{i,j}^{d_s,d_u}(z)$ and $D_{i,j}(z) = D_{i,j}(p_i) + \Delta_{i,j}^{d_u,d_u}(z)$. Equation (2.3) implies

$$\begin{cases} A_{i,j}(G_{x,F}(s)) + B_{i,j}(G_{x,F}(s)) d_s F = d_s \alpha \\ C_{i,j}(G_{x,F}(s)) + D_{i,j}(G_{x,F}(s)) d_s F = d_{\alpha(s)} F_p d_s \alpha \end{cases}$$

from which it follows

$$\begin{aligned} d_s F_p = &\left[ C_{i,j}(G_{x,F}(\alpha^{-1}(s))) + D_{i,j}(G_{x,F}(\alpha^{-1}(s))) d_{\alpha^{-1}(s)} F \right] \cdot \\ &\left[ id + [A_{i,j}(G_{x,F}(\alpha^{-1}(s))) B_{i,j}(G_{x,F}(\alpha^{-1}(s))) d_{\alpha^{-1}(s)} F \right]^{-1} A_{i,j}(G_{x,F}(\alpha^{-1}(s)))^{-1} \end{aligned} \tag{2.4}$$

We now want to estimate $\|dF_p\|_{(C^r)^\star}$. To this end, we need the following trivial computation.

**Lemma 2.2** ([27, Sub-lemma 4.5]).

$$\sup_{|\beta|\le r} a^{-|\beta|}\| \partial^\beta dF_p\|_{C^0} \le \|dF_p\|_{(C^r)^\star} \le e^r (r!)^2 \sup_{|\beta|\le r} a^{-|\beta|}\| \partial^\beta dF_p\|_{C^0}$$

Since $\|\Delta_{i,j}\|_{C^0} \le \delta$, using Lemma 2.2, we can choose $a$ large enough so that there exists a constant $C_r$, depending only on $r$, such that $\|\Delta_{i,j}\|_{(C^r)^\star} \le C_r\delta$, for all $i, j$. Moreover, for every constant $C > 1$ large enough, we can fix $\delta$ small enough and $n_0$ large enough, so that

$$\begin{aligned}\sup_{i,j}\{&\|B_{i,j}\circ G_{x,F}\|_{(C^r)^\star} + \|C_{i,j}\circ G_{x,F}\|_{(C^r)^\star},\\ &\|A_{i,j}^{-1}\circ G_{x,F}\|_{(C^r)^\star},\ \|D_{i,j}\circ G_{x,F}\|_{(C^r)^\star}\} \le \frac{1}{2C}\end{aligned} \tag{2.5}$$

To conclude the proof we need the following

**Lemma 2.3** ([27, Sub-lemma 4.6]). *There exists a constant, depending exclusively on $r$, such that*

$$\|d_{G_{x,F}\circ\alpha^{-1}(\cdot)} f_{i,j}^{-n}\circ\alpha^{-1}\|_{(C^r)^\star} \le C_r\|d_{G_{x,F}(\cdot)} f_{i,j}^{-n}\|_{(C^r)^\star}$$

Using Lemma 2.3, (2.4) and (2.5), we conclude that

$$\begin{aligned}&\|dF_p\|_{(C^r)^\star} = \| \left[C_{i,j}(G_{x,F}(\alpha^{-1}(\cdot))) + D_{i,j}(G_{x,F}(\alpha^{-1}(\cdot)))d_{\alpha^{-1}(\cdot)}F\right]\cdot\\ &\left[id + [A_{i,j}(G_{x,F}(\alpha^{-1}(\cdot)))B_{i,j}(G_{x,F}(\alpha^{-1}(\cdot)))d_{\alpha^{-1}(\cdot)}F\right]^{-1} A_{i,j}(G_{x,F}(\alpha^{-1}(\cdot)))^{-1}\|_{(C^r)^\star} \le\\ &\le C_r\| \left[C_{i,j}(G_{x,F}(\cdot)) + D_{i,j}(G_{x,F}(\cdot))dF\right]\\ &[id + [A_{i,j}(G_{x,F}(\cdot))B_{i,j}(G_{x,F}(\cdot))dF]^{-1} A_{i,j}(G_{x,F}(\cdot))^{-1}\|_{(C^r)^\star} \le\\ &\le \frac{C_r}{4C^2}\frac{(1+\|dF\|_{(C^r)^\star})}{\left(1+\frac{\|dF\|_{(C^r)^\star}}{4C^2}\right)} \le \frac{C_r}{2C^2} \le 1\end{aligned}$$

provided that $C > 1$ is large enough.

The construction of a $C^r$ partition of unity is a standard argument (see for example [44, Theorem 1.4.5]). Q.E.D.

## 2.3 Norms and Banach spaces

We are ready to construct our family of anisotropic Banach spaces. In particular, we are going to define anisotropic norms on spaces of differential forms, and then we obtain the suitable family of anisotropic spaces by completing spaces of differential forms with respect to these norms.

We denote by $\Omega^l(M)$, for each $l = 0, \dots, \dim(M)$, the space of complex smooth differential forms on $M$, namely the set of $C^\infty$ sections of the $l$-exterior algebra of the cotangent bundle $T^*M$ over $M$, with values in $\mathbb{C}$. Given an admissible leaf $W = \psi_i \circ G_{x,F}(\mathcal{B}_{d_s}(0,\rho)) \in \Sigma$, $s \geq 0$ and $l \in \{0, \dots, \dim(M)\}$, we denote by $\Gamma_0^{l,s}(W)$ the space of complex $C^s$ sections of the fiber bundle over $W$, with the fiber space $\wedge^l(T^*M)$ and compact support. In other words, we may think the elements $\Gamma_0^{l,s}(W)$ as $l-$differential forms of class $C^s$, defined on $W$ and vanishing in a neighborhood of $\partial W$. This is exactly the space introduced in [36, Section 3] and, in the definition of the norm, its elements have the role of "test forms". Let $\mathcal{V}^s(W)$ be the space of $C^s$ vector fields defined in a $\dim(M)$ -dimensional neighborhood $U(W) \subseteq U_i$ of $W$.

We want to express forms and vector fields in local coordinates. Given the atlas $\{U_i, \psi_i\}_{i=1}^m$, let $\{\chi_i\}_{i=1}^m$ be a smooth partition of unity subordinate to the atlas, such that $\chi_i|_{\psi_i(\mathcal{B}(0,3\rho))} = 1$. We denote by $\partial_{r_1},\ \partial_{r_2}, \dots,\ \partial_{r_{\dim(M)}}$ a basis for the vector fields on $U_i$ such that $\partial_{x_1} := \psi_i^*(\partial_{r_1}),\ \partial_{x_2} := \psi_i^*(\partial_{r_2}), \dots,\ \partial_{x_{d_s}} := \psi_i^*(\partial_{r_{d_s}}) \in \mathcal{C}^s$ and $\partial_{x_{d_s+1}} := \psi_i^*(\partial_{r_{d_s+1}}),\ \partial_{x_{d_s+2}} := \psi_i^*(\partial_{r_{d_s+2}}), \dots,\ \partial_{x_{d_s+s_u}} := \psi_i^*(\partial_{r_{d_s+d_u}}) \in \mathcal{C}^u$. Without loss of generality, we may suppose that this is an orthonormal basis of vector fields, otherwise one to apply the Gram-Schmidt procedure without essentially affecting the forthcoming arguments. Finally, let $dx_1,\ dx_2,\ \dots,\ dx_d$ be the dual basis of $\partial_{x_1},\ \partial_{x_2}, \dots,\ \partial_{x_d}$, i.e., the corresponding basis of differential forms on $\psi_i(U_i)$. Let $\mathcal{J}_l = \{\overline{j} = (j_1, \dots, j_l) \in \{1, \dots, d\}^l | j_1 < j_2 < \cdots < j_l\}$ the set of ordered $l$-multi-indexes. We adopt the following notation for fields and differential forms: $\partial_{x_{\overline{i}}} := \partial_{x_{i_1}} \wedge \cdots \wedge \partial_{x_{i_l}},\ dx_{\overline{i}} := dx_{i_1} \wedge \cdots \wedge dx_{i_l}$, whence $dx_{\overline{i}}(\partial_{x_{\overline{i}}}) = l!$.

We can decompose every form $h \in \Omega^l(M)$ as $h = \sum_{i=1} h_i$, where $h_i = h\chi_i \in \Omega_0^l(\psi_i(U_i))$, i.e., $h_i$ is a smooth differential form on $\psi_i(U_i)$ with compact support. Moreover, using the local basis, we can write every $h_i$ in coordinates as $h_i = \sum_{\overline{j} \in \mathcal{J}_l} h_i^{\overline{j}} dx_{\overline{j}}$. We define the $C^s$ norm of $h \in \Omega^l(M)$ as

$$\|h\|_{C^s(M)} = \sup_{i=1,\dots,m} \sup_{\overline{j} \in \mathcal{J}_l} \|h_i^{\overline{j}}\|_{C^s(\psi_i(U_i))} = \sup_{i=1,\dots,m} \sup_{\overline{j} \in \mathcal{J}_l} \|h_i^{\overline{j}} \circ \psi_i^{-1}\|_{C^s(U_i)}. \tag{2.6}$$

Similarly, given $\phi \in \Gamma_0^{s,l}(W)$, we can write

$$\phi = \sum_{i=1}^m \phi\chi_i = \sum_{i=1}^m \phi_i = \sum_{i=1}^m \sum_{\overline{j} \in \mathcal{J}_l} \phi_i^{\overline{j}} dx_{\overline{j}}$$

and

$$\|\phi\|_{\Gamma_0^{l,s}(W)} = \sup_{i=1,\ldots,m} \sup_{\bar{j}\in\mathcal{J}_l} \|\phi_i^{\bar{j}}\|_{C^s(\psi_i\circ G_{x,F}(\mathcal{B}_{d_s}(0,\delta)))} = \sup_{i=1,\ldots,m} \sup_{\bar{j}\in\mathcal{J}_l} \|\phi_i^{\bar{j}}\circ\psi_i^{-1}\|_{C^s(G_{x,F}(\mathcal{B}_{d_s}(0,\delta)))}.$$

The last ingredient we need is a scalar product for differential forms. We point out that this scalar product, and consequently the induced norm, depends on the metric. On the other hand, the Banach space we are establishing is independent from the metric. Let us consider the adapted metric $g$ (see Lemma 1.3) and the induced volume form $\omega_0 \in \Omega^{\dim(M)}(M)$. The non-degeneracy condition of $g$ induces an isomorphism $\xi$ between smooth vector fields $\mathcal{V}(M)$ and smooth 1-forms $\Omega^1(M)$ such $g(v,\cdot) = \xi(v)(\cdot)$ for every $v \in \mathcal{V}(M)$. A pointwise scalar product between 1-forms $\omega_1,\ \omega_2 \in \Omega^1(M)$ is $\langle\omega_1,\omega_2\rangle = g(\xi^{-1}(\omega_1),\xi^{-1}(\omega_2))$. Similarly, for $\{\omega_{i,j}\}_{i=1,2;\ j=1,\ldots,l} \subseteq \Omega^1(M)$, it is defined a pointwise scalar product between $l$-forms

$$\langle\omega_{1,1}\wedge\cdots\wedge\omega_{1,l},\omega_{2,1}\wedge\cdots\wedge\omega_{2,l}\rangle = \det\begin{pmatrix} \langle\omega_{1,1},\omega_{2,1}\rangle & \ldots & \langle\omega_{1,l},\omega_{2,1}\rangle \\ \vdots & \ddots & \vdots \\ \langle\omega_{1,1},\omega_{2,l}\rangle & \ldots & \langle\omega_{1,l},\omega_{2,l}\rangle \end{pmatrix}$$

Finally, the scalar product between $l$-forms is the integral of the pointwise scalar product, i.e.,

$$(\omega_1,\omega_2) = \int_M \langle\omega_1,\omega_2\rangle\omega_0, \qquad \omega_1,\omega_2 \in \Omega^l(M)$$

**Remark 2.4.** The scalar product $\langle\cdot,\cdot\rangle$ induces a duality operator between $l$-forms and $(\dim(M)-l)$-forms, the so-called Hodge star operator (see Appendix B for a review of Hodge theory). In effect, given $\omega_1,\omega_2 \in \Omega^l(M)$ there exist unique $\star\omega_1,\star\omega_2 \in \Omega^{d-l}(M)$ such that

$$\omega_2\wedge\star\omega_1 = (-1)^{l(d-l)}\star\omega_1\wedge\omega_2 = \omega_1\wedge\star\omega_2 = \langle\omega_1,\omega_2\rangle\omega_0.$$

We are now ready to define the anisotropic norms.

**Definition 2.5.** *Let $p,q\in\mathbb{N}$. Given $h\in\Omega^l(M)$, we define a seminorm*

$$\|h\|^-_{p,q,l} = \sup_{W\in\Sigma} \sup_{\substack{v_1,\ldots,v_p\in\mathcal{V}^{p+q}(U(W)) \\ \|v_k\|_{C^{p+q}(U(W))}\le 1}} \sup_{\substack{\phi\in\Gamma_0^{p+q,l}(W), \\ \|\phi\|_{\Gamma_0^{p+q,l}(W)}\le 1}} \left|\int_W \langle\phi, L_{v_1}\ldots L_{v_p}h\rangle\omega_W\right|,$$

*where $L_{v_i}$ is the Lie derivative of the $l$-form $h$ w.r.t. the vector field $v_i$ and $\omega_W$ is the measure induced by $g$ on $W$. We define, for every $h\in\Omega^l(M)$, the norm*

$$\|h\|_{p,q,l} = \max_{t\le p}\|h\|^-_{t,q,l}.$$

*Finally, we denote by* $\mathcal{B}^{p,q,l} = \overline{\Omega^l(M)}^{\|\cdot\|_{p,q,l}}$ *the closure of the space of* $l$*-forms w.r.t. such a norm.*

**Remark 2.6.** The following inequalities are trivial consequences of the definition. $\|h\|_{p,q,l} \leq \|h\|_{p+1,q-1,l}$ and $\|h\|_{p,q,l} \leq C\|h\|_{C^p}$, hence $\mathcal{B}^{p+1,q-1,l} \subseteq \mathcal{B}^{p,q,l}$ and $\Omega^l(M) \subseteq \mathcal{B}^{p,q,l}$. Furthermore,

$$\sup_{\substack{v_1,\ldots,v_p \in \mathcal{V}^{p+q}(U(W)) \\ \|v_k\|_{C^{p+q}(U(W))} \leq 1}} \|L_{v_1} \ldots L_{v_p} h\|^-_{0,p+q,l} \leq \|h\|^-_{p,q,l}.$$

**Remark 2.7.** Notice that the Banach space $\mathcal{B}^{p,q,l}$ coincide with the space studied in [20, Section 5] for Anosov map acting on a 2-torus. The authors of [20] were in turn inspired by [36], where these spaces were exploited to analyze dynamical zeta functions of Anosov flows. The following proposition, whose proof recalls [36, Lemma 3.10], shows that we can think these spaces as subspaces of currents, i.e., the continuous dual space of differential forms. For an overview of currents' properties we refer the reader to de Rham's book [23].

Given $p, q \in \mathbb{N}$, let us denote by $\Omega^l_{p+q}(M)$ the space of $C^{p+q}$ $l$-forms equipped with the $C^{p+q}$ norm as defined in (2.6). Let $(\Omega^l_{p+q}(M))^\star$ be its dual space with the weak-$^\star$topology, i.e., the space of currents of dimension $l$, degree $\dim(M) - l$ and regularity $C^{p+q}$.

**Lemma 2.8.** *The space* $\mathcal{B}^{p,q,l}$ *can be identified with a subspace of the space of currents of dimension* $\dim(M) - l$*, degree* $l$ *and regularity* $C^{p+q}$ *on the manifold* $M$*; i.e., there exists an injective bounded linear operator* $\iota \colon \mathcal{B}^{p,q,l} \to (\Omega^l_{p+q}(M))^\star$.

PROOF. Given a smooth differential form $h \in \Omega^l(M)$, we define

$$\iota(h)(g) = (h, g) = \int_M \langle h, g \rangle \omega_0, \text{ for each } g \in \Omega^l_{p+q}(M).$$

Notice that, by Remark 2.4, we may consider $\star g \in \Omega^{\dim(M)-l}_{p+q}(M)$, so that $\iota(h)(g) = \int_M h \wedge \star g$, and the dimension of $h$ is consistent with the definition of the dimension for currents. We can break down $M$ into admissible leaves belonging to $\Sigma$. Thus, recalling the definition of the norm, there exists a constant $C > 0$ such that, for all $g \in \Omega^l_{p+q}(M)$,

$$|\iota(h)(g)| \leq C\|h\|_{p,q,l}\|g\|_{C^{p+q}}$$

As a consequence, the map $\iota$ extends to a bounded linear operator, denoted by the same symbol, $\iota\colon \mathcal{B}^{p,q,l} \to (\Omega^l_{p+q}(M))^\star$.

It remains to show that this operator is injective. Let us consider $h \in \mathcal{B}^{p,q,l}$, such that $\iota(h) \equiv 0$. By definition, there exists a sequence of smooth forms $\{h_n\} \in \Omega^l(M)$ such that $\lim_{n\to+\infty} h_n = h$, which means that $\lim_{n\to+\infty} \|h_n - h\|_{p,q,l} = 0$. We want to prove that $h = 0$, i.e., $\|h\|_{p,q,l} = 0$. Let $W = \psi_i(G_{x,F}(\mathcal{B}_{d_s}(0,\rho))) \in \Sigma$ be an admissible leaf and let $\phi \in \Gamma_0^{p+q,l}(W)$ be a test form. We need to smoothen $\phi$, that we suppose equal zero out of $W$. Given a classical mollifier $\kappa \in C^\infty(\mathbb{R}^{\dim(M)}, \mathbb{R})$ such that $k \geq 0$, $\int_{\mathbb{R}^{\dim(M)}} \kappa = 1$ and $supp(\kappa) \subseteq \mathcal{B}(0,1)$, we consider, for $\epsilon > 0$, $\kappa_\epsilon(x) = \epsilon^{-\dim(M)}\kappa\left(x\epsilon^{-1}\right)$. Since $W$ belongs to $\psi_i(U_i)$, we limit ourselves to consider the chart $(U_i, \psi_i)$. In particular, the coordinate form of $\phi$ becomes $\phi = \sum_{\overline{j}\in\mathcal{J}_l} \phi_i^{\overline{j}} dx_{\overline{j}}$, where $supp(\phi_i^{\overline{j}}) \subseteq \psi_i(U_i)$. We can define

$$\phi_\epsilon(t) = \sum_{\overline{j}\in\mathcal{J}_l} \left( \int_{\mathbb{R}^{\dim(M)}} \kappa_\epsilon(\psi_i^{-1}(t) - y)\phi_i^{\overline{j}}(\psi_i(y))dy \right) dx_{\overline{j}},$$

so that $\phi_\epsilon \in \Omega^l(\psi_i(U_i))$ and

$$\int_W \langle h_n, \phi\rangle \omega_W = \lim_{\epsilon\to 0^+} \int_M \langle h_n, \phi_\epsilon\rangle \omega_0 = \lim_{\epsilon\to 0^+} j(h_n)(\phi_\epsilon),$$

hence

$$\lim_{n\to+\infty} \int_W \langle h_n, \phi\rangle \omega_W = \lim_{n\to+\infty} \lim_{\epsilon\to 0^+} j(h_n)(\phi_\epsilon), \tag{2.7}$$

Moreover, given $p$ vector fields $v_1, \ldots, v_p \in \mathcal{V}^{p+q}(U(W))$ as required in the definition of the norm,

$$\lim_{n\to+\infty} \int_W \langle L_{v_1} \ldots L_{v_p} h_n, \phi\rangle \omega_W = \lim_{n\to+\infty} \lim_{\epsilon\to 0^+} j(L_{v_1} \ldots L_{v_p} h_n)(\phi_\epsilon), \tag{2.8}$$

We claim that, assuming $\epsilon$ small enough, for each $g \in \Omega^l(M)$ and $\phi \in \Gamma_0^q(W)$

$$\left| \int_M \langle g, \phi_\epsilon\rangle \omega_0 \right| \leq C\|g\|_{0,q,l}\|\phi\|_{\Gamma_0^{q,l}(W)}. \tag{2.9}$$

Additionally, for $p > 0$, $v_1, \ldots, v_p \in \mathcal{V}^{p+q}(U(W))$ as in the definition of the norm and for $\epsilon$ small enough,

$$\left| \int_M \langle L_{v_1} \ldots L_{v_p} g, \phi_\epsilon\rangle \omega_0 \right| \leq C\|g\|_{p,q,l}\|\phi\|_{\Gamma_0^{p+q,l}(W)}, \tag{2.10}$$

for each $g \in \Omega^l(M)$ and $\phi \in \Gamma_0^{p+q}(W)$.

As a consequence of (2.9) we obtain that, for $m, n \in \mathbb{N}$,

$$|j(h_n)(\phi_\epsilon) - j(h_m)(\phi_\epsilon)| = \left| \int_M \langle h_n - h_m, \phi_\epsilon \rangle \omega_0 \right| \leq C \|h_n - h_m\|_{0,q,l} \|\phi\|_{\Gamma_0^{q,l}(W)}.$$

Therefore, we can swap the two limits in (2.7) gaining

$$\lim_{n\to+\infty} \int_W \langle h_n, \phi_\epsilon \rangle \omega_W = \lim_{\epsilon\to 0^+} \lim_{n\to+\infty} j(h_n)(\phi_\epsilon) = \lim_{\epsilon\to+\infty} j(h)(\phi_\epsilon) = 0,$$

that is $\|h\|_{0,q,l} = 0$. Similarly, one can prove that $\|h\|_{p,q,l} = 0$ using (2.8) and (2.10). We finally prove the claim. Although a more general result is proven in [36, Lemma D.2.], we give a proof adapted to our context. Let $g = \sum_i \sum_{\overline{j}\in\mathcal{J}_l} g_i^{\overline{j}} dx_{\overline{j}} \in \Omega^l(M)$ and let $\phi \in \Gamma_0^{q,l}(W)$. We compute

$$\begin{aligned}
\left| \int_M \langle g, \phi_\epsilon \rangle \omega_0 \right| &= \left| \int_M \sum_{\overline{j}\in\mathcal{J}_l} g_i^{\overline{j}}(x) \int_{\mathbb{R}^{\dim(M)}} \phi_i^{\overline{j}}(\psi_i(y)) \kappa_\epsilon(\psi_i^{-1}(x) - y) dy \omega_0(dx) \right| = \\
&= \left| \int_M \int_{\mathbb{R}^{\dim(M)}} \sum_{\overline{j}\in\mathcal{J}_l} g_i^{\overline{j}}(x) \phi_i^{\overline{j}}(\psi_i(\psi_i^{-1}(x) - z)) \kappa_\epsilon(z) dz \omega_0(dx) \right| = \\
&= \left| \int_{\mathbb{R}^{\dim(M)}} \kappa_\epsilon(z) \int_M \sum_{\overline{j}\in\mathcal{J}_l} g_i^{\overline{j}}(x) \phi_i^{\overline{j}}(\psi_i(\psi_i^{-1}(x) - z)) \omega_0(dx) dz \right| = (*)
\end{aligned}$$

Assuming $\epsilon$ small enough, for each $z \in \mathcal{B}(0,\epsilon)$, there exist $W_z \in \Sigma$ and $\phi_z \in \Gamma_0^{q,l}(W_z)$ such that $(\phi_z)_i^{\overline{j}}(x) = \phi_i^{\overline{j}}(\psi_i(\psi_i^{-1}(x) - z))$, so that $\|\phi_z\|_{\Gamma_0^{q,l}(W_z)} = \|\phi\|_{\Gamma_0^{q,l}(W)}$, and

$$(*) = \left| \int_{\mathbb{R}^{\dim(M)}} k_\epsilon(z) \int_{W_z} \langle g, \phi_z \rangle \omega_{W_z} dz \right| \leq C \|h\|_{0,q,l} \|\phi\|_{\Gamma_0^{q,l}}.$$

In addition, the same computation shows that, given $p$ vector fields $v_1, \dots, v_p$ as above and for each $\phi \in \Gamma_0^{p+q,l}(W)$

$$\left| \int_M \langle L_{v_1} \dots L_{v_p} g, \phi_\epsilon \rangle \omega_0 \right| \leq C \|L_{v_1} \dots L_{v_p} g\|_{0,p+q,l} \|\phi\|_{\Gamma_0^{p+q,l}(W)} \leq C \|g\|_{p,q,l} \|\phi\|_{\Gamma_0^{p+q,l}(W)},$$

where the last inequality is a consequence of the last property in Remark 2.6. This concludes the proof of the claim. Q.E.D.

## 2.4 Quasi-compactness of the transfer operator

This section is devoted to proof of the spectral properties of the pushforward operator $f_*$, whose action on differential form, extends to a bounded linear operator on $\mathcal{B}^{p,q,l}$. In the field of dynamical systems, $f_*$ falls into the category of transfer operators. The transfer operator approach to statistical properties of dynamical systems was inspired by classical results of statistical mechanics and it was firstly carried out by Ya.G. Sinai [66, 67], D. Ruelle [62, 65] and R. Bowen [15]. Their main idea was to code the system through the construction of a Markov partition, with a suitable transfer matrix, and then to apply Perron-Frobenius theory (we refer the reader to [5] for a complete description of the topic). As already said, the authors of [11] firstly defined anisotropic Banach spaces to tackle directly transfer operators of Anosov systems without coding it. With this new idea, many original, as well as already known results, have been demonstrated applying these techniques. At the same time, their great flexibility gave the opportunity to analyze many other dynamical systems, including partially hyperbolic, non-uniformly hyperbolic and parabolic systems. We only mention papers of C. Liverani and S. Gouëzel [37, 38] or papers of V. Baladi and B. Tsujii[8, 9], which are at the cornerstone of the theory.

When we talk about a transfer operator without specifying it, we generally refer to a linear operator $\mathcal{L}$ acting on the space $L^2(M, \omega_0)$, which is the adjoint, w.r.t. the $L^2(M, \omega_0)$-scalar product, of the composition (Koopman) operator, that is

$$\langle g \circ f, h \rangle_{L^2} = \langle g, \mathcal{L}h \rangle_{L^2}.$$

In particular, given $h \in L^2(M, \omega_0)$, there is an explicit formula

$$\mathcal{L}h(x) = \frac{h \circ f^{-1}(x)}{|\det d_{f^{-1}(x)} f|} \text{ for } \omega_0\text{-a.a. points } x \in M.$$

In the context of Anosov diffeomorphisms, $\mathcal{L}$ does not have good spectral properties when acting on $L^2$, but it behaves very well when it acts on anisotropic Banach spaces of distributions, for example the ones introduced in [37]. In fact, the authors of [37] proved that, under transitivity, $\mathcal{L}$ is quasi-compact (see Definition A.14), 1 is the unique, maximal and simple eigenvalue, whose eigenvector is density of the SRB measure (see [74] for a survey of SRB measures). Additionally, they obtain exponential mixing, as well as the existence of a complete asymptotic expansion for the decay of correlations, of smooth observables [37, Corollary 2.6].

We now notice that, for every $\omega \in \Omega^{\dim(M)}(M)$, can write $\omega = h\omega_0$, where $h \in C^\infty(M)$ and $\omega_0$ is the volume form induced by the metric $g$. For all $x \in M$, for all $v_1, v_2, \ldots, v_d \in T_x M$ it holds

$$\begin{aligned}(f_*\omega)_x(v_1,\ldots,v_d) &= h\circ f^{-1}(x)(\omega_0)_{f^{-1}(x)}(d_x f^{-1}(v_1),\ldots,d_x f^{-1}(v_d)) = \\ &= h\circ f^{-1}(x)|\det d_x f^{-1}|(\omega_0)_x(v_1,\ldots,v_d) = \\ &= (\mathcal{L}h)(x)\omega_0(v_1,\ldots,v_d)\end{aligned} \tag{2.11}$$

In other terms, the action of $f_*$ on smooth $\dim(M)$-forms (i.e., a subset of $\mathcal{B}^{p,q,\dim(M)}$) corresponds to the action of the transfer operator $\mathcal{L}$ on the corresponding density. This means that we can read results of [37] studying how $f_*$ behaves on $\mathcal{B}^{p,q,\dim(M)}$. We are interested in the spectral properties of the pushforward operator acting on $\mathcal{B}^{p,q,l}$ for $l = 0,\ldots,\dim(M)$. In particular, our main theorem will be a consequence of our analysis on the action of $f_*$ on the Banach space of currents $\mathcal{B}^{p,q,d_s}$, where we recall that $d_s$ is the dimension of stable manifolds.

The first ingredient we need to investigate the spectrum of our transfer operator is a quasi-compactness result. We recall that a bounded linear operator $\mathcal{T}$ on the Banach space $\mathcal{B}$ is quasi-compact if the essential spectral $\rho_{ess}(\mathcal{T})$ radius is strictly smaller than the spectral radius $\rho(\mathcal{T})$. Roughly speaking, this means that, for all $t \in (\rho_{ess}(\mathcal{T}), \rho(\mathcal{T}))$, the annulus $\{z \in \mathbb{C}|\ t < z < \rho(\mathcal{T})\}$ contains finitely many eigenvalues of finite multiplicity. The standard procedure to prove quasi-compactness of linear operators is based on the following classical result of Hennion [41], whose proof is recalled in Appendix A-Section 3.

**Theorem 2.9** ([41] Corollary 1)**.** *Let $\mathcal{T}$ be a bounded linear operator on the Banach space $(\mathcal{B}, \|\cdot\|)$. Assume that there exists another norm $|\cdot|$ on $\mathcal{B}$ such that*

1. *the immersion $i\colon (\mathcal{B}, \|\cdot\|) \to (\mathcal{B}, |\cdot|)$ is bounded and compact;*
2. *for all $n \in \mathbb{N}$ there exist $R_n,\ r_n > 0$, with $\liminf_{n\to+\infty} r_n^{1/n} < \rho(T)$ such that*

$$\|\mathcal{T}^n f\| \leq r_n\|f\| + R_n|f|;$$

*then the operator, acting of $(\mathcal{B}, \|\cdot\|)$, is quasi-compact with essential spectral radius $\rho_e(\mathcal{T}) \leq \liminf_{n\to+\infty} r_n^{1/n}$.*

**Lemma 2.10.** *The inclusion $\iota\colon \mathcal{B}^{p,q,l} \to \mathcal{B}^{p-1,q+1,l}$ is compact for each $p \in \mathbb{Z}^+$, $q \in \mathbb{N}$ and for all $l = 1,\ldots,\dim(M)$.*

The above lemma is proved in [37, Lemma 2.1] for the anisotropic Banach space of distributions $\mathcal{B}^{p,q,0}$. We generalize their proof to anisotropic currents.

Proof of Lemma 2.10. By definition, the inclusion $\iota$ is compact if the open unit ball $B \subseteq \mathcal{B}^{p,q,t}$ is relatively compact in $\mathcal{B}^{p-1,q+1,t}$. We use a general criterion[31, Proposition 2.8] to prove that a linear operator $L\colon (\mathcal{B}, \|\cdot\|_{\mathcal{B}}) \to (\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ is compact. Assume that, for all $\epsilon > 0$, there are finitely many continuous linear operators $l_i\colon \mathcal{B} \to \mathbb{R}$ such that $\|x\|_{\mathcal{C}} \le \epsilon \|x\|_{\mathcal{B}} + \sum_i |l_i(x)|$, then $L$ is compact. We recall the proof of this criterion in Appendix A - Lemma A.3.

We fix a chart $(U_i, \psi_i)$, for some $i = 1, \ldots, m$, and $h \in \Omega^l(M)$. Recall that we fixed $r$ in (2.2) as large as we want. Therefore, we may suppose that $p+q < r$, so that the inclusion $i\colon C^r(\mathcal{B}_{d_s}(0, 2\rho, \mathbb{R}^{d_u})) \to C^{p+q}(\mathcal{B}_{d_s}(0, 2\rho, \mathbb{R}^{d_u}))$ is compact. Consequently, for any $\epsilon > 0$, there exists a finite number of full admissible leaves $\widetilde{W_1}, \ldots, \widetilde{W_n}$, such that $\widetilde{W_j} = \psi_i \circ G_{x_j, F_j}(\mathcal{B}_{d_s}(0, 2\rho))$, with $j = 1, \ldots, n$, $x_j \in \mathcal{B}(0, \rho)$ and $F_j \in \mathcal{F}$, and, in addition, for any other $W = \psi_i \circ G_{x,F}(\mathcal{B}(0, \rho))$, there exists $j \in \{1, \ldots, n\}$ such that $|x - x_j| < \epsilon$ and $\|F - F_j\|_{C^{p+q}(\mathcal{B}_{d_s}(0,\rho), \mathbb{R}^{d_u})} < \epsilon$. We now introduce a family of admissible stable leaves $\{\overline{W}_t\}_{t \in [0,1]}$, such that $\overline{W}_0 = W$ and $\overline{W}_1 = W_j$. In particular, let $x_t = x + t(x_j - x)$ and let $\overline{F}_t = F + t(F_j - F)$. We define $\overline{W}_t = \psi_i \circ G_{x_t, \overline{F}_t}(\mathcal{B}_0, \rho)$ and we denote by $\{\Psi_t\}_{t \in [0,1]}$ the family of maps mapping $W$ to $\overline{W}_t$. One can trivially check that $\{\overline{W}_t\}_{t \in [0,1]}$ is actually a family of admissible leaves, i.e they are shortening of full admissible leaves. Consequently, for any $\phi \in \Gamma_0^{p+q,l}(W)$, $\|\phi\|_{\Gamma_0^{p+q,l}(W)} \le 1$, and for $v_1, \ldots, v_{p-1} \in \mathcal{V}^{p+q}(U(W))$, with $\|v_i\|_{C^{p+q}(U(W))} \le 1$,

$$\int_W \langle \phi, L_{v_1} \ldots L_{v_{p-1}} h \rangle \omega_W - \int_{W_j} \langle \overline{\phi}_1, L_{(\overline{v_1})_1} \ldots L_{(\overline{v_{p-1}})_1} h \rangle \omega_{(\overline{W_j})_1} =$$
$$= \int_0^1 \int_{\overline{W}_t} \frac{d}{dt} \langle \overline{\phi}_t, L_{(\overline{v_1})_t} \ldots L_{(\overline{v_{p-1}})_t} h \rangle \omega_{(\overline{W_j})_t} dt$$

where $\overline{\phi}_t = \Psi_t^*(\phi)$ and $(\overline{v_k})_t = \Psi_t^*(v_k)$, for any $k = 1, \ldots, p-1$. Writing last integral in coordinates the r.h.s. becomes

$$\int_0^1 \int_{-\rho}^{\rho} \langle \overline{\phi}_t, \partial_t \, L_{(\overline{v_1})_t} \ldots L_{(\overline{v_{p-1}})_t} h \rangle (x + t(x_j - x) + F(s) + t(F_j(s) - F(s))) \omega_{(\overline{W_j})_t}(ds) dt$$

which is bounded by $C\epsilon \|h\|_{p,q,l}$, since $|x - x_j| < \epsilon$ and $\|F - F_j\|_{C^{p+q}} < \epsilon$ (with a slight abuse of notation, $\partial_t$ denotes the vector field $v = (\psi_i)_*[(x_j - x)\,\partial_t]$.) Accordingly,

$$\|h\|_{p-1,q+1,l} = \max_{k\le p-1} \|h\|^-_{k,q+1,l} \le C\epsilon\|h\|_{p,q,l} +$$
$$+ \max_{k\le p-1} \sup_{j=1,\dots,n} \sup_{\substack{v_1,\dots,v_k\in\mathcal{V}^{k+q+1}(U(W))\\ \|v_i\|_{C^{k+q+1}(U(W))}\le 1}} \sup_{\substack{\phi\in\Gamma_0^{k+q+1,l}(W),\\ \|\phi\|_{\Gamma_0^{k+q+1,l}(W)}\le 1}} \int_{W_j} \langle \phi, L_{v_1}\dots L_{v_k} h\rangle \omega_{W_j}$$

and, up to a small error, we can reduce to consider the integral on a finite number of admissible leaves $W_1,\dots,W_j$. Therefore, assume $W = W_j$. The set of test forms $\Gamma_0^{p+q,l}(W)$, with $\|\phi\|_{\Gamma_0^{p+q,l}(W)} \le 1$ is relatively compact inside $\Gamma_0^{p+q-1,l}(W)$. As a consequence, for any $\epsilon > 0$, there exist $\phi_1,\dots,\phi_N \in \Gamma_0^{p+q-1,l}(W)$, such that for each $\phi \in \Gamma_0^{p+q,l}(W)$, $\|\phi\|_{\Gamma_0^{p+q,l}(W)} \le 1$, it holds $\|\phi - \phi_j\|_{\Gamma_0^{p+q-1,l}(W)} < \epsilon$, for some $j \in \{1,\dots,\dim(M)\}$. Similarly, there exist $v_1,\dots,v_M \in \mathcal{V}^{p+q-1}(U(W))$ such that for each $v \in \mathcal{V}(U(W))$, $\|v\|_{C^{p+q}} \le 1$, it holds $\|v - v_j\|_{C^{p+q-1}} < \epsilon$, for some $j \in \{1,\dots,\dim(M)\}$. It follows that

$$\begin{aligned}\left|\int_W \langle \phi, L_{v_1}\dots L_{v_{p-1}} h\rangle\omega_W - \int_W \langle \phi_j, L_{(v_1)_{j_1}}\dots L_{(v_{p-1})_{j_{p-1}}} h\rangle\omega_W\right| &\le C\epsilon\|h\|_{p-1,q+1,l}\\ &\le C\epsilon\|h\|_{p,q,l}.\end{aligned}$$

Defining

$$l_{j,j_1,\dots,j_{p-1}}(h) = \int_W \langle \phi_j, L_{(v_1)_{j_1}}\dots L_{(v_{p-1})_{j_{p-1}}} h\rangle\omega_W,$$

we obtain

$$\|h\|_{p-1,q+1,l} \le C\epsilon\|h\|_{p,q,l} + \sum_{j,j_1,\dots,j_{p-1}} |l_{j,j_1,\dots,j_{p-1}}(h)| \tag{2.12}$$

for any $h \in \Omega^l(M)$. By density, we can extend (2.12) to $\mathcal{B}^{p,q,l}$ and we can apply Lemma A.3. Q.E.D.

Property *2.* of Hennion's theorem is generally called a Lasota-Yorke type inequality. The following theorem states that such inequalities hold for the pushforward operator when acting on the anisotropic banach spaces $\mathcal{B}^{p,q,l}$. Notice that by property *1.* of Theorem 2.9 $|\cdot| \le \|\cdot\|$, hence the norm $\|\cdot\|$ is stronger than $|\cdot|$. In our context the role of the strong norm is taken by $\|\cdot\|_{p,q,l}$, while the weak norm is $\|\cdot\|_{p-1,q+1,l}$.

**Theorem 2.11.** *$f_*$ acts as a bounded linear operator on the spaces $\mathcal{B}^{p,q,l}$. In particular, for $p \geq 0$ and for $0 < l < \dim(M)$,*

$$\|f_*^n h\|_{p,q,0} \leq C\|h\|_{p,q,0}; \tag{2.13}$$

$$\|f_*^n h\|_{p,q,l} \leq C\lambda^{-|d_s-l|n} e^{h_{top}n}\|h\|_{p,q,l}; \tag{2.14}$$

$$\|f_*^n h\|_{p,q,\dim(M)} \leq C\|h\|_{p,q,\dim(M)}. \tag{2.15}$$

*Moreover, for $p > 0$ and for $0 < l < \dim(M)$,*

$$\|f_*^n h\|_{p,q,0} \leq C\lambda^{-n\min\{p,q\}}\|h\|_{p,q,0} + C\|h\|_{p-1,q+1,0}; \tag{2.16}$$

$$\|f_*^n h\|_{p,q,l} \leq C\lambda^{-n(\min\{p,q\}+|d_s-l|)} e^{h_{top}n}\|h\|_{p,q,l} + C\lambda^{-n|d_s-l|}e^{nh_{top}}\|h\|_{p-1,q+1,l}; \tag{2.17}$$

$$\|f_*^n h\|_{p,q,\dim(M)} \leq C\lambda^{-n\min\{p,q\}}\|h\|_{p,q,\dim(M)} + C\|h\|_{p-1,q+1,\dim(M)}. \tag{2.18}$$

**Remark 2.12.** Notice that (2.14) and (2.17) also work for $l = 0$ and $l = d_s$, but they give weaker bounds than the other four inequalities. In effect, by using Ledrappier-Young entropy formula [19, Theorem D.3.1], it is easy to check that $e^{h_{top}}\lambda^{-d_s} \geq 1$. The reasons behind this mismatch will be clear at the end of the proof, but, just to give an idea, the factor $\lambda^{-|d_s-l|}e^{h_{top}}$ comes up from the estimate on the expansion of $l$-dimensional subspaces of the tangent bundle, under the action of the differential. Hence, there is no expansion on 0-dimensional subspaces when $l = 0$ and, by using the duality (2.11), when $l = \dim(M)$. On the other hand, the bounds of (2.14) and (2.17) work for every Anosov diffeomorphism, but they could be improved on a case by case basis. Since our main results strongly depend on such bounds, better estimates of that factor would improve them.

Proof. Let us prove (2.13) for $p = 0$. Let $h \in C^\infty(M)$, $W \in \Sigma$, $\phi \in \Gamma_0^{q,0}(W) = C_0^q(W)$. The following integral can be split, using the partition of unity $(W_i, \eta_i)_{i\in\mathcal{I}}$ of Lemma 2.1, as

$$\int_W \phi f_*^n h = \int_W \phi h \circ f^{-n} = \int_{f^{-n}W} \phi \circ f^n h \lambda_n^s = \sum_i \int_{W_i} \phi \circ f^n h \lambda_n^s \eta_i,$$

where $\lambda_n^s(x)$ is the Jacobian of the change of variables. As a consequence of Lemma C.1, the $C^q$-norm of $\lambda_n^s$ is bounded by $C|f^n(W_i)|$, where $|f^n(W_i)|$ is the $\omega_{\widetilde{W}}$-measure of $f^n(W_i)$. Moreover, the $C^q$-norm of $\phi \circ f^n$ is bounded by the $C^q$-norm of $\phi$, because the composition with $f^n$ reduces norms along stable manifolds. Thus, since we can assume that the $C^q$-norm of $\eta_i$ is uniformly bounded, the norm of $\phi \circ f^n \lambda_n^s \eta_i$ is bounded by $C|f^n(W_i)|\|\phi\|_{C_0^q(W)}$ , hence

$$\left| \int_W \phi f_*^n h \right| \le C \sum_i |f^n(W_i)| \|\phi\|_{C_0^q(W)} \|h\|_{0,q,l}.$$

Choosing an appropriate covering of $f^{-n}(W)$, without many overlaps, the sum is bounded by a constant and, by density, we get (2.13) with $p = 0$.

Let us tackle together (2.13) for $p \ge 0$ and (2.16). We firstly show that, for $h \in \Omega^l(M)$,

$$\|f_*^n h\|_{t,q,0}^- \le \begin{cases} C\lambda^{-nq} \|h\|_{p,q,0} + C_n \|h\|_{p-1,q+1,0} & \text{if } t < p \\ C\lambda^{-n\min\{p,q\}} \|h\|_{p,q,0} + C_n \|h\|_{p-1,q+1,0} & \text{if } t = p \end{cases} \tag{2.19}$$

so that

$$\|f_*^n h\|_{p,q,0} = \max_{t \le p} \|f_*^n h\|_{t,q,0}^- \le C\lambda^{-n\min\{p,q\}} \|h\|_{p,q,0} + C_n \|h\|_{p-1,q+1,0}. \tag{2.20}$$

The inequality (2.20) does not prove (2.16), as well as (2.13) for $p \ge 0$, because the coefficient of the weak norm does depend on $n$. We proceed by induction on $p$ to remove the dependence on that $n$. We already proved (2.13) for $p = 0$. Assume that (2.13) holds for $p-1$, then we show (2.16) and (2.13) for $p$. Let $\widetilde{\lambda} > \lambda$ be a constant for which the Anosov property (1.1) continues to be true. Then (2.20) implies

$$\|f_*^n h\|_{p,q,0} \le \widetilde{C}\widetilde{\lambda}^{-n\min\{p,q\}} \|h\|_{p,q,0} + \widetilde{C}_n \|h\|_{p-1,q+1,0}.$$

Let $N \in \mathbb{Z}^+$ be a positive integer such that $\widetilde{C}\widetilde{\lambda}^{-N\min\{p,q\}} \le \lambda^{-N\min\{p,q\}}$. Then, for every $n \in \mathbb{Z}^+$, we write $n = Q_n N + R_n$ with $0 \le R_n < N$ and $Q_n \in \mathbb{N}$. Consequently,

$$\begin{aligned} \|f_*^n h\|_{p,q,0} \le & \widetilde{C}\widetilde{\lambda}^{-N\min\{p,q\}} \|f_*^{n-N} h\|_{p,q,0} + \widetilde{C}_N \|f_*^{n-N} h\|_{p-1,q+1,0} \le \\ \le & \widetilde{C}\widetilde{\lambda}^{-N\min\{p,q\}} \|f_*^{n-N} h\|_{p,q,0} + \widetilde{C}_N \|h\|_{p-1,q+1,0} \le \\ \le & \left(\widetilde{C}\widetilde{\lambda}^{-N\min\{p,q\}}\right)^2 \|f_*^{n-2N} h\|_{p,q,0} + \widetilde{C}_N \widetilde{C}\widetilde{\lambda}^{-N\min\{p,q\}} \|f_*^{n-2N} h\|_{p-1,q+1,0} + \\ & + \widetilde{C}_N \|h\|_{p-1,q+1,0} \le \\ \le & \left(\widetilde{C}\widetilde{\lambda}^{-N\min\{p,q\}}\right)^{Q_n} \|f_*^{R_n} h\|_{p,q,0} + \widetilde{C_N} \sum_{i=0}^{Q_N-1} \left(\widetilde{C}\widetilde{\lambda}^{-N\min\{p,q\}}\right)^i \|h\|_{p-1,q+1,0} \le \\ \le & C\lambda^{-(NQ_n+R_n)\min\{p,q\}} \|h\|_{p,q,0} + \widetilde{C_N} \frac{1-\lambda^{-NQ_n\min\{p,q\}}}{1-\lambda^{-N\min\{p,q\}}} \|h\|_{p-1,q+1,0} + \\ & + C_{R_n} \|h\|_{p-1,q+1,0} \le C\lambda^{-n\min\{p,q\}} \|h\|_{p,q,0} + C\|h\|_{p-1,q+1,0}, \end{aligned}$$

where we used (2.20) to estimate $\|f_*^{R_n}h\|_{p,q,0}$ and the inductive hypothesis to estimate $\|f_*^{n-iN}h\|_{p-1,q+1,0}$. This computation also proves (2.13) for $p$. Moreover, both (2.13) and (2.16) extend by density to $\mathcal{B}^{p,q,0}$.

We are left with the proof of (2.20). Let $h \in \Omega^0(M) = C^\infty(M)$, $W \in \Sigma$, $\phi \in \Gamma_0^{t+q,0}(W) = C_0^{t+q}(W)$, such that $\|\phi\|_{C_0^{t+q}(W)} \leq 1$, and $v_1, \dots, v_t \in \mathcal{V}^{t+q}(U(W))$ with $\|v_i\|_{C^{t+q}} \leq 1$. We compute

$$\int_W \phi(x) L_{v_1} \dots L_{v_t} f_*^n h(x) = \int_W \phi(x) v_1 \dots v_t (h \circ f^{-n})(x).$$

By linearity we can assume that $v_j = g_j\, \partial_{x_j}$ where $\partial_{x_j}$ is a coordinate vector field, hence

$$\begin{aligned}\int_W \phi(x) v_1 \dots v_t (h \circ f^{-n})(x) &= \int_W \phi(x) \prod_{j=1}^t (g_j(x)\, \partial_{x_j})(h \circ f^{-n})(x) = \\ &= \sum_i \int_{f^n(W_i)} \phi(x) \eta_i(f^{-n}(x)) \prod_{j=1}^t g_j(x) \prod_{j=1}^t \partial_{x_j} (h \circ f^{-n})(x) + C_n \|h\|_{p-1,q+1,0}\end{aligned}$$

where the last term comes out deriving at least one of the coefficients of the vector fields. With a slight abuse of notations we rewrite $\phi(x)\eta_i(f^{-n}(x)) \prod_{j=1}^d g_j(x)$, which is again a test function, as $\phi(x)$. By [37, Lemma 6.5], given a $C^{t+q}(U(W_i))$ vector field $v$, there exist $C^{t+q}$ vector fields $w^u$ and $w^s$, in a neighborhood $U(W_i)$ of $W_i$, such that $v = w^u + w^s$ and

- for all $x \in f^n(W_i)$ $w^s_x \in T_x f^n(W_i)$;
- $\|w^s\|_{C^{t+q}(U_i)} \leq C_n$;
- $\|w^s \circ f^n\|_{C^{t+q-1}(W_i)} \leq C$;
- $\|d_{f^n(x)} f^{-n} w^u(f^n(x))\|_{C^{t+q}(f^{-n}(U(W)))} \leq C\lambda^{-n}$;

We obtain

$$\int_{f^n(W_i)} \phi(x) \prod_{j=1}^d \partial_{x_j} (h \circ f^{-n})(x) = \sum_{\sigma \in \{s,u\}^t} \int_{f^n(W_i)} \phi(x) \prod_{j=1}^d w_j^{\sigma_j} (h \circ f^{-n})(x).$$

Since $w_j^u w_k^s = w_k^s w_j^u + [w_j^u, w_k^s]$, we can swap two vector fields up to a term which is again $C_n \|h\|_{p-1,q+1,0}$. Thus, we need to estimate terms of the form

$$\int_{f^n(W_i)} \phi(x) \prod_{j=1}^g w_j^s \prod_{j=g+1}^d w_j^u (h \circ f^{-n})(x) = (-1)^g \int_{f^n(W_i)} \prod_{j=1}^g w_j^s \phi(x) \prod_{j=g+1}^d w_j^u (h \circ f^{-n})(x)$$

where we applied an integration by parts. Every vector field $w_j^s$ can be written again in terms of the coordinate vector fields $w_j^s = \sum_{z=1}^d b_{j,z}\,\partial_{x_z}$. As above, if one of the vector fields acts on one of the coefficients we get a term bounded by $C_n\|h\|_{p-1,q+1,o}$. Therefore, we remain with the following terms

$$\int_{f^n(W_i)} \prod_{j=1}^g \partial_{x_j}\,\phi(x) \prod_{j=1}^g b_{j,z_j}(x) \prod_{j=g+1}^d w_j^u(h\circ f^{-n})(x) =$$
$$= \int_{W_i} \prod_{j=1}^g (\partial_{x_j}\,\phi)\circ f^n(x) \prod_{j=1}^g b_{j,z_j}\circ f^n(x) \prod_{j=g+1}^d \widetilde{w}_j^u(h)\lambda_n^s(x)$$

By the third property $\|\prod_{j=1}^g b_{j,z_j}(x)\circ f^n\|_{C^{t+q}} \leq C$. We distinguish two cases If $t=p$ and $g=0$ it holds

$$\left|\int_{W_i} \prod_{j=1}^p \widetilde{w}_j^u(h)\phi\right| \leq C\lambda^{-np}\|h\|_{p,q,o}\|\phi\|_{C_0^{p+q}}.$$

On the other hand, if $t<p$ or $g>0$, let $\overline{\phi} = \prod_{j=1}^g \partial_{x_j}\,\phi \in C_0^{q+t-g}(\widetilde{W})$. We need to smoothen this function through the following standard lemma.

**Lemma 2.13.** *Let $\alpha$ be the bigger integer smaller than $q+t-g$. For $\epsilon>0$, there exists $\phi_\epsilon \in C^{q+t-g+1}$ such that $\|\phi_\epsilon\|_{C^{q+t-g}} \leq C\|\overline{\phi}\|_{C^{q+t-g}}$, $\|\phi_\epsilon\|_{C^{q}+t-g+1} \leq C\epsilon^{-1}\|\overline{\phi}\|_{C^{q}+t-g}$ and $\|\phi_\epsilon - \overline{\phi}\|_{C^\alpha} \leq C\epsilon^{q+t-g-\alpha}\|\overline{\phi}\|$.*

The proof of the above lemma easily follows convolving the function $\overline{\phi}$ with a mollifier of order $\epsilon$ and then computing the norms. In our context we fix $\epsilon$ to be $\lambda^{-(q+t-g)n/(q+t-g-\alpha)}$, so that

$$\|\phi_\epsilon - \overline{\phi}\|_{C^\alpha} \leq C\lambda^{-(q+t-g)n}\|\overline{\phi}\|.$$

This implies that

$$\left|\int_{W_i} \prod_{j=1}^g (\partial_{x_j}\,\phi)\circ f^n(x) \prod_{j=1}^g b_{j,z_j}\circ f^n(x) \prod_{j=g+1}^d \widetilde{w}_j^u(h)\lambda_n^s(x)\right| \leq$$
$$\leq \left|\int_{W_i} (\overline{\phi}-\phi_\epsilon)\circ f^n(x) \prod_{j=1}^g b_{j,z_j}\circ f^n(x) \prod_{j=g+1}^d \widetilde{w}_j^u(h)\lambda_n^s(x)\right| +$$
$$+ \left|\int_{W_i} \phi_\epsilon\circ f^n(x) \prod_{j=1}^g b_{j,z_j}\circ f^n(x) \prod_{j=g+1}^d \widetilde{w}_j^u(h)\lambda_n^s(x)\right|$$

The first term is bounded by

$$C\lambda^{-(q+t-g)n}\lambda^{-(t-g)n}\|\phi\|_{C^{q+t}}\|h\|_{p,q,0} \leq C\lambda^{-qn}\|\phi\|_{C^{q+t}}\|h\|_{p,q,0}$$

where we used that $\lambda^{-(q+t-g)}\lambda^{-(t-g)} \leq \lambda^{-q}$ when $g \leq t$. Since the function $\phi_\epsilon$ is strictly more regular than $\overline{\phi}$, because $t-k<p$, the second term is bounded by $C_n\|h\|_{p-1,q+1,0}$. This concludes the proof of (2.19).

For (2.14) and (2.17), we proceed as above. Hence, we firstly prove (2.14) for $p=0$, then we show the analogous of (2.20) and, finally, we conclude by induction proving that (2.14) for $p-1$ implies (2.17) and (2.14) for $p$. Let us show (2.14) for $p=0$. We consider $h \in \Omega^l(M)$, $W \in \Sigma$ and a test form $\omega \in \Gamma_0^{l,q}(W)$. We need to compute the integral

$$\int_W \langle \omega, f_*^n h\rangle.$$

We fix the local bases for vector fields and forms introduced in section 2.3. Assuming that $W \subseteq \psi_z(U_z,)$, we can write $\omega = \omega\circ\chi_z = \sum_{\overline{j}\in\mathcal{J}_l}\omega_{\overline{j}}dx_{\overline{J}}$ on $\psi_z(\mathcal{B}(0,3\rho))$. It follows that

$$\begin{aligned}\int_W \langle \omega, f_*^n h\rangle &= \int_W \sum_{\overline{j}\in\mathcal{J}_l}\omega_{\overline{j}}\langle dx_{\overline{j}}, f_*^n h\rangle = \sum_{\overline{j}\in\mathcal{J}_l}\int_{f^{-n}(W)}\omega_{\overline{j}}\circ f^n\langle dx_{\overline{j}}, f_*^n h\rangle\circ f^n\lambda_n^s = \\ &= \sum_{\overline{j},\overline{k}\in\mathcal{J}_l}\sum_i\int_{W_i} h_{\overline{k}}\omega_{\overline{j}}\circ f^n\langle dx_{\overline{j}}, f_*^n dx_{\overline{k}}\rangle\circ f^n\lambda_n^s\eta_i.\end{aligned}$$

where $h = h\circ\chi_i = \sum_{\overline{k}\in\mathcal{J}_l}h_{\overline{k}}dx_{\overline{k}}$, on $\psi_i(\mathcal{B}(0,3\rho)) \supseteq W_i$. The $C^q$-norm of the functions $\langle dx_{\overline{j}}, f_*^n dx_{\overline{k}}\rangle\circ f^n\lambda_n^s$ is bounded by $C\lambda^{-|d_s-l|n}$ (see Lemma C.2). Thus,

$$\left|\int_W \langle\omega, f_*^n h\rangle\right| \leq C\lambda^{-|d_s-l|n}|f^{-n}(W)|\|\omega\|_{\Gamma_0^{l,q}}\|h\|_{0,q,l}$$

By classical results (see for instance Theorem D.7), the volume growth of $f^{-n}(W)$ fulfills $|f^{-n}(W)| \sim e^{h_{top}n}|W|$, where $h_{top}$ is the topological entropy of $f$. By density, we get (2.14) for $p=0$.

Next, let us prove that, for $h \in \Omega^l(M)$,

$$\|f_*^n h\|_{p,q,l} \leq C\lambda^{-n\min\{p,q\}-n|d_s-l|}e^{nh_{top}}\|h\|_{p,q,l} + C_n\|h\|_{p-1,q+1,l}. \tag{2.21}$$

Let $h \in \Omega^l(M)$, $W \in \Sigma$, $\omega \in \Gamma_0^{p+q}(W)$ and let $v_1,\dots,v_t \in \mathcal{V}^{t+q}(U(W))$ be $t$ vector fields such that $\|v_i\|_{C^{p+q}(U(W))} \leq 1$. As above, we can write $\omega = \sum_{\overline{j}\in\mathcal{J}_l}\omega_{\overline{j}}dx_{\overline{j}}$ and we compute

$$\int_W \langle \omega, L_{v_1} \ldots L_{v_p} f_*^n h \rangle = \sum_i \int_{f^n(W_i)} \sum_{\bar{j},\bar{k} \in \mathcal{J}_l} \omega_{\bar{j}} \langle dx_{\bar{j}}, L_{v_1} \ldots L_{v_p} [(h_{\bar{k}} \circ f^{-n}) f_*^n (dx_{\bar{k}})] \rangle \eta_i \circ f^{-n}$$

$$= \sum_i \int_{f^n(W_i)} \sum_{\bar{j},\bar{k} \in \mathcal{J}_l} \sum_{A \subseteq \{1,\ldots,p\}} \omega_{\bar{j}} \prod_{a \in A} L_{v_a}(h_{\bar{k}} \circ f^{-n}) \langle dx_{\bar{j}}, \prod_{a \in A^c} L_{v_a}(f_*^n(dx_{\bar{k}})) \rangle \eta_i \circ f^{-n}$$

where $h = \sum_{\bar{k} \in \mathcal{J}_l} h_{\bar{k}} dx_{\bar{k}}$ on $W_i$, while the product of derivatives is ordered. If $A = \{1, \ldots, p\}$ the terms

$$\sum_i \int_{f^n(W_i)} \sum_{\bar{j},\bar{k} \in \mathcal{J}_l} \omega_{\bar{j}} \prod_{a=1}^{p} L_{v_a}(h_{\bar{k}} \circ f^{-n}) \langle dx_{\bar{j}}, f_*^n(dx_{\bar{k}}) \rangle \eta_i \circ f^{-n}$$

can be treated putting together the proofs of (2.14), for $p = 0$, and (2.16). Hence, it is bounded by $C\lambda^{-n\min\{p,q\}}\lambda^{-n|d_s-l|}e^{nh_{top}}\|h\|_{p,q,l} + C\|h\|_{p-1,q+1,l}$. Every other term has the form

$$\sum_i \int_{f^n(W_i)} \sum_{\bar{j},\bar{k} \in \mathcal{J}_l} \omega_{\bar{j}} \prod_{a \in A} L_{v_a}(h_{\bar{k}} \circ f^{-n}) \langle dx_{\bar{j}}, \prod_{a \in A^c \neq \varnothing} L_{v_a}(f_*^n(dx_{\bar{k}})) \rangle.$$

Since there are $t < p$ derivatives acting on $(h_{\bar{k}} \circ f^{-n})$, (2.14), for $p = 0$, and (2.16) imply that these terms are bounded by $C_n\|h\|_{p-1,q+1,l}$ and this concludes the proof of (2.21). Finally, by the same inductive procedure used to prove (2.13) and (2.16), one can prove (2.14) for $p > 0$ and (2.17). Moreover, by density, these inequalities extend to $\mathcal{B}^{p,q,l}$. Notice that in this case we cannot expect that coefficient in front of the weak norm in (2.17) is uniformly bounded. On the contrary, we have just proved that it cannot grow more that $\lambda^{-n|d_s-l|}e^{h_{top}}$.

We are left with the proof of (2.15) and (2.18). We have already noticed in (2.11) the duality, induced by the volume form $\omega_0$, between $d_s$-forms and functions. Accordingly, (2.15) and (2.18) hold true in $\Omega^{d_s}(M)$ for $f_*$ if and only if the same are satisfied by the transfer operator $\mathcal{L}$ acting on functions $C^\infty(M) = \Omega^0(M)$. The authors of [37, Lemma 2.2] proved Lasota-Yorke inequalities for $\mathcal{L}$. By density, we conclude that (2.15) and (2.18) extend to $\mathcal{B}^{p,q,\dim(M)}$.

Q.E.D.

**Remark 2.14.** The inequalities (2.17) and (2.14), for $l = d_s$, also implies the following inequality that we are going to use later (see the proof of Lemma 3.9). In fact, for $p>0$, $q>0$, $l=d_s$ and $\omega\in\mathcal{B}^{p,q,d_s}$, one can easily prove by induction that

$$\|f_*^{pn}\omega\|_{p,q,d_s}\leq Ce^{pnh_{top}}\lambda^{-n}\|h\|_{p,q,d_s}+Ce^{pnh_{top}}\|h\|_{0,p+q,d_s}, \tag{2.22}$$

where the $p$ in $f_*^{pn}$ represents the same parameter $p$ of the norm. In effect, (2.17) gives (2.22) for $p=1$. Assume (2.22) true up to $p-1$. Then, by using (2.14), (2.17) and the property $\|\cdot\|_{p-1,q+1,d_s}\leq\|\cdot\|_{p,q,d_s}$, we obtain

$$\begin{aligned}\|f_*^{pn}h\|_{p,q,d_s}\leq&Ce^{nh_{top}}\lambda^{-n}\|f_*^{(p-1)n}h\|_{p,q,d_s}+Ce^{nh_{top}}\|f_*^{(p-1)n}h\|_{p-1,q+1,d_s}\leq\\ \leq&Ce^{pnh_{top}}\lambda^{-n}\|h\|_{p,q,d_s}+Ce^{2nh_{top}}\lambda^{-n}\|f_*^{(p-2)n}h\|_{p-1,q+1,d_s}+\\ &+Ce^{pnh_{top}}\|h\|_{0,p+q,d_s}\leq\\ \leq&Ce^{pnh_{top}}\lambda^{-n}\|h\|_{p,q,d_s}+Ce^{pnh_{top}}\|h\|_{0,p+q,d_s},\end{aligned}$$

which proves (2.22).

**Corollary 2.15.** *The spectral radius and the essential spectral radius of $f_*\colon\mathcal{B}^{p,q,l}\to\mathcal{B}^{p,q,l}$ fulfill*

$$\rho(f_*|_{\mathcal{B}^{p,q,l}})\leq\begin{cases}1 & \textit{if } l=0 \textit{ or } l=d\\ \lambda^{-|d_s-l|}e^{h_{top}} & \textit{if } 0<l<d\end{cases}$$

$$\rho_{ess}(f_*|_{\mathcal{B}^{p,q,l}})\leq\begin{cases}\lambda^{-\min\{p,q\}} & \textit{if } l=0 \textit{ or } l=d\\ \lambda^{-\min\{p,q\}}\lambda^{-|d_s-l|}e^{h_{top}} & \textit{if } 0<l<d\end{cases}$$

Proof. The estimates on spectral radii follow by (2.13), (2.14) and (2.15), using Lemma A.9. The estimates on the essential spectral radii are consequence of Hennion's theorem 2.9 whose hypotheses are satisfied by Lemma 2.10 and Theorem 2.11. Q.E.D.

Once we have established this spectral picture a natural question may arise: how does the spectrum of $f_*$ acting on $\mathcal{B}^{p,q,l}$, denoted by $\sigma(f_*|_{\mathcal{B}^{p,q,l}})$, depends on $p$ and $q$? The following lemma answers this question, at least for the spectrum we are interested in.

**Lemma 2.16.** *Let $\mathcal{B}^{p,q,l}$ and $\mathcal{B}^{p',q',l}$ be two anisotropic Banach spaces of currents for some parameters $p,q,p',q'\in\mathbb{N}$. Assume that $\rho_{ess}(f_*|_{\mathcal{B}^{p',q',l}})\leq\rho_{ess}(f_*|_{\mathcal{B}^{p,q,l}})$. Then*

$$\sigma(f_*|_{\mathcal{B}^{p,q,l}})\cap\{z\in\mathbb{C}:\ |z|>\rho_{ess}(f_*|_{\mathcal{B}^{p,q,l}})\}=\sigma(f_*|_{\mathcal{B}^{p',q',l}})\cap\{z\in\mathbb{C}:\ |z|>\rho_{ess}(f_*|_{\mathcal{B}^{p,q,l}})\}.$$

*Moreover, the corresponding generalized eigenspaces coincide and they are included in $\mathcal{B}^{p,q,l}\cap\mathcal{B}^{p',q',l}$.*

PROOF. This is a consequence of Lemma A.15, where $\mathcal{B}_0 = \Omega^l(M)$, $\mathcal{B}_1 = \mathcal{B}^{p,q,l}$, $\mathcal{B}_2 = \mathcal{B}^{p',q',l}$ and $\mathcal{B} = \mathcal{B}^{\min\{p,p'\},\max\{q,q'\},l}$. Q.E.D.

# Chapter 3

# The spectrum of the pushforward operator

We now want to investigate the spectrum of the pushforward operator acting on anisotropic Banach spaces $\mathcal{B}^{p,q,l}$, $0 \leq l \leq \dim(M)$. In particular, we are going to use information about the spectrum of $f_*\colon \mathcal{B}^{p,q,d_s} \to \mathcal{B}^{p,q,d_s}$, where $d_s$ is the dimension of the stable bundle, to prove our main theorem.

From now on, we always assume $p$ and $q$ large enough, so that there exists $\nu \in (0,1)$ such that

$$\max_{i=0,\ldots,d_s} \max\{\lambda^{-\min\{p\pm i, q\mp i\}-|d_s-l|} e^{h_{top}}, \lambda^{-\min\{p\pm i, q\mp i\}}\} < \nu < 1, \tag{3.1}$$

for any $i = 0, \ldots, \dim(M)$. Corollary 2.15 and (3.1) ensure that the essential spectral radius $f_*$ acting on the Banach spaces we are interested in is bounded by $\nu$. Results of this chapter reflect the reasoning of [20, Sections 5.5.2 and 5.5.3] for the 2-dimensional case. Since the dimension of the stable bundle in their case was 1, the authors of that paper were interested in the action of $f_*$ on $\mathcal{B}^{p,q,1}$. The key point of their idea was to relate some eigenvalues of $f_*\colon \mathcal{B}^{p,q,1} \to \mathcal{B}^{p,q,1}$ to the action of the dynamics on de Rham cohomology. This is why in this chapter we also recall some basic notions of de Rham cohomology, we define the anisotropic de Rham cohomology (Section 3.2) and we prove the connection between the standard and the anisotropic de Rham cohomology (Section 3.3). Before approaching the cohomological aspects, we analyze the peripheral spectrum of $f_*\colon \mathcal{B}^{p,q,d_s} \to \mathcal{B}^{p,q,d_s}$ and we construct the measure of maximal entropy. This is the content of the following section. Most of the results of Section 3.1 are inspired by [38, Section 4, 5], where the authors treat a generalization of Anosov diffeomorphisms as well as other invariant measures. They also use different Banach spaces, which in some sense contain our anisotropic Banach spaces. For all these reasons, we rewrite the proofs we need in our simplified setting.

## 3.1 Peripheral spectrum and measure of maximal entropy

By peripheral spectrum we mean the set of eigenvalues of $f_*|_{\mathcal{B}^{p,q,d_s}}$ of maximal modulus. Corollary 2.15 only tells us that the spectral radius $\rho(f_*|_{\mathcal{B}^{p,q,d_s}})$ is bounded by $e^{h_{top}}$. On the other hand, the following lemma proves that this upper bound is actually attained.

**Lemma 3.1.** *The spectral radius of $f_*|_{\mathcal{B}^{p,q,d_s}}$ is exactly $e^{h_{top}}$.*

Before giving the proof of Lemma 3.1, we introduce the following $d_s$-differential form $\omega_\Sigma \in \Omega^{d_s}(M)$, that we are going to use along this section. We need a $d_s$-form which gives positive volume to every admissible stable leaf $W \in \Sigma$, in the sense that $\int_W \omega_\Sigma > 0$, for any $W \in \Sigma$. The first idea would be to consider the volume $\omega_W$ induced by $\omega_0$ on every admissible leaf $W \in \Sigma$, but this is only a $d_s$-form on $W$ and not on $M$. On the other hand, on every chart $(U_i, \psi_i)$, one can easily define a $d_s$-differential form $u_i$ which gives positive volume to every leaf with tangent space in the Euclidean stable cone bundle $\zeta^s$. Consequently, $\psi_i^* u_i \in \Omega^{d_s}(\psi_i(U_i))$ gives positive volume to every $W \in \psi_i(U_i)$. Finally, by using the partition of unity $\{\chi_i\}_{i=1}^m$, we define $\omega_\Sigma = \sum_{i=1}^m \chi_i \psi_i^*(u_i) \in \Omega^{d_s}(M)$.

Proof of Lemma 3.1. We have already used that

$$e^{h_{top}} = \limsup_{n\to+\infty} \left( \sup_{W\in\Sigma} |f^{-n}(W)| \right)^{\frac{1}{n}},$$

where $|f^{-n}(W)|$ is the volume of $f^{-n}(W)$ w.r.t. the measure induced by the Riemannian volume $\omega_0$ on $f^{-n}(W)$ (see Theorem D.7). Let $\omega_\Sigma \in \Omega^{d_s}(M)$ the differential form defined above which gives positive volume to every admissible stable leaf of $\Sigma$. Then, given any $W \in \Sigma$, let $\{W_i\}_{i=1}^l$ be the covering of $f^{-n}(W)$, constructed in

Lemma 2.1. By compactness of every $W_j$, there exists $C_j$ such that

$$\int_{W_j} \omega_{W_j} \leq C_j \int_{W_j} \omega_\Sigma,$$

where $\omega_{W_j}$ is the volume form induced by $\omega_0$ on $W_j$. As a consequence,

$$\begin{aligned}|f^{-n}(W)| \leq \sum_{j=1}^{l} |W_j| \leq \sum_{j=1}^{l} C_j \int_{W_j} \omega_\Sigma \leq \sum_{j=1}^{l} C_j \int_{f^n(W_j)} f_*^n \omega_\Sigma \\ \leq C \int_W f_*^n \omega_\Sigma \leq C \|f_*^n \omega_\Sigma\|_{p,q,d_s} \leq C \|f_*^n\|_{\mathcal{B}^{p,q,d_s} \to \mathcal{B}^{p,q,d_s}},\end{aligned}$$

hence

$$e^{h_{top}} = \limsup_{n \to +\infty} \left( \sup_{W \in \Sigma} |f^{-n}(W)| \right)^{\frac{1}{n}} \leq \limsup_{n \to +\infty} \|f_*^n\|_{\mathcal{B}^{p,q,d_s} \to \mathcal{B}^{p,q,d_s}}^{\frac{1}{n}} = \rho(f_*|_{\mathcal{B}^{p,q,d_s}}),$$

where last equality follows by Lemma A.9. Corollary 2.15 gives $\rho(f_*|_{\mathcal{B}^{p,q,d_s}}) \leq e^{h_{top}}$, hence we conclude that $\rho(f_*|_{\mathcal{B}^{p,q,d_s}}) = e^{h_{top}}$. Q.E.D.

As a consequence of Lemma 3.1 and the quasi-compactness of $f_*$, we can write

$$f_* = \sum_{i=0}^{N} (z_i e^{h_{top}}) \Pi_i + R, \tag{3.2}$$

where every $z_i$ is a complex number of modulus 1, the operator $\Pi_i$ is the finite rank projection on the eigenspace corresponding to the eigenvalue $z_i e^{h_{top}}$ and $R$ is a quasicompact linear operator whose spectral radius is strictly smaller than $e^{h_{top}}$. Moreover, $\Pi_i \circ \Pi_j = \delta_{i,j} \Pi_i$ and $\Pi_i \circ R = R \circ \Pi_i = 0$. Notice that, as a consequence of (2.14), the operator $e^{-nh_{top}} f_*^n$ is bounded for all $n$ and there cannot be Jordan blocks for eigenvalues of modulus $e^{h_{top}}$.

**Remark 3.2.** Up to now, we have not considered orientability issues. We have just assumed that $M$ is an orientable manifold, but we have never made any assumption about the orientation of the stable/unstable foliation. In effect, up to considering a finite covering of $M$ we can always assume that these two foliation are oriented. Moreover, we can also suppose that $f$ preserves the orientation of both foliations. Otherwise, it would be enough to consider $f^2$ in place of $f$. Accordingly, from now on, we assume to work with a diffeomorphism $f$ preserving the orientation of the oriented stable and unstable manifolds.

The rest of this section is devoted to the proof of the following proposition.

**Proposition 3.3.** *$e^{h_{top}}$ is the unique simple maximal eigenvalue of $f_*$ acting on $\mathcal{B}^{p,q,d_s}$. Let $\bar\omega \in \mathcal{B}^{p,q,d_s}$ be a corresponding eigenvector and let $\bar t \in (\mathcal{B}^{p,q,d_s})'$ be the dual eigenvector such $\bar t(\bar\omega) = 1$. The continuous linear operator $\phi \mapsto \bar t(\phi\bar\omega)$, defined on $C^{p+q}$-functions, extends to a bounded linear operator on $C^0$-functions, i.e., it is a measure. In particular, $\mu_{BM}(\cdot) = \bar t(\cdot\ \bar\omega)$ is a positive measure and it is the unique measure of maximal entropy.*

Proof. Let us consider the $d_s$-differential form $\omega_\Sigma$ that gives positive volume to every admissible leaf $W \in \Sigma$ and that is defined before the proof of Lemma 3.1. We set $\bar\omega = \Pi_1\omega_\Sigma$, where $\Pi_1$ is the eigenprojector related to the eigenvalue $e^{h_{top}}$ of (3.2). Accordingly, $f_*\bar\omega = e^{h_{top}}\bar\omega$. Notice that, a priori, $\bar\omega$ could be null, because we do not know yet that $e^{h_{top}}$ is an eigenvalue of $f_*$. On the other hand, we are going to prove that $\bar\omega$ is actually nonzero.

Next lemma recalls [38, Lemma 4.9] adapting it to our setting.

**Lemma 3.4.** *Let $\omega \in \mathcal{B}^{p,q,d_s}$ be an eigenvector for the eigenvalue $z_i e^{h_{top}}$ such that $|z_i| = 1$. Then $\omega$ gives a measure on every admissible leaf $W \in \Sigma$. Moreover, every such $\omega$ is absolutely continuous with bounded density w.r.t. the measure defined by $\bar\omega$.*

Proof of Lemma 3.4. We firstly show that, given $W \in \Sigma$ and $\phi \in \Gamma_0^{q,d_s}(W)$, it holds

$$\left|\int_W \langle\phi,\omega\rangle\omega_W\right| \le C\|\phi\|_{\Gamma_0^{0,d_s}}. \tag{3.3}$$

In fact, since $\Omega^{d_s}(M)$ is dense in $\mathcal{B}^{p,q,d_s}$, $\Pi_{z_i}$ is continuous and $\Pi_{z_i}\Omega^{d_s}(M)$ is closed, we get $\Pi_{z_i}\Omega^{d_s}(M) = \Pi_{z_i}\mathcal{B}^{p,q,d_s}$. Thus, there exists a smooth form $\widetilde\omega \in \Omega^{d_s}(M)$ such that $\Pi_{z_i}\widetilde\omega = \omega$ and, by (3.2),

$$\int_W \langle\phi,\omega\rangle\omega_W = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}(z_i e^{h_{top}})^{-k}\int_W \langle\phi, f_*^k\widetilde\omega\rangle\omega_W. \tag{3.4}$$

Therefore,

$$\begin{aligned}\left|\int_W \langle\phi,\omega\rangle\omega_W\right| &\le \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}e^{-kh_{top}}\left|\int_W \langle\phi, f_*^k\widetilde\omega\rangle\omega_W\right| \\ &\le \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}e^{-kh_{top}}\|\phi\|_{\Gamma_0^{0,d_s}}\|f_*^k\widetilde\omega\|_{0,0,d_s} \le C\|\phi\|_{\Gamma_0^{0,d_s}}\|\widetilde\omega\|_{0,0,d_s}.\end{aligned}$$

where last inequalities follows by the smoothness of $\widetilde{\omega}$ and by (2.14).

We point out that, since $\omega \in \mathcal{B}^{p,q,d_s}$, then above integral can always be estimated by the $C^q$-norm of $\phi$. The lemma we have just proved shows that for eigenvectors corresponding to maximal eigenvalues, the bound is given by the $C^0$ norm of $\phi$.

Next, let us consider a function $\psi \in C^0_0(W)$, then $\psi\omega_W \in \Gamma_0^{0,d_s}$ and we define

$$\int_W \psi \mathcal{M}_W(\omega) = \int_W \langle \psi\omega_W, \omega\rangle \omega_W.$$

As a consequence of (3.3),

$$\left| \int_W \psi \mathcal{M}_W(\omega) \right| \leq C \|\psi\|_{C^0(W)},$$

hence $\omega$ defines a measure on any admissible leaf $W \in \Sigma$. When $\omega = \bar{\omega} = \Pi_1 \omega_\Sigma$, the equality (3.4) implies that $\mathcal{M}_W(\bar{\omega})$ is a nonnegative measure. In addition, for every $\psi \in C^q_0(W)$,

$$\left| \int_W \psi \mathcal{M}_W(\omega) \right| = \left| \int_W \langle \psi\omega_W, \omega\rangle \omega_W \right| = \left| \int_W \langle \psi\omega_W, \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} (z_i e^{h_{top}})^{-k} f^k_* \omega\rangle \omega_W \right| \leq$$
$$\leq \int_W \left| \langle \psi\omega_W, \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-kh_{top}} f^k_* \omega\rangle \right| \omega_W = \int_W |\langle \psi\omega_W, \Pi_1\omega\rangle| \omega_W \leq$$
$$\leq C \int_W |\langle \psi\omega_W, \bar{\omega}\rangle| \omega_W \leq C \int_W |\psi| \mathcal{M}_W(\bar{\omega})$$

and, since above inequality extends to continuous functions by density, the measure $\mathcal{M}_W(\omega)$ is absolutely continuous w.r.t. $\mathcal{M}_W(\bar{\omega})$ with bounded density.

Q.E.D.

Let $\omega \in \Omega^{d_s}$ and let $W, W' \in \Sigma$ be admissible leaves, whose intersection is again a $d_s$-dimensional submanifold. In addition, assume that the orientations of $W$ and $W'$ agree on $W \cap W'$. Then, for every function $\psi \in C^q_0(W)$, the operator $\mathcal{M}_W(\omega)$, such that

$$\int_W \psi \mathcal{M}_W(\omega) = \int_W \langle \psi\omega_W, \omega\rangle \omega_W,$$

is a bounded operator on $C^q_0(W)$. Moreover, for $\psi \in C^q_0(W \cap W')$

$$\int_W \psi \mathcal{M}_W(\omega) = \int_W \langle \psi\omega_W, \omega\rangle \omega_W = \int_W \langle \psi i^*_W \omega_0, \omega\rangle i^*_W \omega_0 =$$
$$= \int_{W'} \langle \psi i^*_{W'} \omega_0, \omega\rangle i^*_{W'} \omega_0 = \int_{W'} \langle \psi\omega_{W'}, \omega\rangle \omega_{W'} = \int_{W'} \psi \mathcal{M}_{W'}(\omega),$$

where $i_W$, resp. $i_{W'}$, are the embeddings of $W$, resp $W'$, in $M$. Accordingly, every $\omega \in \Omega^{d_s}$ defines an element of the dual of $C_0^q(\mathcal{S})$, where $\mathcal{S}$ can be obtained by gluing elements of $\Sigma$. By density, if $\omega \in \mathcal{B}^{p,q,l}$ is an eigenvector such that $\Pi_{z_i}\omega = \omega$, then $\omega$ induces a measure, denoted by $\mathcal{M}(\omega)$, on every oriented stable manifold of $M$.

The following lemma takes up [38, Lemma 4.10] by adapting it to our context.

**Lemma 3.5.** *The function $\omega \mapsto \mathcal{M}(\omega)$, defined on the eigenspace $\Pi_{z_i}\mathcal{B}^{p,q,l}$ corresponding to the eigenvalue $z_i e^{h_{top}}$, is injective. Additionally, $\bar{\omega}$ is nonzero.*

Proof of Lemma 3.5. Assume that $\mathcal{M}(\omega) = 0$ for some $\omega \in \Pi_{z_i}\mathcal{B}^{p,q,d_s}$. We firstly show that $\|\omega\|_{0,q,d_s} = 0$. Along the proof of Lemma 2.10, we proved that there exists a constant $C > 0$ such that, for any $\epsilon > 0$, there are a finite number of admissible leaves $W_1, \dots, W_k \in \Sigma$, such that for every $W \in \Sigma$ and for any $\phi \in \Gamma_0^{q,d_s}$, there is at least one $W_j$ such that

$$\left| \int_W \langle \phi, \omega \rangle \omega_W - \int_{W_j} \langle \bar{\phi}_1, \omega \rangle \omega_{W_j} \right| \le C\epsilon \|\phi\|_{\Gamma_0^{q,d_s}}, \tag{3.5}$$

where $\bar{\phi}_1 = \Psi_1^* \phi$ (see the proof of Lemma 2.10). Since $W_j$ is a compact oriented $d_s$-dimensional manifold endowed with a volume form $\omega_{W_j}$, there exists $\bar{\psi}_1 \in C_0^q(W_j)$ such that $\bar{\phi}_1 = \bar{\psi}_1 \omega_{W_j}$. Consequently, if $W_j$ is contained in a stable manifold, then

$$\int_{W_j} \langle \bar{\phi}_1, \omega \rangle \omega_{W_j} = \int_{W_j} \bar{\psi}_1 \mathcal{M}(\omega) = 0$$

and above inequality becomes

$$\left| \int_W \langle \phi, \omega \rangle \omega_W \right| \le C\epsilon \|\phi\|_{\Gamma_0^{q,d_s}}.$$

More generally, there exist $n_0 \in \mathbb{N}$ and a sequence $\{\epsilon_n\}_{n>n_0}$, going exponentially fast to zero, such that, for each $n > n_0$ and for any full admissible leaf $\widetilde{W} \in \widetilde{\Sigma}$,

$$f^{-n}(W) \subseteq \cup_{i=1}^{l} W_i^{(n)} \subseteq f^{-n}(\widetilde{W}),$$

as stated in Lemma 2.1. In addition, every $W_i^{(n)}$ is $\epsilon_n$-closed to some leaf contained in a stable manifold in the sense of (3.5). Next, given $W \in \Sigma$ and $\phi \in \Gamma_0^{q,d_s}(W)$, there exists $\psi \in C_0^q(W)$ such that $\phi = \psi\omega_W$. We compute

$$\begin{aligned}
&\left|\int_W \langle \phi, \omega\rangle \omega_W\right| = \left|e^{-nh_{top}} \int_W \langle \phi, f_*^n\omega\rangle \omega_W\right| = \\
=&\left|e^{-nh_{top}} \int_W (-1)^{l(\dim(M)-l)} \langle \star (f^n)^* \star \phi, \omega\rangle \circ f^{-n} \det(df^{-n})\omega_W\right| \leq \\
\leq& e^{-nh_{top}} \sum_i \left|\int_{W_i^{(n)}} (-1)^{l(\dim(M)-l)} \langle \star (f^n)^* \star \phi, \omega\rangle \det(df^{-n}) \circ f^n \lambda_n^s \omega_{W_i}\right| = (\diamondsuit)
\end{aligned}$$

Notice that $\langle \star(f^n)^* \star \phi, \omega\rangle = \psi \circ f^n \langle \star (f^n)^* \star \omega_W, \omega\rangle$, while $(-1)^{l(\dim(M)-l)} \star (f^n)^* \star \omega_W = (\lambda_n^s)^{-1} \det(df^n)\omega_W$. In effect, by definition $\omega_W \wedge \star\omega_W = \omega_0$, resp. $\omega_{W_i^{(n)}} \wedge \omega_{W_i^{(n)}} = \omega_0$, on $W$, resp. on $W_i^{(n)}$, hence

$$\lambda_n^s \omega_{W_i^{(n)}} \wedge (f^n)^* \star \omega_W = (f^n)^*\omega_W \wedge (f^n)^* \star \omega_W = \det(df^n)\omega_0 = \det(df^n)\omega_{W_i^{(n)}} \wedge \star\omega_{W_i^{(n)}},$$

that is $\langle \omega_{W_i^{(n)}}, (-1)^{l(\dim(M)-l)} \star (f^n)^* \star \omega_W\rangle = \langle \omega_{W_i^{(n)}}, (\lambda_n^s)^{-1} \det(df^n)\omega_{W_i^{(n)}}\rangle$ and that proves the second equality. Continuing the calculation

$$(\diamondsuit) = e^{-nh_{top}} \sum_i \left|\int_{W_i^{(n)}} \psi \circ f^n \langle \omega_{W_i^{(n)}}, \omega\rangle \omega_{W_i^{(n)}}\right| \leq C\epsilon_n e^{-nh_{top}} \#\{W_i^{(n)}\} \|\psi \circ f^n\|_{C^q}$$

Since $|e^{-nh_{top}} \#\{W_i^{(n)}\}| \leq C$ and $\epsilon_n$ decays exponentially to zero, we obtain that $\|\omega\|_{0,q,d_s} = 0$. Let us proceed by induction on $p$ in order to prove that $\|\omega\|_{p,q,d_s} = 0$. Assume that the result is true up to $p-1$. Then, using the Lasota-Yorke inequality (2.17),

$$\|\omega\|_{p,q,d_s} = e^{-nh_{top}} \|f_*^n\omega\|_{p,q,d_s} \leq C\lambda^{-n\min\{p,q\}} \|\omega\|_{p,q,d_s} \xrightarrow[n\to+\infty]{} 0$$

We conclude that $\omega = 0$, that is $\omega \mapsto \mathcal{M}(\omega)$ is injective. Finally, assume by contradiction that $\bar\omega = 0$, then, by injectivity $\mathcal{M}(\bar\omega) = 0$. Thus, for any other eigenvector $\omega$ corresponding to an eigenvalue of modulus $e^{h_{top}}$, it must hold that $\mathcal{M}(\omega) = 0$, because $\mathcal{M}(\omega)$ is absolutely continuous with respect to $\mathcal{M}(\omega)$ by Lemma 3.4. We conclude that any such $\omega = 0$, hence the spectral radius of $f_*$ acting on $\mathcal{B}^{p,q,d_s}$ is strictly smaller than $e^{h_{top}}$ and this contradicts Lemma 3.1.

Q.E.D.

The following step consists in proving that, assuming $f$ topologically transitive, $e^{h_{top}}$ is a simple eigenvalue and it is the unique maximal eigenvalue of $f_*|_{\mathcal{B}^{p,q,d_s}}$.

Before proceeding with the proof, we need to recall some basic notions regarding continuous leafwise measure and their properties. We avoid to rewrite the proofs of results about this concept, but refer the reader to the survey [38, Section 9], where they are proved in great generality.

**Definition 3.6.** *Let $X$ be a locally compact space. We assume that there exists an atlas $\{U, \phi_U\}$ such that $U \subseteq X$ is open and it is homeomorphic to $\mathcal{B}_d(0,1) \times K_U$, for some locally compact space $K_U$, under the homeomorphism $\phi_U$. In addition, we assume that the changes of coordinates fulfill $\phi_U \circ \phi_V^{-1}(x,y) = (a(x,y), b(y))$, i.e., they map leaves to leaves. A continuous leafwise measure $m$ is a family of Radon measures, each one defined on a leaf, such that, for every continuous function $\psi$ supported on the chart $(U, \phi_U)$, the integral*

$$I_\psi(y) = \int_{\phi_U^{-1}(\mathcal{B}_d(0,1)\times\{y\})} \phi \; dm$$

*is a continuous function of $y$.*

Suppose that there exists a family of metrics on $X$, each one defined on a leaf, such that they also vary continuously with the leaf. Let $T\colon X \to X$, be a continuous, leaves preserving homeomorphism, which uniformly expands distances on every leaf (i.e., there exists $\delta > 0$ and $C > 1$, such that $d_W(Tx, Ty) \geq C d_W(x,y)$, whenever $x, y$ belong to the leaf $W$ and $d_W(x,y) < \delta$). We can now recall the result we need in order to prove the subsequent Lemma 3.8.

**Proposition 3.7.** *[38, Proposition 9.1, Proposition 9.4]*
*Let $m$ be a nonnegative continuous leafwise measure and let $m'$ be another complex continuous leafwise measure. Assume that there exists $C > 0$ for which $|m'| \leq Cm$ on every leaf. Moreover, suppose that $T^*m = m$ and $T^*m' = \gamma m'$, with $|\gamma| = 1$. Finally, assume that $T$ is topologically mixing and that given any open set $O$ of a leaf, there exists $x \in O$ with dense orbit. Then there is a $c \in \mathbb{C}$ such that $m' = cm$, hence $\gamma = 1$ or $m' = 0$.*

**Lemma 3.8.** *Under the assumption that $f$ is topologically transitive, $e^{h_{top}}$ is the unique maximal eigenvalue of $f_*|_{\mathcal{B}^{p,q,d_s}}$. In addition, $e^{h_{top}}$ is simple.*

Proof of Lemma 3.8. We already know that $\bar{\omega} \neq 0$ and $f_*\bar{\omega} = e^{h_{top}}\bar{\omega}$, hence $e^{h_{top}}$ is an eigenvalue of $f_*|_{\mathcal{B}^{p,q,d_s}}$. Let $\omega \in \mathcal{B}^{p,q,d_s}$ be any other eigenvector with corresponding eigenvalue $\gamma e^{h_{top}}$, such that $|\gamma| = 1$. As a consequence of Lemma 3.4, $\omega$ is a continuous leafwise measure on each stable manifold and $|\mathcal{M}(\omega)| \le C\mathcal{M}(\bar{\omega})$. We check that the other hypotheses of Proposition 3.7 hold true. Firstly, an Anosov diffeomorphism is topologically transitive if and only if it is topologically mixing [17, Theorem 5.10.3]. Next, setting $T = f^{-1}$, then $T$ is uniformly expanding on stable leaves of $f$. Moreover, defining $T^*\mathcal{M}(\omega) = \mathcal{M}(T^*\omega)$, we get $T^*\mathcal{M}(\bar{\omega}) = e^{h_{top}}\mathcal{M}(\bar{\omega})$ and $T^*\mathcal{M}(\omega) = \gamma e^{h_{top}}\mathcal{M}(\omega)$. It remains to prove that every open set $O$ contained in a stable manifold admits a point $x \in O$ with dense orbit. Let $x$ be a point in $O$. By topological transitivity, there exists a $y$ close to $x$, with dense orbit. By classical results (see Theorem 1.10), the local stable manifold centered at $x$, $W^s_\epsilon(x)$, and the local unstable manifold centered at $y$, $W^u_\epsilon(y)$, intersect in exactly one point $z = [x, y] = W^s_\epsilon(x) \cap W^u_\epsilon(y)$. Accordingly, $z \in O$ and its orbit is dense. By Proposition 3.7, we conclude that $\mathcal{M}(\omega) = c\mathcal{M}(\bar{\omega})$, which in turn implies that $\omega = c\bar{\omega}$ and $\gamma = 1$.

Q.E.D.

It remains to prove that the eigenvectors corresponding to the unique eigenvalue $e^{h_{top}}$ defines a positive invariant measure and this is the measure of maximal entropy. Let $\bar{\omega} \in \mathcal{B}^{p,q,d_s}$ be, as above, the eigenvector for which $f_*\bar{\omega} = e^{h_{top}}\bar{\omega}$. Let $\bar{t} \in (\mathcal{B}^{p,q,d_s})'$ the unique element of the dual space of $\mathcal{B}^{p,q,d_s}$ such that[1] $f'_*\bar{t} = e^{h_{top}}\bar{t}$ and $\bar{t}(\bar{\omega}) = 1$.

**Lemma 3.9.** *The linear operator* $\mu_{BM} = \bar{t}(\,\cdot\,\bar{\omega})$, *actually defined on* $C^{p+q}(M)$ *functions, extends to a bounded linear operator on continuous functions, i.e., it is a measure. In addition, for every* $\psi \in C^0(M)$, $\mu_{BM}(\psi \circ f) = \mu_{BM}(\psi)$ *and* $\mu_{BM}$ *is a positive probability measure.*

Proof of Lemma 3.9. As above, we adapt to our setting the proofs of [38, Lemma 6.1, Lemma 6.2]. Notice that, for every $\omega \in \mathcal{B}^{p,q,l}$ and for every function $\psi \in C^{p+q}(M)$, the product $\psi\omega \in \mathcal{B}^{p,q,l}$. Moreover, for every $\omega \in \mathcal{B}^{p,q,d_s}$ it holds

$$|\bar{t}(\omega)| \le C\|\omega\|_{0,p+q,d_s}. \tag{3.6}$$

In effect, using (2.22),

$$\begin{aligned}|\bar{t}(\omega)| =& e^{-pnh_{top}}|(f'_*)^{pn}\bar{t}(\omega)| = e^{-pnh_{top}}|\bar{t}(f^{pn}_*\omega)| \le e^{-pnh_{top}}\|f^{pn}_*\omega\|_{p,q,d_s} \le \\ \le& C\lambda^{-n}\|\omega\|_{p,q,d_s} + C\|\omega\|_{0,p+q,d_s}\end{aligned}$$

[1]We recall that the dual action of $f_*$ is the linear operator $f'_*\colon (\mathcal{B}^{p,q,d_s})' \to (\mathcal{B}^{p,q,d_s})'$ such that, for each $t \in (\mathcal{B}^{p,q,d_s})'$ and for each $\omega \in \mathcal{B}^{p,q,d_s}$, $f'_*(t)(\omega) = t(f_*\omega)$.

Taking the limit for $n$ going to $\infty$ we get (3.6). Next, by (3.3), we obtain that $\|\psi\bar{\omega}\|_{0,p+q,d_s} \leq C\|\psi\|_{C^0}$. Thus,

$$|\bar{t}(\psi\bar{\omega})| \leq C\|\psi\|_{C^0}$$

and $\bar{t}(\,\cdot\,\bar{\omega})$ extends to a bounded operator on continuous functions. Furthermore,

$$\mu_{BM}(\psi\circ f) = \bar{t}(\psi\circ f\bar{\omega}) = e^{h_{top}}\bar{t}(f^*(\psi\bar{\omega})) = e^{h_{top}}(f^*)'\bar{t}(\psi\bar{\omega}) = \bar{t}(\psi\bar{\omega}) = \mu_{BM}(\psi),$$

which proves that $\mu_{BM}$ is $f$-invariant.

Let us prove that $\mu_{BM}$ is a positive measure. By the spectral decomposition (3.2), we can write

$$\lim_{n\to+\infty} e^{-nh_{top}} f_*^n\omega = \pi_1(\omega)\bar{\omega},$$

whenever $\omega \in \mathcal{B}^{p,q,d_s}$, where $\pi_1$ is a linear form on $\mathcal{B}^{p,q,l}$. Moreover, since $\bar{t}(\bar{\omega}) = 1$,

$$\pi_1(\omega) = \pi_1(\omega)\bar{t}(\bar{\omega}) = \bar{t}(\pi_1(\omega)\bar{\omega}) = \bar{t}(\lim_{n\to+\infty} e^{-nh_{top}} f_*^n\omega) = \lim_{n\to+\infty} e^{-nh_{top}}\bar{t}(f_*^n\omega) = \bar{t}(\omega)$$

Accordingly,

$$\lim_{n\to+\infty} e^{-nh_{top}} f_*^n\omega = \bar{t}(\omega)\bar{\omega}$$

Given $\phi, \psi \geq 0$ and a leaf $W \in \Sigma$, we get

$$0 \leq \lim_{n\to+\infty} \int_W \langle \phi\omega_W, e^{-nh_{top}} f_*^n(\psi\omega_\Sigma)\rangle\omega_W = \bar{t}(\psi\omega_\Sigma)\int_W \phi\mathcal{M}_W(\bar{\omega}) \tag{3.7}$$

Lemma (3.4) shows that $\mathcal{M}(\omega)$ is a nonnegative and nonzero measure. Consequently, we can choose $W$ and $\phi > 0$ so that last integral of (3.7) is strictly positive. This shows that, for every $\psi \geq 0$, $\bar{t}(\psi\omega_\Sigma) \geq 0$. Thus, for $\psi \geq 0$, we get

$$\begin{aligned}\mu_{BM}(\psi) =& \bar{t}(\psi\bar{\omega}) = \lim_{n\to+\infty} e^{-nh_{top}}\bar{t}(\psi f_*^n\omega_\Sigma) = \lim_{n\to+\infty} e^{-nh_{top}}\bar{t}(f_*^n(\psi\circ f^n\omega_\Sigma)) = \\ &= \lim_{n\to+\infty} \bar{t}(\phi\circ f^n\omega_\Sigma) \geq 0,\end{aligned}$$

which shows that $\mu_{BM}$ is a positive measure. Finally, since $\bar{t}(\bar{\omega}) = 1$, we conclude that $\mu_{BM}$ is a probability measure.

Q.E.D.

It remains to prove that $\mu_{BM}$ is the unique measure of maximal entropy. What follows is proved in full generality in [38, Theorem 6.4], where the authors showed that the measure given by the maximal eigenvalue of their transfer operator maximizes the variational principle. Since we are studying the action of pushforward operator, our case can be read by their paper considering a null potential. As above, we give the proofs adapting them to our case. The following Lemma 3.10 is a streamlined version of [38, Proposition 6.3].

**Lemma 3.10.** *Given $n \in \mathbb{N}$, $x \in M$ and $\epsilon > 0$, we denote by*

$$B_n(x,\epsilon) = \{y \in M |\ d(f^{-i}(y), f^{-i}(x)) < \epsilon,\ \textit{for } i = 0, 1, \dots, n-1\}$$

*the dynamical ball centered at $x$, of length $n$ and radius $\epsilon$. Then, there exist two constants $c_\epsilon, C_\epsilon > 0$ such that*

$$c_\epsilon e^{-nh_{top}} \leq \mu_{BM}(B_n(x,\epsilon)) \leq \overline{\mu_{BM}(B_n(x,\epsilon))} \leq C_\epsilon e^{-nh_{top}}$$

Proof of Lemma 3.10. Let $\phi \in C^{p+q}$ be a compactly supported function such that $0 \leq \phi \leq 1$, $\mathrm{supp}(\phi) \subseteq B_n(x,\epsilon)$ and $\phi|_{B_n(x,\epsilon/2)} = 1$. We show that

$$c_\epsilon e^{-nh_{top}} \leq \mu_{BM}(\phi) \leq C_\epsilon e^{-nh_{top}},$$

which implies the lemma. Let $W \in \Sigma$, $\psi \in \Gamma_0^q(W)$, with $\|\psi\|_{\Gamma_0^q(W)} \leq 1$. Then, writing $\psi = \psi_0 \omega_W$ for some $\psi \in C_0^q(W)$, and proceeding as in the proof of Lemma 3.5, we get

$$\begin{aligned}\int_W \langle \psi, \phi\bar{\omega}\rangle \omega_W =& e^{-nh_{top}} \int_W \psi_0 \phi \langle \omega_W, f_*^n \bar{\omega}\rangle \omega_W = e^{-nh_{top}} \int_W \psi_0 \phi \langle \omega_W, \bar{\omega}\rangle \circ f^{-n} (\lambda_n^s)^{-1} \omega_W = \\ =& e^{-nh_{top}} \sum_j \int_{W_j} \rho_j \psi_0 \circ f^n \phi \circ f^n \langle \omega_W, \bar{\omega}\rangle \omega_{W_j}\end{aligned}$$

The number of $W_j$ on which the integral is nonzero is uniformly bounded, because $\phi$ is supported in $B_n(x,\epsilon)$. Accordingly,

$$|\mu_{BM}(\phi)| = |\bar{t}(\phi\bar{\omega})| \leq C \|\phi\bar{\omega}\|_{0,q,d_s} \leq C \sup_{W,\psi} \left| \int_W \langle \psi, \phi\bar{\omega}\rangle \omega_W \right| \leq C_\epsilon e^{-nh_{top}}$$

To estimate the other inequality, firstly notice that $\mathcal{M}(\bar{\omega})$ gives strictly positive measure to any open piece of stable leaf. If it were not the case, there would be a ball $\mathcal{B} = \mathcal{B}_{d_s}(x,\delta)$, contained in a stable manifold, such that $\mathcal{M}(\bar{\omega})(\mathcal{B}) = 0$. By invariance, $\mathcal{M}(\bar{\omega})$ also assigns zero measure to $f^{-n}(\mathcal{B})$. By reasoning as in the proof of Lemma 3.8, using the topological mixing property, $f^{-n}(\mathcal{B})$ will meet a point $z \in W^s(x)$ with dense orbit. Finally, since $\mathcal{M}(\bar{\omega})$ is a continuous leafwise measure and since the orbit of $z$ is dense, we conclude that $\mathcal{M}(\bar{\omega}) = 0$, which contradicts Lemma 3.5.

Next we prove that, if $W$ is a piece of stable manifold and $W$ contains a point $y$ with $d(x, y) < \epsilon/10$ and $d(y, \delta W) \geq \epsilon$, then

$$\int_W \phi\mathcal{M}(\omega) \geq c_\epsilon e^{-nh_{top}}.$$

In fact, $f^{-n}(W)$ contains a $d_s$-dimensional ball $\mathcal{B}$ of radius $\epsilon/10$, which is contained in $f^{-n}(B_n(x, \epsilon/2))$. Consequently,

$$\begin{aligned}\int_W \phi\mathcal{M}(\bar{\omega}) &= \int_W \phi e^{-nh_{top}}\mathcal{M}(f_*^n\bar{\omega}) = \\ &= \int_{f^{-n}(W)} \phi \circ f^n e^{-nh_{top}}\mathcal{M}(\bar{\omega}) \geq e^{-nh_{top}} \int_{\mathcal{B}} \mathcal{M}(\bar{\omega}) \geq c_\epsilon e^{h_{top}},\end{aligned}$$

where, in the last inequality, we used that $\mathcal{M}(\bar{\omega})$ assigns positive measure to $\mathcal{B}$. Notice that, by compactness, the constant $c_\epsilon$ does not depend on the leaf $W$.

Topological mixing also implies the following fact: for every $\delta > 0$ there exists $M$, depending on $\epsilon$ and $\delta$, such that, for each $m \geq M$, there is a constant $C$, which depends on $\epsilon$, $\delta$ and $m$, such that, for every connected $W$ contained in a stable manifold, with $\mathrm{diam}(W) \geq 2\delta$, it holds

$$\int_{f^{-m}(W)} \phi\mathcal{M}(\bar{\omega}) \geq Ce^{-nh_{top}}. \tag{3.8}$$

Finally, we prove that, for a full admissible stable leaf $\widetilde{W}$, contained in a stable manifold, there exists $C$, depending on $\epsilon$ and $\widetilde{W}$, such that, if $p$ is large enough,

$$e^{-ph_{top}} \int_{\widetilde{W}} f_*^p(\phi\mathcal{M}(\omega)) \geq Ce^{-nh_{top}} \tag{3.9}$$

Let $L$ be a positive integer such that $M \leq L \leq p$. Let $\{W_j\}$ be the subdivision of $f^{-p}(W)$ as described by Lemma 2.1. Then

$$\int_{\widetilde{W}} f_*^p(\phi\mathcal{M}(\bar{\omega})) = \int_{f^{-p+L}(\widetilde{W})} f_*^p(\phi\mathcal{M}(\bar{\omega})) \geq C\sum_j \int_{B_j} f_*^L(\phi\mathcal{M}(\bar{\omega})) = C\sum_j \int_{f^{-L}(B_j)} \phi\mathcal{M}(\bar{\omega})),$$

where $B_j$ is a $d_s$-dimensional ball of radius $2\delta$ contained in $f^L(W_j)$. To every integral on $f^{-L}(B_j)$ we can apply (3.8) and, since the sum growth as $e^{ph_{top}}$, we obtain that

$$\int_{\widetilde{W}} f_*^p(\phi\mathcal{M}(\bar{\omega})) \geq Ce^{ph_{top}}e^{-nh_{top}},$$

which implies (3.9). Since

$$\lim_{p\to+\infty} e^{-ph_{top}} f_*^p(\phi\bar{\omega}) = \bar{t}(\phi\bar{\omega}),$$

considering a $W$ which satisfies (3.9), we conclude that

$$\mu_{BM}(\phi) \geq \bar{t}(\phi\bar{\omega}) \geq \lim_{p\to+\infty} e^{-ph_{top}} \int_{\widetilde{W}} f_*^p(\phi\mathcal{M}(\bar{\omega})) \geq c_\epsilon e^{-nh_{top}}.$$

Q.E.D.

Next result is the last ingredient for the proof of Proposition 3.3.

**Lemma 3.11.** *The measure $\mu_{BM}$ is the unique measure of maximal entropy, i.e., it is the Bowen-Margulis measure of the system.*

PROOF OF 3.11. Recall that, for every invariant measure $\mu$ for $f$, the metric entropy $h_\mu(f)$ measure the average information given by the knowledge of the present state, assuming to know arbitrarily long past. The Variational Principle (Theorem D.6) states that

$$\sup\{h_\mu(f) |\ \mu \text{ is a } f\text{-invariant measure}\} = h_{top}.$$

For a basic overview of these topics we refer to [47, Sections 4.3-4.5]. Next, notice that the spectral decomposition (3.2) implies that $\mu_{BM}$ is mixing, hence ergodic. Thus, by the local entropy theorem [16], since $\mu_{BM}$ is a $f$-invariant, ergodic, probability measure, we obtain

$$\lim_{\epsilon\to+\infty} \limsup_{n\to+\infty} -\frac{1}{n}\log(\mu_{BM}(B_n(x,\epsilon))) = \lim_{\epsilon\to+\infty} \liminf_{n\to+\infty} -\frac{1}{n}\log(\mu_{BM}(B_n(x,\epsilon))) = h_{\mu_{BM}}(f)$$

By Lemma 3.10 we conclude that $h_{top} = h_{\mu(BM)}(f)$, hence $\mu_{BM}$ is a measure of maximal entropy. The proof that this $\mu_{BM}$ is the unique measure of maximal entropy easily follows repeating the proof of [47, Theorem 20.3.7]

Q.E.D.

This concludes the proof of Proposition (3.3).

Q.E.D.

## 3.2 Anisotropic de Rham cohomology and spectrum

We recall that the space of $C^\infty$ (complex) differential forms $\Omega^l(M)$ with the exterior derivative $d\colon \Omega^l(M) \to \Omega^{l+1}(M)$ is a cochain complex, i.e., $d \circ d = 0$. $\omega \in \Omega^l(M)$ is closed if $d\omega = 0$, while $\omega$ is exact if there exists $u \in \Omega^{l-1}(M)$ such that $du = \omega$. Since $d \circ d = 0$, exact forms are a vector subspace of closed forms. Accordingly, it makes sense to define the de Rham cohomology group with complex coefficients $H^l_{dR}(M, \mathbb{C}) = H^l_{dR}(M)$ as the quotient of closed $l$-forms w.r.t. exact $l$-forms. The pushforward $f_*$ of a $C^\infty$-diffeomorphism $f$ on $M$ preserves closed and exact forms, hence it induces a linear map from the cohomology group $H^l_{dR}(M)$ to itself defined by $f_\#[\omega] = [f_*\omega]$.

Next lemma give us the possibility to extend these ideas to our anisotropic Banach spaces.

**Lemma 3.12.** *The exterior derivative extends to a continuous operator, denoted by the same letter, $d\colon \mathcal{B}^{p,q,l} \to \mathcal{B}^{p-1,q+1,l+1}$. It holds true again that $d \circ d = 0$.*

Proof. Consider $h \in \Omega^l(M)$, $W \in \Sigma$, $\phi \in \Gamma_0^{p+q-1,l+1}$ and $v_1, \dots, v_{p-1} \in \mathcal{V}^{p+q-1}(U(W))$. If $W \subseteq \psi_i(U_i)$, then we can write, using coordinates, $h = h \circ \chi_i = \sum_{\overline{j} \in \mathcal{J}_l} h_{\overline{j}} dx_{\overline{j}}$ on $\psi_i(\mathcal{B}(0, 3\rho))$. Accordingly,

$$\left| \int_W \langle \phi, L_{v_1} \dots L_{v_{p-1}} dh \rangle \omega_W \right| = \left| \int_W \langle \phi, L_{v_1} \dots L_{v_{p-1}} d \left( \sum_{\overline{j} \in \mathcal{J}_l} h_{\overline{j}} dx_{\overline{j}} \right) \rangle \omega_W \right| =$$

$$= \left| \sum_{\overline{j} \in \mathcal{J}_l} \sum_{s=1}^{\dim(M)} \int_W \langle \phi, L_{v_1} \dots L_{v_{p-1}} \, \partial_{x_s} h_{\overline{j}} dx_{\overline{j}} \wedge dx_s \rangle \omega_W \right| \le C \|\phi\|_{\Gamma_0^{p+q-1,l+1}} \|h\|_{p,q,l}.$$

We conclude that $d$ extends to a bounded operator $d\colon \mathcal{B}^{p,q,l} \to \mathcal{B}^{p-1,q+1,l+1}$. Let us prove $d \circ d = 0$. Recall that, given two differential forms $h, g \in \Omega^l(M)$, $h$ behaves as a current in the following way:

$$i(h)(g) = (h, g) = \int_M \langle h, g \rangle \omega_0 = \int_M h \wedge \star g.$$

Consequently, given $h \in \mathcal{B}^{p,q,l}$ and a sequence $h_n \in \Omega^l(M)$ converging to $h$ in the $\|\cdot\|_{p,q,l}$-norm, then $dh_n$ converges to $dh$ in $\mathcal{B}^{p-1,q+1,l+1}$ and, for each $g \in \Omega^{l+1}(M)$,

$$i(dh)(g) = \lim_{n \to +\infty} i(dh_n)(g) = \lim_{n \to +\infty} i(h_n)(\delta g) = i(h)(\delta g),$$

where $\delta$, defined in Definition B.3, is the dual operator of $d$ (Lemma B.4). Accordingly, since $\delta \circ \delta = 0$, we conclude that, for $h \in \mathcal{B}^{p,q,l}$, $i(d \circ dh)(g) = i(h)(\delta \circ \delta g) = 0$, for any $g \in \Omega^{l+2}(M)$, hence $d \circ d = 0$.

Q.E.D.

We say, by analogy, that a current $\omega \in \mathcal{B}^{p,q,l}$ is closed if $d\omega = 0$, while it is exact if there exists $u \in \mathcal{B}^{p+1,q-1,l-1}$ such that $du = \omega$. As a consequence of Lemma 3.12, we define the anisotropic De Rham cohomology $\widetilde{H}^{p,q,l}_{dR}(M)$ as the quotient of closed currents w.r.t. exact currents of $\mathcal{B}^{p,q,l}$. Since $f_*\colon \mathcal{B}^{p,q,l} \to \mathcal{B}^{p,q,l}$ and $d\colon \mathcal{B}^{p+1,q-1,l-1} \to \mathcal{B}^{p,q,l}$ are both continuous linear operators, $f_*$ sends closed currents in closed currents and exact currents in exact currents. Consequently, it induces a linear map on $\widetilde{H}^{p,q,l}_{dR}(M)$ such that $f_\#[\omega] = [f_*\omega]$.

Next proposition relates the spectrum of $f_*$ acting on anisotropic Banach spaces and the spectrum $f_\#$ on anisotropic de Rham cohomology. We just consider the spectrum of $f_*$ contained in the set $\{z \in \mathbb{C} \mid |z| > \nu\}$, where $\nu$ is the bound defined in (3.1). Accordingly, we only consider discrete spectrum.

**Proposition 3.13.**

$$\sigma(f_*|_{\mathcal{B}^{p,q,l}}) \cap \{|z| > \nu\} \subseteq \Big[\sigma(f_*|_{\mathcal{B}^{p+1,q-1,l-1}}) \cup \sigma(f_\#|_{\widetilde{H}^{p,q,l}_{dR}}) \cup \sigma(f_*|_{\mathcal{B}^{p-1,q+1,l+1}})\Big] \cap \{|z| > \nu\},$$

Proof. Let $\omega \in \mathcal{B}^{p,q,l}$ be an eigenvector of the pushforward operator $f_*$ corresponding to the eigenvalue $\mu$, with $|\mu| > \nu$, that is $f_*\omega = \mu\omega$. If $\omega$ is not closed, then $f_*d\omega = df_*\omega = \mu d\omega$, i.e., $d\omega \neq 0$ is an eigenvector for $f_*$ in $\mathcal{B}^{p-1,q+1,l+1}$. This proves that $\mu \in \sigma(f_*|_{\mathcal{B}^{p-1,q+1,l+1}})$. On the other hand, if $\omega$ is closed, i.e., $d\omega = 0$ we need to distinguish two cases. If $\omega$ is not exact, then it defines a nontrivial cohomology class $[\omega] \in \widetilde{H}^{p,q,l}_{dR}$ and, by definition, $f_\#[\omega] = [f_*\omega] = [\mu\omega] = \mu[\omega]$, hence $\mu \in \sigma(f_\#|_{\widetilde{H}^{l}_{dR}})$. Finally, if $\omega$ is exact, then there exists $q \in \mathcal{B}^{p+1,q-1,l-1}$ such that $\omega = dq$. It follows that $f_*dq = df_*q = \mu dq$. This is not enough to conclude that $\mu \in \sigma(f_*|_{\mathcal{B}^{p+1,q-1,l-1}})$, because it only gives that $d(f_*q - \mu q) = 0$, hence $f_*q = \mu q + v$ with $v \in \mathcal{B}^{p+1,q-1,l-1}$ closed. On the other hand, if the operator $f_* - \mu\mathrm{id}$ was invertible on closed $(l-1)$-currents of $\mathcal{B}^{p+1,q-1,l-1}$, then $u = (f_* - \mu\mathrm{id})^{-1}v$ and $u$ would be closed. But $du = \omega \neq 0$, hence we obtain that $f_* - \mu\mathrm{id}$ cannot invertible on closed currents of $\mathcal{B}^{p+1,q-1,l-1}$. Accordingly, $\mu$ must be an eigenvalue with a closed eigenvector in $\mathcal{B}^{p+1,q-1,l-1}$, because the spectrum of $f_*|_{\mathcal{B}^{p+1,q-1,l-1}}$ in $\{z \in \mathbb{C}|\ |z| > \nu\}$ is discrete.

Q.E.D.

We can also prove that the spectrum of the action on anisotropic cohomology out of the ball of radius $\nu$ is included into the spectrum of $f_*$.

**Proposition 3.14.**

$$\sigma(f_\#|_{\widetilde{H}^{p,q,l}_{dR}}) \cap \{|z| > \nu\} \subseteq \sigma(f_*|_{\mathcal{B}^{p,q,l}}) \cap \{|z| > \nu\}$$

Proof. If $[\omega] \in \widetilde{H}^{p,q,l}_{dR}$ is an eigenvector of $f_\#$ of modulus greater than $\nu$, then $f_\#[\omega] = \mu[\omega]$ and $[\omega] \neq 0$. If $f_*\omega = \mu\omega$, then $\mu \in \sigma(f_*|_{\mathcal{B}^{p,q,l}})$. On the other hand, there exists $u \in \mathcal{B}^{p+1,q-1,l-1}$ such that

$$f_*\omega = \mu\omega + du.$$

We can proceed as above looking for a current $\omega' = \omega + du'$, with $u' \in \mathcal{B}^{p+1,q-1,l-1}$, so that $[\omega] = [\omega']$, and with $f_*\omega' = \mu\omega'$. Last equality means

$$\mu\omega + \mu du' = \mu\omega' = f_*\omega' = f_*\omega + f_*du' = \mu\omega + du + f_*du',$$

hence $(f_* - \mu\mathrm{id})du' = -du$. If $(f_* - \mu\mathrm{id})$ is invertible on exact currents of $\mathcal{B}^{p,q,l}$, then the desired $u' = (f_* - \mu\mathrm{id})^{-1}u$ and $\mu \in \sigma(f_*|_{\mathcal{B}^{p,q,l}})$. Conversely, if $(f_* - \mu\mathrm{id})$ is not invertible, on exact currents of $\mathcal{B}^{p,q,l}$, since the spectrum is discrete in $\{|z| > \nu\}$, there exists $d\bar{u} \in \mathcal{B}^{p,q,l}$ such that $f_*d\bar{u} = \mu d\bar{u}$. Q.E.D.

In the particular case $l = d_s$ we can identify the spectrum of $f_*$ with the spectrum of the action on anisotropic cohomology out of the ball of radius $\lambda^{-1}e^{h_{top}}$.

**Corollary 3.15.**

$$\sigma(f_*|_{\mathcal{B}^{p,q,d_s}}) \cap \{z \in \mathbb{C} : |z| > \lambda^{-1}e^{h_{top}}\} = \sigma(f_\#|_{\widetilde{H}^{p,q,d_s}_{d_R}}) \cap \{z \in \mathbb{C} : |z| > \lambda^{-1}e^{h_{top}}\}$$

Proof. It follows by Proposition 3.13, Proposition 3.14 and Corollary 2.15, because the spectral radius of $f_*$ on $\mathcal{B}^{p+1,q-1,l-1}$ and on $\mathcal{B}^{p-1,q+1,l+1}$ is bounded by $\lambda^{-1}e^{h_{top}}$.

Q.E.D.

## 3.3 Connection with the standard de Rham cohomology

In view of Corollary 3.15, it makes sense to study the spectrum of $f_{\#}$ on the anisotropic de Rham cohomology group $\widetilde{H}^{p,q,d_s}_{dR}(M)$. The first attempt to solve this problem could be trying to show that anisotropic de Rham cohomology is actually isomorphic to the standard de Rham cohomology recalled at the beginning of Section 3.2. In fact, it is well-known by classical results of G. de Rham [23] that the de Rham cohomology for currents is isomorphic to the de Rham cohomology for differential forms. On the other hand we are working with a linear subspace $\mathcal{B}^{p,q,l}$ of the dual space of $C^{p+q}$ differential forms $(\Omega^l_{p+q}(M))^\star$(see Lemma 2.8) , which is in turn a subspace of the space of currents $(\Omega^l(M))^\star$, i.e., the dual of $C^\infty$ differential forms. Therefore, we have fewer closed currents and fewer exact currents than the full space of currents and, a priori, there is no relation between our cohomology and the standard cohomology.

The authors of [20, Section 5.7] showed that this isomorphism exists when $d_s = 1$ and it is enough to prove our main result for Anosov diffeomorphisms of the 2-torus. Their strategy also works for our Anosov diffeomorphisms on higher dimensional manifolds whenever $d_s = 1$, but unfortunately, the extension to other cases requires a bit of work. A motivation for which their proof fails is given in Remark 3.24.

In order to overcome this obstacle, we firstly need to introduce an intermediate version of our anisotropic Banach spaces.

**Definition 3.16.** *Let $\omega \in \Omega^l(M)$ be a $C^\infty$ differential form and let $p, q \in \mathbb{N}$. We define the following norm*

$$|\omega|_{p,q,l} = \|\omega\|_{p,q,l} + \|d\omega\|_{p,q,l+1},$$

*where $\|\cdot\|_{p,q,l}$ is the norm of Definition 2.5. Let us denote by $\mathcal{C}^{p,q,l} = \overline{\Omega^l(M)}^{|\cdot|_{p,q,l}}$ the closure of the space of $l$-forms w.r.t. this norm.*

The following proposition collects all the properties we need about this new anisotropic Banach spaces.

**Proposition 3.17.** *The following properties hold* *a)* $\mathcal{B}^{p+j,q-j,l} \subseteq \mathcal{C}^{p,q,l} \subseteq \mathcal{B}^{p,q,l}$ *for any* $p, q \in \mathbb{N}$ *and for each* $j = 1, \ldots, q$;

*b)* $\omega \in \mathcal{C}^{p,q,l}$ *if and only if* $\omega \in \mathcal{B}^{p,q,l}$ *and* $d\omega \in \mathcal{B}^{p,q,l+1}$;

*c) The exterior derivative extends to a bounded linear operator* $d\colon \mathcal{C}^{p,q,l} \to \mathcal{C}^{p,q,l+1}$ *and* $d \circ d = 0$.

*d)* $f_*$ *extends to a bounded linear operator* $f_*\colon \mathcal{C}^{p,q,l} \to \mathcal{C}^{p,q,l}$. *The spectral radius* $\rho(f_*|_{\mathcal{C}^{p,q,l}}) \leq \lambda^{-|d_s-l|} e^{h_{top}}$, *while, for* $p$ *and* $q$ *large enough, the essential spectral radius* $\rho_{ess}(f_*|_{\mathcal{C}^{p,q,l}}) \leq \nu$, *where* $\nu$ *is the same as in* (3.1).

Proof. By definition $\|\omega\|_{p,q,l} \leq \|\omega\|_{p,q,l} + \|d\omega\|_{p,q,l+1} = |\omega|_{p,q,l}$, hence $\mathcal{C}^{p,q,l} \subseteq \mathcal{B}^{p,q,l}$. Next, $|\omega|_{p,q,l} = \|\omega\|_{p,q,l} + \|d\omega\|_{p,q,l+1} \leq \|\omega\|_{p,q,l} + \|d\|\|\omega\|_{p+1,q-1,l} \leq C\|\omega\|_{p+1,q-1,l}$, where the first inequality follows by the continuity of $d\colon \mathcal{B}^{p+1,q-1,l} \to \mathcal{B}^{p,q,l+1}$ (see Lemma 3.12), while the second one is a consequence of the inclusion $\mathcal{B}^{p+1,q-1,l} \subseteq \mathcal{B}^{p,q,l}$ (see Remark 2.6). Accordingly, $\mathcal{B}^{p+1,q-1,l} \subseteq \mathcal{C}^{p,q,l} \subseteq \mathcal{B}^{p,q,l}$ and, using again Remark 2.6, we get *a)*. *b)* is a trivial consequence of Definition 3.16. To prove *c)* notice that $|d\omega|_{p,q,l+1} = \|d\omega\|_{p,q,l+1} \leq \|\omega\|_{p,q,l} + \|d\omega\|_{p,q,l+1} = |\omega|_{p,q,l}$, hence $d\colon \mathcal{C}^{p,q,l} \to \mathcal{C}^{p,q,l+1}$ is bounded. $d \circ d = 0$ easily follows by the inclusions *a)* and by Lemma 3.12. Finally, let us prove *d)*. The statement about the spectral radius is a consequence of the inclusion $\mathcal{C}^{p,q,l} \subseteq \mathcal{B}^{p,q,l}$ and Corollary 2.15. Moreover, by using the Lasota-Yorke inequality (2.17), we obtain

$$\begin{aligned}
&|f_*^n\omega|_{p,q,l} = \|f_*^n\omega\|_{p,q,l} + \|f_*^n d\omega\|_{p,q,l+1} \leq \\
\leq &C\lambda^{-n(|d_s-l|+\min\{p,q\})} e^{nh_{top}} \|\omega\|_{p,q,l} + C\lambda^{-n|d_s-l|} e^{nh_{top}} \|\omega\|_{p-1,q+1,l} + \\
&+ C\lambda^{-n(|d_s-l-1|+\min\{p,q\})} e^{nh_{top}} \|d\omega\|_{p,q,l+1} + C\lambda^{-n|d_s-l-1|} e^{nh_{top}} \|d\omega\|_{p-1,q+1,l+1} \leq \\
\leq &C \max\{\lambda^{-n|d_s-l|}, \lambda^{-n|d_s-l-1|}\} \lambda^{-n\min\{p,q\}} e^{nh_{top}} |\omega|_{p,q,l} + \\
&+ C \max\{\lambda^{-n|d_s-l|}, \lambda^{-n|d_s-l-1|}\} e^{nh_{top}} |\omega|_{p-1,q+1,l}
\end{aligned}$$

Hence, using again Hennion's Theorem, we conclude that the essential spectral radius is bounded by $\max\{\lambda^{-|d_s-l|}, \lambda^{-|d_s-l-1|}\}\lambda^{-\min\{p,q\}} e^{h_{top}}$, which is smaller than $\nu$ for $p$ and $q$ large enough.

Q.E.D.

Property *c)* of Proposition 3.17 gives the following cochain complex

$$0 \xrightarrow{d} \mathcal{C}^{p,q,0} \xrightarrow{d} \mathcal{C}^{p,q,1} \xrightarrow{d} \mathcal{C}^{p,q,2} \to \ldots \xrightarrow{d} \mathcal{C}^{p,q,\dim(M)-1} \xrightarrow{d} \mathcal{C}^{p,q,\dim(M)} \xrightarrow{d} 0,$$

hence we can define $\bar{H}^{p,q,l}_{dR}(M)$ as the quotient of closed currents w.r.t. exact currents of $\mathcal{C}^{p,q,l}$. Collecting information about the spectrum of $f_*$ acting on different versions of anisotropic Banach space we obtain the following result.

**Corollary 3.18.** *Let $p, q \in \mathbb{N}$ be large enough. Then*

1. $\sigma(f_*|_{\mathcal{C}^{p,q,l}}) \cap \{|z| > \nu\} = \sigma(f_*|_{\mathcal{B}^{p\pm i,q\mp i,l}}) \cap \{|z| > \nu\}$ *for any* $i = 0, \dots, \dim(M)$;
2. $\sigma(f_*|_{\mathcal{C}^{p,q,l}}) \cap \{|z| > \nu\} \subseteq \left[\sigma(f_*|_{\mathcal{C}^{p,q,l-1}}) \cup \sigma(f_\#|_{\bar{H}^{p,q,l}_{dR}}) \cup \sigma(f_*|_{\mathcal{C}^{p,q,l+1}})\right] \cap \{|z| > \nu\}$;
3. $\sigma(f_\#|_{\bar{H}^{p,q,l}_{dR}}) \cap \{|z| > \nu\} \subseteq \sigma(f_*|_{\mathcal{C}^{p,q,l}}) \cap \{|z| > \nu\}$;
4. $\sigma(f_*|_{\mathcal{C}^{p,q,d_s}}) \cap \{z \in \mathbb{C} \mid |z| > \lambda^{-1}e^{h_{top}}\} = \sigma(f_\#|_{\bar{H}^{p,q,d_s}_{dR}}) \cap \{z \in \mathbb{C} \mid |z| > \lambda^{-1}e^{h_{top}}\}$

Proof. Equality *1.* is a consequence of Lemma A.15 with $\mathcal{B}_0 = \Omega^l(M)$, $\mathcal{B}_1 = \mathcal{C}^{p,q,l}$, $\mathcal{B}_2 = \mathcal{B}^{p\pm i,q\mp i,l}$ and $\mathcal{B} = \mathcal{B}^{\max\{p,p\pm i\},\min\{q,q\mp i\},l}$. *2.*, resp. *3.* can be proved by repeating the proof of Proposition 3.13, resp. Proposition 3.14, with $\mathcal{B}^{p,q,l}$ replaced by $\mathcal{C}^{p,q,l}$ and $\widetilde{H}^{p,q,d_s}_{dR}$ replaced by $\bar{H}^{p,q,d_s}_{dR}$. Finally, *2.*, *3.* and *d)* of Proposition 3.17 imply *4.*

Q.E.D.

**Remark 3.19.** Notice that one may directly study the action of $f_*$ on $\mathcal{C}^{p,q,l}$, without considering the original anisotropic Banach spaces $\mathcal{B}^{p,q,l}$. There are several reasons that have led us to our choice. In fact, proofs of Lasota-Yorke inequalities (Lemma 2.13) and compact inclusion(Lemma 2.10), as well as the inclusion into currents (Lemma 2.8) turn out to be simpler from a technical point of view. Secondly, these spaces have been largely investigated in recent years [37, 38, 36] and we have picked up some ideas from the literature. Lastly, we found the issue when we started treating anisotropic cohomology for $d_s > 1$ and we discovered that without our trick the proof of the isomorphism Theorem 3.21 does not work with $\widetilde{H}^{p,q,l}_{dR}$, that is the cohomology obtained with $\mathcal{B}^{p,q,l}$ (see Remark 3.24 below).

We are now ready to prove the isomorphism between the anisotropic de Rham cohomology $\bar{H}^{p,q,l}_{dR}$ and the standard de Rham cohomology. It is well known, in the fields of algebraic topology and differential geometry, that the standard (complex) de Rham cohomology is isomorphic to the (complex) Čech cohomology. In effect, denoting with $H^*_{\check{C}}(M, \mathbb{C}) = H^*_{\check{C}}(M)$ the Čech cohomology with complex coefficients, it holds

**Theorem 3.20** (De Rham isomorphism theorem)**.** *There exists a natural isomorphism between the standard de Rham cohomology and the Čech cohomology*

$$H^*_{dR}(M) \cong H^*_{\check{C}}(M)$$

There are several different proofs of above result. The most elegant one, due to André Weil [72] (see also [57], [12]), inspires our proof of the following isomorphism theorem. In fact, the careful reader may also be able to reconstruct the proof of Theorem 3.20 from the one of Theorem 3.21.

**Theorem 3.21.** *Let $p, q \in \mathbb{N}$ be large enough. There exists a natural isomorphism between the anisotropic de Rham cohomology and the Čech cohomology*

$$\bar{H}^{p,q,*}_{dR}(M) \cong H^*_{\check{C}}(M).$$

*By Theorem* (3.20)*, we obtain*

$$\bar{H}^{p,q,*}_{dR}(M) \cong H^*_{dR}(M).$$

Before giving the proof of Theorem (3.21), we recall the basic facts about Čech cohomology (for a complete treatment of the topic see [40, 57]).

Let $\mathcal{U} = \{U_a\}_{a \in \mathcal{A}}$ be a contractible open covering of the manifold $M$, i.e., we suppose that every finite nonempty intersection

$$U_{a_1} \cap U_{a_2} \cap \cdots \cap U_{a_n} \neq \varnothing$$

is contractible (homotopic to a point). We denote by $(a_0, \ldots, a_k) := U_{a_0} \cap \cdots \cap U_{a_k}$. Let $\check{C}_k(M,\mathcal{U})$ be the complex vector space generated by elements $(a_0, \ldots, a_k) \neq \varnothing$; the elements in $\check{C}_k(M,\mathcal{U})$ are called (Čech) $k$-chains. A (Čech) $k-$cochain $c$ is an element of the dual of $\check{C}_k(M,\mathcal{U})$ such that, for every permutation $\sigma$ of the indexes $\{0, \ldots, k\}$,

$$c(a_0, \ldots, a_k) = \operatorname{sgn}(\sigma) c(a_{\sigma(0)}, \ldots, a_{\sigma(k)}).$$

Let $\check{C}^k(M,\mathcal{U})$ be the complex vector space of all $k-$cochains. We define the coboundary operator

$$\delta \colon \check{C}^k(M,\mathcal{U}) \to \check{C}^{k+1}(M,\mathcal{U})$$

such that

$$(\delta c)(a_0, \ldots, a_{k+1}) = \sum_{j=0}^{k+1} (-1)^j c(a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{k+1}).$$

A straightforward computation (see Lemma 3.22) shows that $\delta^2 = \delta \circ \delta = 0$, hence the couple $(\check{C}^*(M,\mathcal{U}), \delta)$ is a cochain complex. Recall that a $k$-cochain $c \in \check{C}^k(M,\mathcal{U})$ is a $k$-cocycle if $\delta(c) = 0$, and it is a $k$-coboundary if there exists $c' \in \check{C}^{k-1}(M,\mathcal{U})$ such that $\delta(c') = c$. As always, every $k$-coboundary is a $k$-cocycle, because of $\delta^2 = 0$. Therefore, we define the Čech $k$-cohomology group $H^k_{\check{C}}(M,\mathcal{U})$ as the quotient of $k$-cocycles with respect to $k$-coboundaries. A priori, this new cohomology depends on the covering $\mathcal{U}$ of the manifold $M$. It will be clear from the proof of Theorem 3.21 that it is actually independent of that choice and it make sense to write $H_{\check{C}}(M)$ in place of $H_{\check{C}}(M,\mathcal{U})$.

Without loss of generality we can assume that the sets $\{V_i = \psi_i(U_i)\}_{i=1}^m$ is a contractible covering of the manifold $M$. Let $\phi_i$ be a smooth partition of unity subordinated to the covering $\{V_i\}_{i=1}^m$. We can also suppose that $\{int(supp(\phi_k))\}_{k=1}^m$, where $int(supp(\phi_k))$ is the interior of the support of $\phi_k$, is a contractible open covering. If $\omega$ is a differential form on $M$, then the restriction to $V_i$ is well-defined. On the other hand, in our case $\omega$ is a current and the restriction of $\omega$ to the subset $V_i$ can be defined as $\omega\phi_i$. A straightforward computation shows that $\omega\phi_i \in \mathcal{C}^{p,q,l}$.

We are now ready to present the proof of Theorem 3.21.

Proof of of Theorem 3.21. Let us introduce the following notations. $\mathcal{C}^{p,q,l}(M)$ denotes the above anisotropic Banach space $\mathcal{C}^{p,q,l}$. Let $\mathcal{C}^{p,q,l,k}(\mathcal{U})$ be the vector space of linear functions

$$\omega\colon \check{C}_k(M,\mathcal{U}) \to \mathcal{C}^{p,q,l},$$

such that, for every permutation $\sigma$ of the set $\{0,\dots,k\}$,

$$\omega(a_{\sigma(0)},\dots,a_{\sigma(k)}) = \operatorname{sgn}(\sigma)\omega(a_0,\dots,a_k). \tag{3.10}$$

We extend the operators $\delta$ and $d$ defining:

$$\begin{aligned}
\bar{\delta}\colon \mathcal{C}^{p,q,l,k}(\mathcal{U}) \to\ & \mathcal{C}^{p,q,l,k+1}(\mathcal{U})\\
\omega \mapsto\ & \bar{\delta}\omega\colon \check{C}_{k+1}(M,\mathcal{U}) \longrightarrow \mathcal{C}^{p,q,l}\\
& (\bar{\delta}\omega)(a_0,a_1\dots,a_{k+1}) = \sum_{j=0}^{k+1}(-1)^j\omega(a_0,\dots,a_{j-1},a_{j+1},\dots,a_{k+1})\\
\bar{d}\colon \mathcal{C}^{p,q,l,k}(\mathcal{U}) \to\ & \mathcal{C}^{p,q,l+1,k}(\mathcal{U})\\
\omega \mapsto\ & \bar{d}\omega\colon \check{C}_k(M,\mathcal{U}) \longrightarrow \mathcal{C}^{p,q,l+1}\\
& (\bar{d}\omega)(a_0,\dots,a_k) = d(\omega(a_0,\dots,a_k))
\end{aligned}$$

Moreover, we define

$$\begin{aligned} i\colon \mathcal{C}^{p,q,l} &\to \mathcal{C}^{p,q,l,0} \\ \omega &\mapsto i(\omega)\colon \check{C}_0(M,\mathcal{U}) \to \mathcal{C}^{p,q,l} \\ &\qquad i(\omega)(a_0) = \omega \\ j\colon \check{C}^k(M,\mathcal{U}) &\to \mathcal{C}^{p,q,0,k} \\ c &\mapsto j(c)\colon \check{C}_k(M,\mathcal{U}) \to \mathcal{C}^{p,q,0} \\ &\qquad j(c)(a_0,\dots,a_k) = c(a_0,\dots,a_k) \end{aligned}$$

**Lemma 3.22.** *The following equalities hold true.*

$$\delta\circ\delta = \bar\delta\circ\bar\delta = d\circ d = \bar d\circ\bar d = \bar\delta\circ i = \bar d\circ j = 0,$$

$$\bar d\circ\bar\delta = \bar\delta\circ\bar d,\ \ j\circ\delta = \bar\delta\circ j,\ \ i\circ d = \bar d\circ i.$$

Proof of Lemma 3.22. For every $(a_0,\dots,a_{k+2})\in\check{C}_{k+2}(M,\mathcal{U})$

$$\begin{aligned} &\bar\delta^2\omega(a_0,\dots,a_{k+2}) = \sum_{t=0}^{k+2}(-1)^t\bar\delta\omega(a_0,\dots,a_{t-1},a_{t+1},\dots,a_{k+2}) = \\ &= \sum_{t=0}^{k+2}(-1)^t\sum_{s=0}^{t-1}(-1)^s\omega(a_0,\dots,a_{s-1},a_{s+1},\dots,a_{t-1},a_{t+1},\dots,a_{k+2}) + \\ &+ \sum_{t=0}^{k+2}(-1)^t\sum_{s=t+1}^{k+2}(-1)^{s-1}\omega(a_0,\dots,a_{t-1},a_{t+1},\dots,a_{s-1},a_{s+1},\dots,a_{k+2}) = 0, \end{aligned}$$

because each term of the first sum appears in the second one with opposite sign. The same computation, with $\omega\in\mathcal{B}^{p,q,l,k}$ replaced by $c\in\check{C}^k(M,\mathcal{U})$, gives $\delta\circ\delta=0$. The equality $d\circ d=0$ is a trivial consequence of Lemma 3.12 and, by definition, $(\bar d^2\omega)(a_0,\dots,a_k) = d^2(\omega(a_0,\dots,a_k)) = 0$. Next, $(\bar\delta\circ i(\omega))(a_0,a_1) = i(\omega)(a_1) - i(\omega)(a_0) = \omega-\omega = 0$. Moreover, $\bar d\circ j(c)(a_0,\dots,a_k) = d(c(a_0,\dots,a_k)) = 0$, because $c(a_0,\dots,a_k)$ is a constant smooth function on $M$. This proves the first line of equalities. Let us show commutation properties.

$$\begin{aligned} &(\bar d\circ\bar\delta\omega)(a_0,\dots,a_{k+1}) = d(\delta\omega(a_0,\dots,a_{k+1})) = \\ &= \sum_{t=0}^{k+1}(-1)^t d(\omega(a_0,\dots,a_{t-1},a_{t+1},\dots,a_{k+1})) = \bar\delta\circ\bar d\omega(a_0,\dots,a_{k+1}). \end{aligned}$$

Similarly,

$$(j \circ \delta(c))(a_0, \ldots, a_{k+1}) = \delta(c)(a_0, \ldots, a_{k+1}) = \sum_{t=0}^{k+1} (-1)^t c(a_0, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{k+1}) =$$

$$= \sum_{t=0}^{k+1} (-1)^t j(c)(a_0, \ldots, a_{t-1}, a_{t+1}, \ldots, a_{k+1}) = \bar{\delta} \circ j(c)(a_0, \ldots, a_{k+1}).$$

Finally,

$$(\bar{d} \circ i)(\omega)(a_0) = d(i(\omega)(a_0)) = d\omega = (i \circ d)(\omega).$$

This concludes the proof of Lemma 3.23.

Q.E.D.

As a consequence of Lemma 3.22, we obtain the following commutative diagram. In particular, we underline that the first column contains the anisotropic de Rham cohomology complex $(\mathcal{C}^{p,q,*}(M), d)$, while the bottom row represents the Čech cohomology complex $(\check{C}^*(M,\mathcal{U}), \delta)$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & & \vdots & \\
\mathcal{C}^{p,q,l}(M) \arrow[u, "d"'] \arrow[r, "i"] & \mathcal{C}^{p,q,l,0}(\mathcal{U}) \arrow[u, "\bar{d}"'] \arrow[r, "\bar{\delta}"] & \mathcal{C}^{p,q,l,1}(\mathcal{U}) \arrow[u, "d"'] \arrow[r, "\bar{\delta}"] & \cdots \arrow[r, "\bar{\delta}"] & \mathcal{C}^{p,q,l,k}(\mathcal{U}) \arrow[u, "\bar{d}"'] \arrow[r, "\bar{\delta}"] & \cdots \\
\vdots \arrow[u, "d"'] & \vdots \arrow[u, "\bar{d}"'] & \vdots \arrow[u, "\bar{d}"'] & & \vdots \arrow[u, "\bar{d}"'] & \\
\mathcal{C}^{p,q,1}(M) \arrow[u, "d"'] \arrow[r, "i"] & \mathcal{C}^{p,q,1,0}(\mathcal{U}) \arrow[u, "\bar{d}"'] \arrow[r, "\bar{\delta}"] & \mathcal{C}^{p,q,1,1}(\mathcal{U}) \arrow[u, "\bar{d}"'] \arrow[r, "\bar{\delta}"] & \cdots \arrow[r, "\bar{\delta}"] & \mathcal{C}^{p,q,1,k}(\mathcal{U}) \arrow[u, "\bar{d}"'] \arrow[r, "\bar{\delta}"] & \cdots \\
\mathcal{C}^{p,q,0}(M) \arrow[u, "d"'] \arrow[r, "i"] & \mathcal{C}^{p,q,0,0}(\mathcal{U}) \arrow[u, "\bar{d}"'] \arrow[r, "\bar{\delta}"] & \mathcal{C}^{p,q,0,1}(\mathcal{U}) \arrow[u, "\bar{d}"'] \arrow[r, "\bar{\delta}"] & \cdots \arrow[r, "\bar{\delta}"] & \mathcal{C}^{p,q,0,k}(\mathcal{U}) \arrow[u, "\bar{d}"'] \arrow[r, "\bar{\delta}"] & \cdots \\
& \check{C}^0(M,\mathcal{U}) \arrow[u, "j"'] \arrow[r, "\delta"] & \check{C}^1(M,\mathcal{U}) \arrow[u, "j"'] \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & \check{C}^k(M,\mathcal{U}) \arrow[u, "j"'] \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

The key idea of our proof is a zigzag argument which links closed non-exact elements of the first column to a closed non-exact elements of the bottom row, and vice versa. The next Lemma 3.23 is the tool we exploit to go through vertical arrows. In fact, we recall that, by classical results of differential geometry, every closed differential form is locally exact. This is an informal way of writing Poincaré's lemma. The following statement generalizes Poincaré's lemma to the context of our anisotropic Banach spaces. In particular, this is analogous to [20, Lemma 5.15] where the authors proved that every closed 1-currents is, in a way, locally exact.

**Lemma 3.23.** *Let $h \in \mathcal{C}^{p,q,0}$ be a closed 0-current. Then, for any $k = 1, \dots, m$, there exists $c_k \in \mathbb{C}$, such that $\phi_k \cdot (h - c_k) = 0$. Let $\omega \in \mathcal{C}^{p,q,l}(M)$ be an $l$-current, for $l > 0$.If $(d\omega)\phi_k = 0$, then there exists $u_k \in \mathcal{C}^{p,q,l-1}(M)$ such that*

$$d(u_k \phi_k) = \omega\phi_k + (-1)^{l-1} u_k \wedge d\phi_k.$$

**Remark 3.24.** Lemma 3.23 represents the main difference between this thesis and [20, Section 5] and it justifies the introduction of the spaces $\mathcal{C}^{p,q,l}$. In effect, they only considered closed elements of $\mathcal{B}^{p,q,1}$ and such currents are locally exact with potential in $\mathcal{B}^{p+1,q-1,0}$. In general, if $\omega \in \mathcal{B}^{p,q,l}$ is a closed current for $l > 1$, then this is locally exact, but we cannot expect that the potential is more regular than $\omega$. Notice that the same issue holds for differential forms. In fact, if $\omega$ is a closed $C^r$ 1-form on a star-shaped domain $U$, then $\omega = du$ where $u \in C^{r+1}(U)$. On the other hand, if $\omega$ is a closed $C^r$ $l$-form on $U$ for some $l > 1$, then $\omega = du$ for some $(l-1)$-form $u$ of class $C^r$ and coefficients of $u$ are $C^{r+1}$ exclusively in some directions.

We postpone the proof of Lemma 3.23 to Appendix C and we describe its application to our problem. We use the above diagram and Lemma 3.23 to define an isomorphism

$$\Phi\colon \bar{H}^{p,q,l}_{dR}(M) \to H_{\check{C}}(M,\mathcal{U}).$$

Let $[\omega] \in \bar{H}^{p,q,l}_{dR}(M)$ be an element of the anisotropic de Rham cohomology, where $\omega \in \mathcal{C}^{p,q,l}$ is closed. By definition, $i(\omega) \in \mathcal{C}^{p,q,l,0}(\mathcal{U})$, $(\bar{d} \circ i(\omega)) = d\omega = 0$ and $(\bar{\delta} \circ i(\omega)) = 0$, by Lemma 3.22. Therefore, by Lemma 3.23, for any $a_0 = 1, \dots, m$, there exists $u^{(l-1)}_{a_0} \in \mathcal{C}^{p,q,l-1}$ such that $d(u^{(l-1)}_{a_0}\phi_{a_0}) = \omega\phi_{a_0} + (-1)^{l-1} u^{(l-1)}_{a_0} \wedge d\phi_{a_0}$. Let us define $u^{(l-1)} \in \mathcal{C}^{p,q,l-1,0}(\mathcal{U})$, such that $u^{(l-1)}(a_0) = u^{(l-1)}_{a_0}$. Consequently, $(\bar{d}u^{(l-1)})(a_0)\phi_{a_0} = (du^{(l-1)}_{a_0})\phi_{a_0} = \omega\phi_{a_0}$. Next, we consider $\bar{\delta}u^{(l-1)} \in \mathcal{C}^{p,q,l-1,1}(\mathcal{U})$. Since the diagram commutes,

$$\begin{aligned}(\bar{d} \circ \bar{\delta} u^{(l-1)})(a_0, a_1)\phi_{a_0}\phi_{a_1} &= (\bar{\delta} \circ \bar{d} u^{(l-1)})(a_0, a_1)\phi_{a_0}\phi_{a_1} = \\ =(du^{(l-1)}_{a_1})\phi_{a_0}\phi_{a_1} - (du^{(l-1)}_{a_0})\phi_{a_0}\phi_{a_1} &= \omega\phi_{a_0}\phi_{a_1} - \omega\phi_{a_0}\phi_{a_1} = 0.\end{aligned}$$

As a consequence, $d(\bar{\delta}u^{(l-1)}(a_0,a_1))\phi_{a_0}\phi_{a_1} = 0$, hence, by Lemma 3.23 there exists $u^{(l-2)}_{a_0,a_1} \in \mathcal{C}^{p,q,l-2}$ such that $(du^{(l-2)}_{a_0,a_1})\phi_{a_0}\phi_{a_1} = \bar{\delta}u^{(l-1)}(a_0,a_1)\phi_{a_0}\phi_{a_1}$ and

$$d(u^{(l-2)}_{a_0,a_1}\phi_{a_0}\phi_{a_1}) = \bar{\delta}u^{(l-1)}(a_0,a_1)\phi_{a_0}\phi_{a_1} + (-1)^{l-2}u^{(l-2)}_{a_0,a_1} \wedge d(\phi_{a_0}\phi_{a_1}).$$

As above, we define $u^{(l-2)} \in \mathcal{C}^{p,q,l-2,2}(\mathcal{U})$ such that $u^{(l-2)}(a_0,a_1) = u^{(l-2)}_{a_0,a_1}$. We point out that the choice of $u^{(l-2)}(a_0,a_1)$ is not unique, but it is unique up to elements $v \in \mathcal{C}^{p,q,l-2}$ such that $(dv)\phi_{a_0}\phi_{a_1} = 0$. Therefore, we set $u^{(l-2)}(a_0,a_1) = u^{(l-2)}_{a_0,a_1}$ for $a_0 < a_1$, while we impose that $u^{(l-2)}(a_1,a_0) = -u^{(l-2)}_{a_0,a_1}$. In fact, this agrees with (3.10) and

$$\begin{aligned} du^{(l-2)}(a_0,a_1)\phi_{a_0}\phi_{a_1} &= \bar{\delta}u^{(l-1)}(a_0,a_1)\phi_{a_0}\phi_{a_1} = (u^{(l-1)}(a_1) - u^{(l-1)}(a_0))\phi_{a_0}\phi_{a_1} = \\ = -\,(u^{(l-1)}(a_0) - u^{(l-1)}(a_1))\phi_{a_0}\phi_{a_1} &= -\bar{\delta}u^{(l-1)}(a_1,a_0)\phi_{a_0}\phi_{a_1} = -du^{(l-2)}(a_1,a_0)\phi_{a_0}\phi_{a_1}. \end{aligned}$$

Repeating the argument, we consider $\bar{\delta}u^{(l-2)}$. Then, for any $a_0,a_1,a_2 \in \{1,\dots,m\}$,

$$\begin{aligned} &d(\bar{\delta}u^{(l-2)}(a_0,a_1,a_2))\phi_{a_0}\phi_{a_1}\phi_{a_2} = (\bar{d} \circ \bar{\delta}u^{(l-2)})(a_0,a_1,a_2)\phi_{a_0}\phi_{a_1}\phi_{a_2} = \\ =&(\bar{\delta} \circ \bar{d}u^{(l-2)})(a_0,a_1,a_2)\phi_{a_0}\phi_{a_1}\phi_{a_2} = (du^{l-2}_{a_1,a_2} - du^{l-2}_{a_0,a_2} + du^{l-2}_{a_0,a_1})\phi_{a_0}\phi_{a_1}\phi_{a_2} = \\ =&(\bar{\delta}u^{(l-1)}(a_1,a_2) - \bar{\delta}u^{(l-1)}(a_0,a_2) + \bar{\delta}u^{(l-1)}(a_0,a_1))\phi_{a_0}\phi_{a_1}\phi_{a_2} = \\ =&(\bar{\delta} \circ \bar{\delta}u^{(l-1)})(a_0,a_1,a_2)\phi_{a_0,a_1,a_2} = 0, \end{aligned}$$

where, in the last line, we used $\bar{\delta}\circ\bar{\delta} = 0$. Consequently, there exists $u^{(l-3)}_{a_0,a_1,a_2} \in \mathcal{C}^{p,q,l-3}$ such that $(du^{(l-3)}_{a_0,a_1,a_2})\phi_{a_0}\phi_{a_1}\phi_{a_2} = \bar{\delta}u^{(l-2)}(a_0,a_1,a_2)\phi_{a_0}\phi_{a_1}\phi_{a_2}$ and

$$d(u^{(l-3)}_{a_0,a_1,a_2}\phi_{a_0}\phi_{a_1}\phi_{a_2}) = \bar{\delta}u^{(l-1)}(a_0,a_1)\phi_{a_0}\phi_{a_1}\phi_{a_2} + (-1)^{l-3}u^{(l-3)}_{a_0,a_1,a_2} \wedge d(\phi_{a_0}\phi_{a_1}\phi_{a_2}).$$

We fix a representative $u^{(l-3)}(a_0,a_1,a_2) = u^{(l-3)}_{a_0,a_1,a_2}$, for $a_0 < a_1 < a_2$, and we define $u^{(l-3)}(a_{\sigma(0)},a_{\sigma(1)},a_{\sigma(2)}) = \operatorname{sgn}(\sigma)u^{(l-3)}_{a_0,a_1,a_2}$, for any permutation $\sigma$ of $\{0,1,2\}$. After $l$ steps, using the same procedure, we obtain $u^{(0)} \in \mathcal{C}^{p,q,0,l-1}(\mathcal{U})$ such that, for any $a_0,\dots,a_{l-1} \in \{1,\dots,m\}$,

$$\begin{aligned} d(u^{(0)}(a_0,\dots,a_{l-1})\phi_{a_0}\dots\phi_{a_{l-1}}) =&\bar{\delta}u^{(1)}(a_0,\dots,a_{l-1})\phi_{a_0}\dots\phi_{a_{l-1}}+ \\ &+u^{(0)}(a_0,\dots,a_{l-1}) \wedge d(\phi_{a_0}\dots\phi_{a_{l-1}}), \\ d(u^{(0)}(a_0,\dots,a_{l-1}))\phi_{a_0}\dots\phi_{a_{l-1}} =&\bar{\delta}u^{(1)}(a_0,\dots,a_{l-1})\phi_{a_0}\dots\phi_{a_{l-1}} \end{aligned}$$

and

$$(u^{(0)}(a_{\sigma(0)},\dots,a_{\sigma(l-1)})) = \operatorname{sgn}(\sigma)(u^{(0)}(a_0,\dots,a_{l-1}))$$

for any permutation $\sigma$ of $\{0,\ldots,l-1\}$. Using again the commutation property of the diagram, we obtain that

$$d(\bar{\delta}u^{(0)}(a_0,\ldots,a_l))\phi_{a_0}\ldots\phi_{a_l}=0,$$

for any $a_0,\ldots,a_l\in\{1,\ldots,m\}$. Thus, by Lemma 3.23, there exists $c_{a_0,\ldots,a_l}\in\mathbb{C}$ such that

$$(\bar{\delta}u^{(0)}(a_0,\ldots,a_l)-c_{a_0,\ldots,a_l})\phi_{a_0}\ldots\phi_{a_l}=0.$$

Consequently, we can choose $c\colon\check{C}_l(M,\mathcal{U})\to\mathbb{C}$ fixing $c(a_0,\ldots,a_l)=c_{a_0,\ldots,a_l}$, whenever $a_0<a_1<\cdots<a_l$, and define $c(a_{\sigma(0)},\ldots,a_{\sigma(l)})=\operatorname{sgn}(\sigma)c_{a_{\sigma(0)},\ldots,a_{\sigma(l)}}$, for each permutation $\sigma$ of $\{0,\ldots,l\}$. Thus, $c\in\check{C}^l(M,\mathcal{U})$ and $j(c)(a_0,\ldots,a_l)\phi_{a_0}\ldots\phi_{a_l}=\bar{\delta}u^{(0)}(a_0,\ldots,a_l)\phi_{a_0}\ldots\phi_{a_l}$. Since

$$\begin{aligned}j\circ\delta(c)(a_0,\ldots,a_{l+1})\phi_{a_0}\ldots\phi_{a_{l+1}}&=\bar{\delta}\circ j(c)(a_0,\ldots,a_{l+1})\phi_{a_0}\ldots\phi_{a_{l+1}}=\\=&\bar{\delta}\circ\bar{\delta}u^{(0)}(a_0,\ldots,a_{l+1})\phi_{a_0}\ldots\phi_{a_{l+1}}=0\end{aligned}$$

and $j$ is injective, $\delta(c)(a_0,\ldots,a_{l+1})\phi_{a_0}\ldots\phi_{a_{l+1}}=0$, hence $\delta(c)(a_0,\ldots,a_{l+1})=0$. Accordingly, $c$ is a coboundary and we set $\Phi([\omega])=[c]$.

Now, we need to prove that $\Phi$ is well-defined, that is, if $[\widetilde{\omega}]=[\omega]\in\bar{H}^{p,q,l}_{dR}(M)$, then it must be true that $[c_\omega]=\Phi([\omega])=\Phi([\widetilde{\omega}])=[c_{\widetilde{\omega}}]$. In fact, if $[\omega]=[\widetilde{\omega}]$, then there exists $q\in\mathcal{C}^{p,q,l-1}$ such that $dq=\omega-\widetilde{\omega}$. Clearly, $i(\omega)$ and $i(\widetilde{\omega})$ are cohomologous, i.e., $i(\omega)-i(\widetilde{\omega})=i(\omega-\widetilde{\omega})=i(dq)=\bar{d}(i(q))$. Let $u^{(l-1)},\widetilde{u}^{(l-1)}\in\mathcal{C}^{p,q,l-1,0}(\mathcal{U})$ such that, for any $a_0\in\{1,\ldots,m\}$, $du^{(l-1)}(a_0)\phi_{a_0}=i(\omega)\phi_{a_0}$ and $d\widetilde{u}^{(l-1)}(a_0)\phi_{a_0}=i(\widetilde{\omega})\phi_{a_0}$, as determined above. It holds true that

$$\bar{d}(\widetilde{u}^{(l-1)}-u^{(l-1)})(a_0)\phi_{a_0}=\bar{d}(i(\widetilde{\omega}-\omega))(a_0)\phi_{a_0}=\bar{d}\circ\bar{d}(i(q))(a_0)\phi_{a_0}=0,$$

hence, by Lemma 3.23, there is $q^{(l-1)}\in\mathcal{C}^{p,q,l-2,0}(\mathcal{U})$ such that

$$\bar{d}q^{(l-1)}(a_0)\phi_{a_0}=(\widetilde{u}^{(l-1)}-u^{(l-1)})(a_0)\phi_{a_0},$$

for each $a_0\in\{1,\ldots,m\}$. Iterating the previous procedure, we obtain that

$$(\widetilde{u}^{(0)}-u^{(0)})(a_0,\ldots,a_{l-1})\phi_{a_0}\ldots\phi_{a_{l-1}}=j(q^{(0)})(a_0,\ldots,a_{l-1})\phi_{a_0}\ldots\phi_{a_{l-1}},$$

for some $q^{(0)}\in\check{C}^{l-1}(M,\mathcal{U})$. Moreover,

$$\delta\widetilde{u}^{(0)}(a_0,\ldots,a_l)\phi_{a_0}\ldots\phi_{a_l}-j(c_{\widetilde{\omega}})(a_0,\ldots,a_l)\phi_{a_0}\ldots\phi_{a_l}=0$$

and

$$\delta u^{(0)}(a_0,\dots,a_l)\phi_{a_0}\dots\phi_{a_l} - j(c_\omega)(a_0,\dots,a_l)\phi_{a_0}\dots\phi_{a_l} = 0.$$

Accordingly,

$$(j(c_{\widetilde{\omega}}) - j(c_\omega))(a_0,\dots,a_l)\phi_{a_0}\dots\phi_{a_l} = \bar{\delta}(\widetilde{u}^{(0)} - u^{(0)})(a_0,\dots,a_l)\phi_{a_0}\dots\phi_{a_l} =$$
$$=\bar{\delta}\circ j(q^{(0)})(a_0,\dots,a_l)\phi_{a_0}\dots\phi_{a_l} = j\circ\delta(q^{(0)})(a_0,\dots,a_l)\phi_{a_0}\dots\phi_{a_l},$$

hence, by injectivity of $j$,

$$(c_{\widetilde{\omega}} - c_\omega)(a_0,\dots,a_l) = \delta q^{(0)}(a_0,\dots,a_l),$$

i.e., $c_\omega$ and $c_{\widetilde{\omega}}$ are cohomologous.

We now need to show that $\Phi$ is invertible. In particular, we construct the inverse map following the strategy of the direct map. Next result plays the role of Lemma 3.23 for the rows of the abelian diagram, i.e., it is a Poincaré's lemma for $\bar{\delta}$.

**Lemma 3.25.** *Let $h \in \mathcal{C}^{p,q,l,0}(\mathcal{U})$, such that $\bar{\delta}h = 0$. Then there exists $\omega \in \mathcal{C}^{p,q,l}$ such that $i(\omega) = h$. Let $\omega \in \mathcal{C}^{p,q,l,k}(\mathcal{U})$ for some $k > 0$.If $\bar{\delta}\omega = 0$, there exists $u \in \mathcal{C}^{p,q,l,k-1}(\mathcal{U})$ such that $\bar{\delta}u = \omega$.*

As above, we postpone the proof of Lemma 3.23 to Appendix C and we apply it to conclude this proof. Let $[c] \in H^l_{\check{C}}(M,\mathcal{U})$ be a Čech cohomology class. Since $\delta(c) = 0$, it holds $\bar{\delta}\circ j(c) = j\circ\delta(c) = 0$. Consequently, Lemma 3.25 implies that there exists $v^{(0)} \in \mathcal{C}^{p,q,0,l-1}$ such that $\bar{\delta}v^{(0)} = j(c)$. Considering $\bar{d}v^{(0)}$, we get $\bar{\delta}\circ\bar{d}v^{(0)} = \bar{d}\circ\bar{\delta}v^{(0)} = \bar{d}\circ j(c) = 0$. Thus, we can apply again Lemma 3.25 to find a $v^{(1)} \in \mathcal{C}^{p,q,1,l-2}$ such that $\bar{\delta}v^{(1)} = \bar{d}v^{(0)}$. Iterating this argument, after $l$ steps, we obtain a $v^{(l)} \in \mathcal{C}^{p,q,l-1,0}$ such that $\bar{\delta}v^{(l)} = \bar{d}v^{(l-1)}$. Since $\bar{\delta}\circ\bar{d}v^{(l)} = \bar{d}\circ\bar{\delta}v^{(l)} = \bar{d}\circ\bar{d}v^{(l-1)} = 0$, we can apply again Lemma 3.25 and we conclude that there exists $\omega \in \mathcal{C}^{p,q,l}$ such that $i(\omega) = \bar{d}v^{(l)}$. One can easily check, using the same method described in the definition of $\Phi$, that $\omega$ is a closed current and it is unique up to exact currents. As a consequence, we can define $\Psi\colon H^l_{\check{C}}(M,\mathcal{U}) \to \bar{H}^{p,q,l}_{dR}(M)$ such that $\Psi([c]) = [\omega]$.

It remains to show that $\Phi$ and $\Psi$ are each other's inverses to get the required isomorphism. Let $[\omega] \in \bar{H}^{p,q,l}_{dR}(M)$, where $\omega \in \mathcal{C}^{p,q,l}$. Then we associated to $\omega$ a sequence $(u^{(s)})_{s=0,\dots,l-1}$ of elements $u^{(s)} \in \mathcal{C}^{p,q,s,l-1-s}(\mathcal{U})$ such that

$$d(u^{(s)}(a_0,\dots,a_{l-1-s}))\phi_{a_0}\dots\phi_{a_{l-1-s}} = \bar{\delta}u^{(s+1)}(a_0,\dots,a_{l-1-s})\phi_{a_0}\dots\phi_{a_{l-1-s}}.$$

On the other hand, the construction of $\Psi$ produces a sequence $(v^{(s)})_{s=0,\dots,l-1}$ such that $v^{(s)} \in \mathcal{C}^{p,q,s,l-1-s}$ such that $\bar\delta(v^{(s+1)}) = \bar d v^{(s)}$, that is

$$\bar\delta v^{(s+1)}(a_0,\dots,a_{l-1-s})\phi_{a_0}\dots\phi_{a_{l-1-s}} = d(v^{(s)}(a_0,\dots,a_{l-1-s}))\phi_{a_0}\dots\phi_{a_{l-1-s}}.$$

Without loss of generality we can assume $v^{(s)} = u^{(s)}$, so that $\Phi\circ\Psi = \Psi\circ\Phi = \mathrm{id}$. This concludes the proof of Theorem 3.21.

Q.E.D.

## 3.4 Proof of Theorems 1.21, 1.22 and Corollary 1.23

We are ready to prove our main theorem and its corollary. In particular, Theorem 1.21 trivially follows by Theorem 1.22. Accordingly, let us prove the stronger Theorem 1.22.

Proof of Theorem 1.22. As a consequence of Theorem 3.21 and Corollary 3.18

$$\begin{aligned}\sigma(f_*|_{\mathcal{C}^{p,q,d_s}}) \cap \{z\in\mathbb{C} : |z| > \lambda^{-1}e^{h_{top}}\} &= \sigma(f_\#|_{\bar H^{p,q,d_s}_{dR}}) \cap \{z\in\mathbb{C} : |z| > \lambda^{-1}e^{h_{top}}\}\\ &= \sigma(f_\#|_{H^{d_s}_{dR}}) \cap \{z\in\mathbb{C} : |z| > \lambda^{-1}e^{h_{top}}\}\end{aligned}$$

Since $e^{h_{top}}$ is a simple maximal eigenvalue of $f_*|_{\mathcal{B}^{p,q,d_s}}$, it holds that $\Lambda_1$, the maximal eigenvalue of $f_\#|_{H^{d_s}_{dR}(M)}$, is $e^{h_{top}}$, while the second one fulfills $|\Lambda_2| < e^{h_{top}}$. In addition, by Proposition 3.3, the eigenvector $\bar\omega$ and the dual eigenvector $\bar t$ related to $e^{h_{top}}$ defines the measure of maximal entropy $\mu_{BM}$. Furthermore, for any other eigenvalue $\Lambda_i$, with $|\Lambda_i| > \lambda^{-1}e^{h_{top}}$, we set a Jordan basis $\{\bar\omega_{i,k}\}_{k=1}^{N_i}$, such that $f_*(\bar\omega_{i,1}) = \Lambda_i\bar\omega_{i,1}$ and $f_*(\bar\omega_{i,k}) = \Lambda_i\bar\omega_{i,k} + \bar\omega_{i,k-1}$, for $k = 2,\dots,N_i$. Let $\{\bar t_{i,k}\}_{k=1}^{N_i}$ be the dual Jordan basis, such that $\bar t_{i,k}(\bar\omega_{i,j}) = \delta_{k,j}$. Notice that $f_*'\bar t_{i,N_i} = \Lambda_i\bar t_{i,N_i}$ and $f_*'\bar t_{i,k} = \Lambda_i\bar t_{i,k} + \bar t_{i,k+1}$, for $k = 1,\dots,N_i-1$. We point out that every $\Lambda_i$ represents a single Jordan, because eigenvalues are counted according to their algebraic multiplicity. We obtain that

$$f_* = e^{h_{top}}\bar\omega\otimes\bar t + \sum_{i=2}^{m}\left[\Lambda_i\left(\sum_{j=1}^{N_i}\bar\omega_{i,j}\otimes\bar t_{i,j}\right) + \sum_{j=1}^{N_i-1}\bar\omega_{i,j}\otimes\bar t_{i,j+1}\right] + \mathcal{Q},$$

where $\mathcal{Q}$ is a linear operator such that $\|\mathcal{Q}\|_{(\mathcal{C}^{p,q,d_s})'} \le \lambda^{-1}e^{h_{top}}$. As a consequence,

$$f_*^n = e^{h_{top}}\bar\omega\otimes\bar t + \sum_{i=2}^{m}\sum_{k=0}^{N_i-1}\binom{n}{k}\Lambda_i^{n-k}\left(\sum_{j=1}^{N_i-k}\bar\omega_{i,j}\otimes\bar t_{i,j+k}\right) + \mathcal{Q}^n,$$

where $\binom{n}{k} = 0$ for $k > n$. Then, for any $\phi, \psi \in C^\infty(M)$,

$$\int_M \phi\psi \circ f^n d\mu_{BM} = \mu_{BM}(\phi\psi \circ f^n) = \bar{t}(\phi\psi \circ f^n \bar{\omega}) = e^{-nh_{top}}\bar{t}(f_*^n(\phi\psi \circ f^n \bar{\omega})) =$$
$$= e^{-nh_{top}}\bar{t}(\psi(f_*^n(\phi\bar{\omega})) = \bar{t}(\phi\bar{\omega})\bar{t}(\psi\bar{\omega}) +$$
$$+ \sum_{i=2}^m \sum_{k=0}^{N_i-1} \binom{n}{k} \Lambda_i^{n-k} e^{-nh_{top}} \left( \sum_{j=1}^{N_i-k} \bar{t}_{i,j+k}(\phi\bar{\omega})\bar{t}(\psi\bar{\omega}_{i,j}) \right) + e^{-nh_{top}}\bar{t}(\psi \mathcal{Q}^n(\phi\bar{\omega}))$$

By defining a finite number of bilinear forms $\{c_{\Lambda_i,k}(\cdot,\cdot)\}_{\substack{i=2,\dots,m \\ k=0,\dots,N_i-1}}$ such that

$$\sum_{k=0}^{N_i-1} \binom{n}{k} \Lambda_i^{-k} \left( \sum_{j=1}^{N_i-k} \bar{t}_{i,j+k}(\phi\bar{\omega})\bar{t}(\psi\bar{\omega}_{i,j}) \right) = \sum_{k=0}^{N_i-1} n^k c_{\Lambda_i,k}(\phi,\psi),$$

we get

$$\int_M \phi\psi \circ f^n d\mu_{BM} = \mu_{BM}(\phi)\mu_{BM}(\psi) +$$
$$+ \sum_{i=2}^m \sum_{k=0}^{N_i-1} (\Lambda_i e^{-h_{top}})^n n^k c_{\Lambda_i,k}(\phi,\psi) + e^{-nh_{top}}\bar{t}(\psi \mathcal{Q}^n(\phi\bar{\omega})),$$

hence

$$\left| \int_M \phi\psi \circ f^n d\mu_{BM} - \int_M \phi d\mu_{BM} \int_M \psi d\mu_{BM} - \right.$$
$$\left. - \sum_{i=2}^m \sum_{k=0}^{N_i-1} (\Lambda_i e^{-h_{top}})^n n^k c_{\Lambda_i,k}(\phi,\psi) \right| \le C\lambda^{-n} \|\phi\|_{C^r} \|\psi\|_{C^r}$$

Q.E.D.

Proof of Corollary 1.23. Proposition 1.18 implies that $f$ is topologically conjugate to a hyperbolic automorphism of the torus $F\colon \mathbb{T}^{\dim(M)} \to \mathbb{T}^{\dim(M)}$. Accordingly, since $F$ is topologically transitive [56], the same goes for $f$ and we can apply Theorem 1.22. It is enough to show that the second highest eigenvalue $\Lambda_2$ of $F_\#|_{H^{d_s}_{dR}(M)}$ satisfies $|\Lambda_2| < \lambda^{-1}e^{h_{top}}$. Since the action of the dynamics on de Rham cohomology is invariant under topological conjugacy, it holds $f_\# = F_\#$. But $F$ is linear, hence, with a slight abuse of notation we can write $F^{-1} = F_\#|_{H^1_{dR}(M)}$. Notice that this equality makes no sense, because $F^{-1}$ acts on $\mathbb{T}^{\dim(M)}$, while $F_\#$ acts on $H^1_{dR}(\mathbb{T}^{\dim(M)})$. On the other hand, fixing a basis of $\mathbb{R}^{\dim(M)}$, $F^{-1}$ is induced by a $\dim(M) \times \dim(M)$ matrix $A \in GL_{\dim(M)}(\mathbb{Z})$ with $\det(A) = \pm 1$, and the same

$A$ is the matrix associated to $F_\#$ w.r.t. the canonical basis of $H^1_{dR}(\mathbb{T}^{\dim(M)})$, that is $\{[dx_1],[dx_2],\dots,[dx_{\dim(M)}]\}$. Next, assume that $\sigma_1=\{\nu_1,\nu_2,\dots,\nu_{\dim(M)}\}$ is the spectrum of $f_\#$ acting on $H^1_{dR}(\mathbb{T}^{\dim(M)})$. Then, we can assume that

$$|\nu_1|\le|\nu_2|\le\cdots\le|\nu_{d_u}|<1<|\nu_{d_u+1}|\le|\nu_{d_u+2}|\le\cdots\le|\nu_{\dim(M)}|$$

and it holds true that $\left|\prod_{i=1}^{\dim(M)}\nu_i\right|=1$. Moreover, the spectrum $\sigma_l$ of $F_\#$ acting on $H^l_{dR}(\mathbb{T}^{\dim(M)})$ can be determined by multiplying $l$ eigenvalues of $\sigma_1$, i.e., $\sigma_l=\{\prod_{i\in I}\nu_i|\ I\subseteq\{1,\dots,\dim(M)\},\ |I|=l\}$. Since $e^{h_{top}}$ is the maximal eigenvalue of $F_\#|_{H^{d_s}_{dR}(M)}$, we have

$$e^{h_{top}}=\prod_{i=d_u+1}^{\dim(M)}\nu_i=\prod_{i=1}^{d_s}\nu_i^{-1}.$$

Notice that this equality agrees with Ledrappier-Young entropy formula [19, Theorem D.3.1]

$$h_{top}=\sum_{i=d_s}^{\dim(M)}\ln\nu_i=\left(\sum_{i=1}^{d_u}\ln\nu_i\right)^{-1}.$$

In addition, we deduce that

$$\Lambda_2=\nu_{d_u}\prod_{i=d_u+2}^{\dim(M)}\nu_i,$$

while the maximal eigenvalues of $F_\#$ acting on $H^{d_s-1}_{dR}(M)$, resp. $H^{d_s+1}_{dR}(M)$, is

$$\zeta_{d_s-1}=\prod_{i=d_u}^{\dim(M)}\nu_i,\ \text{resp. }\zeta_{d_s+1}=\prod_{i=d_u+2}^{\dim(M)}\nu_i.$$

Furthermore, $|\zeta_{d_s-1}|\le\lambda^{-1}e^{h_{top}}$ and $|\zeta_{d_{s+1}}|\le\lambda^{-1}e^{h_{top}}$. In fact, by 3.18, $\zeta_{d_s-1}$, resp. $\zeta_{d_s+1}$, is an eigenvalue of $f_*|_{\mathcal{C}^{p,q,d_s-1}}$, resp. $f_*|_{\mathcal{C}^{p,q,d_s+1}}$, and $\rho(f_*|_{\mathcal{C}^{p,q,d_s\pm1}})\le\lambda^{-1}e^{h_{top}}$. Finally, noticing that

$$|\Lambda_2|<\min\{|\zeta_{d_s-1}|,|\zeta_{d_s+1}|\}\le\lambda^{-1}e^{h_{top}},$$

we conclude that

$$|\Lambda_2|<\lambda^{-1}e^{h_{top}}.$$

Q.E.D.

**Remark 3.26.** We recall the argument of [20] which works for Anosov diffeomorphisms on the 2-dimensional torus (see Remark 0.3) and that cannot be extended to higher-dimensional spaces. Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a $C^\infty$ Anosov diffeomorphism of the 2-torus. Then, by Corollary 1.23,

$$\left|\int_{\mathbb{T}^2} (\phi \circ f^n)\psi d\mu_{BM} - \int_{\mathbb{T}^2} \phi d\mu_{BM} \int_{\mathbb{T}^2} \psi d\mu_{BM}\right| \le Ce^{-nh_{top}} \|\phi\|_{C^r} \|\psi\|_{C^r}$$

On the other hand, authors of [20] proved that there exists a $\kappa \in (0,1)$ such that

$$\left|\int_{\mathbb{T}^2} (\phi \circ f^n)\psi d\mu_{BM} - \int_{\mathbb{T}^2} \phi d\mu_{BM} \int_{\mathbb{T}^2} \psi d\mu_{BM}\right| \le C\kappa e^{-nh_{top}} \|\phi\|_{C^r} \|\psi\|_{C^r},$$

excluding Ruelle-Pollicott resonances of modulus $e^{-h_{top}}$, i.e., eigenvalues of $f_*|_{\mathcal{C}^{p,q,1}}$ of modulus 1. In fact, by Corollary 2.15, $\rho(f_*|_{\mathcal{C}^{p,q,0}}) \le 1$, $\rho(f_*|_{\mathcal{C}^{p,q,1}}) \le e^{h_{top}}$ and $\rho(f_*|_{\mathcal{C}^{p,q,2}}) \le 1$. Assume that $\omega \in \mathcal{C}^{p,q,1}$ satisfies $f_*\omega = \nu\omega$ with $|\nu| = 1$. Then $\nu$ does not belong to the spectrum of the action on cohomology $f_\#|_{H^1_{dR}(M)}$, which contains $e^{\pm h_{top}}$. Accordingly, either $\omega$ is not closed or $\omega$ is exact. In the first case, $d\omega \neq 0$ and $f_*d\omega = \nu d\omega$. As a consequence of [20, Lemma 5.14], the Hodge operator, actually defined on differential forms, extends to a bounded isomorphism

$$\begin{aligned} \star\colon \mathcal{C}^{p,q,0} &\to \mathcal{C}^{p,q,2} \\ h &\mapsto \star h = h\omega_0 \end{aligned}$$

In addition, $\mathcal{L}h = f_* \star h$ as in (2.11). Consequently, $d\omega = h\omega_0$, where $h = \star d\omega$ and $\mathcal{L}h = \nu h$. [37, Theorem 2.3] states that 1 is the unique maximal eigenvalue of $\mathcal{L}$ and the corresponding eigenvector is the density of the SRB-measure $\mu_{SRB}$. Thus, $\mathcal{L} \star d\omega = \star d\omega$ and $d\omega = \mu_{SRB}$. We conclude that $\int_M \mu_{SRB} = \int_M d\omega = 0$ and this contradicts the property that $\mu_{SRB}$ is a positive measure, hence $\omega$ must be closed. Assume $\omega = dh$ for some $h \in \mathcal{C}^{p,q,0}$. Then $f_*h = \nu h + c$ with $dc = 0$. In particular, Lemma 3.23 implies that $c$ is a constant. Let $\mu^-_{SRB}$ be the SRB-measure for $f^{-1}$. Then we can write $h = a + g$ where $a$ is a constant and $\int_M g\mu^-_{SRB} = 0$. Since $f_*a = a$ and the space of null $\mu^-_{SRB}$-measures is $f_*$-invariant, we obtain that

$$c + \nu a + \nu g = f_*g + f_*a.$$

By integrating w.r.t. $\mu^-_{SRB}$, we conclude that $c = a(1-\nu)$, hence $f_*g = \nu g$. By [20, Lemma 5.17], 1 is the unique maximal eigenvalue of $f_*|_{\mathcal{C}^{p,q,0}}$ and the corresponding eigenvector is constant. We obtain that $f_*g = g$ and $g$ is constant, that is $g = 0$. We conclude that $h = c$ and $0 = dh = \omega$, which contradicts the assumptions. Accordingly, $f_*|_{\mathcal{C}^{p,q,1}}$ does not have eigenvalues of modulus 1.

# Appendix A

# A toolbox of Functional Analysis

This appendix contains basic concepts of Functional Analysis that we use in this thesis. It is not our intention to provide a complete description of the topics, but we refer the interested reader to [29, 30].

## A.1 Linear operators on Banach spaces

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and be $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ be Banach spaces, i.e., a complete normed spaces. We denote by $L(\mathcal{B}, \mathcal{C})$ the vector space of bounded, hence continuous, linear operator from $\mathcal{B}$ to $\mathcal{C}$, namely the set of linear maps $\mathcal{T}\colon \mathcal{B} \to \mathcal{C}$ such that $\sup_{0\neq x\in\mathcal{B}} \frac{\|\mathcal{T}(x)\|_{\mathcal{C}}}{\|x\|_{\mathcal{B}}} < +\infty$. $L(\mathcal{B}, \mathcal{C})$ becomes a Banach space once equipped with the norm

$$\|\mathcal{T}\| = \sup_{0\neq x\in\mathcal{B}} \frac{\|\mathcal{T}(x)\|_{\mathcal{C}}}{\|x\|_{\mathcal{B}}}.$$

Moreover, one may prove that, if $(\mathcal{B}, \|\cdot\|)$ is a Banach algebra, i.e., $\|xy\| \leq \|x\|\|y\|$, for all $x, y \in \mathcal{B}$, then also $(L(\mathcal{B}, \mathcal{B}), \|\cdot\|)$ is a Banach algebra w.r.t. composition. Given $\mathcal{T} \in L(\mathcal{B}, \mathcal{C})$, we denote by $\ker(T) = \{x \in \mathcal{B} : \ \mathcal{T}(x) = 0\}$ and $\mathrm{im}(\mathcal{T}) = \{\mathcal{T}(x) : \ x \in \mathcal{B}\}$ the kernel and the range of $\mathcal{T}$, respectively. Notice that, by continuity, $\ker(\mathcal{T})$ is a closed subspace of $\mathcal{B}$.

**Definition A.1.** *A bounded linear operator $\mathcal{T} \in L(\mathcal{B}, \mathcal{C})$ is compact if $\overline{\mathcal{T}(\mathcal{B}_{\mathcal{B}}(0,1))}$ is compact, i.e., if the image of the unit ball in $\mathcal{B}$ is relatively compact in $\mathcal{C}$.*

**Remark A.2.** We underline that $\mathcal{T}$ is compact if and only if $\overline{\mathcal{T}(B)}$ is compact for any bounded set $B \subseteq \mathcal{B}$. In effect, it is clear that, if the second condition is true, then

$\mathcal{T}$ is compact. On the other hand, if $\mathcal{T}$ is compact, then, for any bounded set $B \subseteq \mathcal{B}$, there is $R > 0$ such that $B \subseteq \mathcal{B}(0, R)$. Consequently, $\mathcal{T}(B) \subseteq R\mathcal{T}(\mathcal{B}(0,1))$. Since, by assumption, $\overline{\mathcal{T}(\mathcal{B}(0,1))}$ is compact and $\overline{\mathcal{T}(B)}$ is closed, then $\overline{\mathcal{T}(B)}$ is compact.

The following is a useful criterion to prove that a linear operator is compact.

**Lemma A.3.** *[31, Proposition 2.8] Let $\mathcal{T} \in L(\mathcal{B}, \mathcal{C})$ be a bounded linear operator. Assume that for all $\epsilon > 0$ there are finitely many continuous linear maps $l_i \colon \mathcal{B} \to \mathbb{R}$, $i = 1, \dots, m$, such that $\|\mathcal{T}(x)\|_{\mathcal{C}} \leq \epsilon \|x\|_{\mathcal{B}} + \sum_i |l_i(x)|$ for each $x \in \mathcal{B}$, then $\mathcal{T}$ is compact.*

PROOF. Let $(x_n)_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathcal{B}}(0,1)$ be a sequence inside the unit ball of $\mathcal{B}$. Up to considering a subsequence, we can assume that $(l_i(x_n))_{n\in\mathbb{N}}$ converges for every $i = 1, \dots, m$. Accordingly,

$$\limsup_{m,n\to+\infty} \|\mathcal{T}(x_n) - \mathcal{T}(x_m)\|_{\mathcal{C}} \leq 2\epsilon,$$

hence we can extract a Cauchy subsequence $(\mathcal{T}(x_{n_k}))$. Since $\mathcal{C}$ is a Banach space, $(\mathcal{T}(x_{n_k}))$ converges and we conclude that $\mathcal{T}$ is compact. Q.E.D.

## A.2 Bochner integral and functions with values in a Banach space

In the following we need to integrate functions on measure spaces $(M, \Sigma, \mu)$ with values in a Banach space $(\mathcal{B}, \|\cdot\|)$. The Bochner integral (see also [73]) plays exactly this role. As in the definition of Lebesgue integral, we firstly want to integrate simple functions. Let $A_1, \dots, A_m$ be disjoint elements of the $\sigma$-algebra $\Sigma$ and let $b_1, \dots, b_m \in \mathcal{B}$. If $\mu(A_i) < +\infty$ for each $i = 1, \dots, m$, then we say that the simple function is integrable and we define the integral

$$\int_M \sum_{i=1}^m \mathbb{1}_{A_i}(x) b_i \mu(dx) = \sum_{i=1}^m \mu(A_i) b_i.$$

Let us endow $\mathcal{B}$ of the Borel $\sigma$-algebra induced by the norm.

**Definition A.4.** *Let $f\colon M \to \mathcal{B}$ be a measurable function. We say that $f$ is Bochner integrable, if there is a sequence $(\phi_n)_{i\in\mathbb{N}}$ such that*

$$\lim_{i\to+\infty}\int_M \|f-\phi_i\|d\mu = 0,$$

*where every $\phi_i$ is a simple integrable function. In that case we define*

$$\int_M f d\mu = \lim_{i\to+\infty}\int_M \phi_i d\mu$$

**Lemma A.5.** *Bochner integral is well-defined, i.e., it does not depends on the sequence of simple integrable functions we choose.*

PROOF. Let $(\psi_i)_{i\in\mathbb{N}}$ be another sequence of simple integrable functions. Then for all $\epsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$

$$\left\|\int_M \psi_i - \phi_i d\mu\right\| = \left\|\int_M \psi_i - f + f - \phi_i d\mu\right\| \leq \left\|\int_M \psi_i - f d\mu\right\| + \left\|\int_M \phi_i - f d\mu\right\| < 2\epsilon.$$

Therefore,

$$\lim_{i\to+\infty}\int_M \phi_i d\mu = \lim_{i\to+\infty}\int_M \psi_i d\mu.$$

Q.E.D.

We also need to extend the definition of analytic functions to maps of $C^0(\mathbb{C},\mathcal{B})$.

**Definition A.6.** *Let $U \subseteq \mathbb{C}$ be an open set and let $f \in C^0(\mathbb{C},\mathcal{B})$. We say that $f$ is analytic on $U$ if, for each $z_0 \in U$ there is a neighborhood $U(z_0) \subseteq U$ containing $z_0$ and a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that*

$$f(z) = \sum_{n\in\mathbb{N}} a_n(z-z_0)^n$$

*for every $z \in U(z_0)$.*

The following proposition collects properties of analytic functions with values in $\mathbb{C}$ that are also true for analytic functions with values in $\mathcal{B}$. We leave the proof to the reader, since it reflects the proof for functions with values in $\mathbb{C}$.

**Proposition A.7.** *Let $f \in C^0(\mathbb{C},\mathcal{B})$ and suppose that $f$ is analytic in a simply connected open set $U \subseteq \mathbb{C}$. Let $\gamma$ be a simple smooth closed curve inside $U$ and let $\bar{z}$ be a point in the interior of the region bounded by $\gamma$. Then*

$$\int_\gamma f(z)dz = 0; \tag{A.1}$$

$$f(\bar{z}) = \frac{1}{2\pi i}\int_\gamma (\zeta-\bar{z})^{-1} f(\zeta)d\zeta. \tag{A.2}$$

## A.3 Specturm and essential spectrum

**Definition A.8.** *Let $\mathcal{T} \in L(\mathcal{B}, \mathcal{B})$ be a bounded linear operator. We say that $z \in \mathbb{C}$ belongs to the spectrum $\sigma(\mathcal{T})$ of $\mathcal{T}$, if the operator $(z\mathrm{id} - \mathcal{T})$ does not admit a bounded inverse. Otherwise, $z \in \mathbb{C}$ belongs to the resolvent $\mathcal{R}(\mathcal{T})$ of $\mathcal{T}$. If $z \in \mathcal{R}(\mathcal{T})$, we denote by $\mathcal{R}(z, \mathcal{T}) = (z\mathrm{id} - \mathcal{T})^{-1}$ the resolvent operator.*

**Lemma A.9.** *Given $\mathcal{T} \in L(\mathcal{B}, \mathcal{B})$, the limit*

$$\rho(\mathcal{T}) = \lim_{n\to+\infty} \|\mathcal{T}^n\|^{\frac{1}{n}}$$

*exists, it is finite and*

$$\rho(\mathcal{T}) = \sup_{z\in\sigma(\mathcal{T})} |z|.$$

*Accordingly, we call $\rho(\mathcal{T})$ the spectral radius of $\mathcal{T}$.*

Proof. We firstly notice that the sequence $a_n = \ln \|\mathcal{T}^n\|$ is subadditive, that is $a_{n+m} \le a_m + a_n$. In fact, since $L(\mathcal{B}, \mathcal{B})$ is a Banach algebra,

$$a_{n+m} = \ln \|\mathcal{T}^{n+m}\| \le \ln(\|\mathcal{T}^n\|\|\mathcal{T}^m\|) \le \ln \|\mathcal{T}^n\| + \ln \|\mathcal{T}^m\| \le a_n + a_m.$$

We now show that, for a subadditive sequence, $(a_n)_{n\in\mathbb{Z}^+}$ the limit $\lim_{n\to+\infty} \frac{a_n}{n}$ exists and it equals $\inf_{n\in\mathbb{N}} \frac{a_n}{n}$. Let us denote by $A = \inf_{n\in\mathbb{N}} \frac{a_n}{n}$. For all $\epsilon > 0$, let $n \in \mathbb{Z}^+$ such that $a_n < n(A+\epsilon)$, and let $A_n = \max_{1\le i\le n} a_n$. Then, for $m \ge n$, we write $m = qn + r$, where $0 \le r \le n-1$, and we obtain

$$\frac{a_m}{m} \le \frac{qa_n}{m} + \frac{A_n}{m} < \frac{qn(A+\epsilon)}{m} + \frac{A_n}{m} \xrightarrow[m\to+\infty]{} A + \epsilon.$$

As a consequence, the limit

$$\lim_{n\to+\infty} \|\mathcal{T}^n\|^{\frac{1}{n}} = \lim_{n\to+\infty} \exp\left(\frac{1}{n}\ln\|\mathcal{T}^n\|\right) = \inf_{n\in\mathbb{Z}^+} \exp\left(\frac{1}{n}\ln\|\mathcal{T}^n\|\right)$$

exists. Moreover, since $\mathcal{T}$ is bounded and $L(\mathcal{B}, \mathcal{B})$ is a Banach algebra,

$$\rho(\mathcal{T}) = \lim_{n\to+\infty} \|\mathcal{T}^n\|^{\frac{1}{n}} \le \|\mathcal{T}\| < +\infty.$$

We now prove the second equality. Let us recall that, by Neumann's theorem,

$$(\mathrm{id} - \mathcal{T})^{-1} = \sum_{i=0}^{+\infty} \mathcal{T}^i$$

in case the series at the right hand side converges in $L(\mathcal{B},\mathcal{B})$. In fact, considering the partial sum $\mathcal{S}_n = \sum_{i=0}^{n} \mathcal{T}^i$ we have

$$(\mathrm{id} - \mathcal{T})\mathcal{S}_n = \sum_{i=0}^{n} \mathcal{T}^i - \sum_{i=1}^{n+1} \mathcal{T}^i = \mathrm{id} - \mathcal{T}^{n+1} \xrightarrow[n\to+\infty]{} \mathrm{id},$$

because $\mathcal{T}^{n+1} \xrightarrow[n\to+\infty]{} 0$ as a consequence of the convergence of the series. Notice that a sufficient (not necessary) condition for the convergence of Neumann's sum is $\|\mathcal{T}\| < 1$. It is actually sufficient that there exist $\bar{n} \in \mathbb{N}$ and $a \in (0,1)$ such that, for any $n \geq \bar{n}$, $\|z^{-n}\mathcal{T}^n\| < a^n < 1$. We can write

$$(z\mathrm{id} - \mathcal{T})^{-1} = z^{-1}(\mathrm{id} - z^{-1}\mathcal{T})^{-1} = z^{-1}\sum_{n=0}^{+\infty} z^{-n}\mathcal{T}^n$$

Moreover, for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that, for all $n \geq n_\epsilon$, it holds $\|\mathcal{T}^n\| < (\rho(\mathcal{T}) + \epsilon)^n$. Accordingly, the Neumann's series converges whenever $\|z^{-n}\mathcal{T}^n\| < |z|^{-n}(\rho(\mathcal{T}) + \epsilon)^n < a^n < 1$, for instance when $|z| > \rho(\mathcal{T}) + 2\epsilon$. Since $\epsilon$ is arbitrary small, we conclude that

$$\sup_{z\in\sigma(\mathcal{T})} |z| \leq \rho(\mathcal{T})$$

To prove the equality, assume by contradiction that $\sup_{z\in\sigma(\mathcal{T})} |z| < r < \rho(\mathcal{T})$. Let $\gamma$ be the boundary of $\mathcal{B}(0,r) \subset \mathbb{C}$. Then, we compute

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma z^n(z\mathrm{id} - \mathcal{T})^{-1}dz &= \mathcal{T}^n + \frac{1}{2\pi i}\int_\gamma (z^n\mathrm{id} - \mathcal{T}^n)(z - \mathcal{T})^{-1}dz = \\ &= \mathcal{T}^n + \frac{1}{2\pi i}\int_\gamma \sum_{i=0}^{n-1} z^i\mathcal{T}^{n-i}dz = \mathcal{T}^n.\end{aligned}$$

where we used (A.1) in the first equality and (A.2) in the last equality. Accordingly, $\|\mathcal{T}^n\| \leq Cr^n$ for all $n \in \mathbb{N}$, hence $\rho(\mathcal{T}) \leq r$ against the assumption. Q.E.D.

**Lemma A.10.** *Let $P \in L(\mathcal{B},\mathcal{B})$ be a projection operator, which means that $P^2 = P$. Then $\mathcal{B} = \ker(P) \oplus \mathrm{im}(P)$. Furthermore, both $\ker(P)$ and $\mathrm{im}(P)$ are closed.*

Proof. If $x \in \mathcal{B}$, then $x = (\mathrm{id} - P)x + Px$ and $P(\mathrm{id} - P)x = Px - P^2x = 0$, i.e., $(\mathrm{id} - P)x \in \ker(P)$. In addition, if $x \in \ker(P) \cap \mathrm{im}(P)$, then $Py = x$ and $0 = P(x) = P^2(y) = P(y) = x$. Thus, $\ker(P) \cap \mathrm{im}(P) = \{0\}$. In addition, $\ker(P) = P^{-1}(0)$ and $P$ is continuous, hence $\ker(P)$ is closed. The same is true for $\mathrm{im}(P)$, because it is easy to show that $\mathrm{im}(P) = \ker(\mathrm{id} - P)$. In effect, $(\mathrm{id} - P)Px = Px - P^2x = Px - Px = 0$, that is $\mathrm{im}(P) \subseteq \ker(\mathrm{id} - P)$. On the other hand, if $(\mathrm{id} - P)x = 0$, then $Px = x$, i.e., $x \in \mathrm{im}(P)$ and $\mathrm{im}(P) \supseteq \ker(\mathrm{id} - P)$. Q.E.D.

The following result allows us to decompose a bounded linear operator in accordance with a splitting of the spectrum.

**Proposition A.11.** *Let $\mathcal{T} \in L(\mathcal{B}, \mathcal{B})$ be a bounded linear operator. Suppose that the spectrum of $\mathcal{T}$ splits into two disjoint nonempty subsets $\sigma(\mathcal{T}) = \sigma_1 \cup \sigma_2$. Let $\gamma$ be a simple smooth closed curve on $\mathcal{R}(\mathcal{T})$ such that $\sigma_1$ and $\sigma_2$ are divided by $\gamma$. In particular, suppose that $\sigma_1$ is contained in the interior of $\gamma$. We define the linear operator*

$$P = \frac{1}{2\pi i}\int_\gamma (z\mathrm{id} - \mathcal{T})^{-1} dz.$$

*Then $P$ is a projection, it does not depends on $\gamma$ and it commutes with $\mathcal{T}$, i.e., $\mathcal{T}P = P\mathcal{T}$. Moreover, $\ker(P)$ and $\mathrm{im}(P)$ are $\mathcal{T}$ invariant linear subspaces of $\mathcal{B}$, $\sigma(\mathcal{T}|_{\mathrm{im}(\mathcal{T})}) = \sigma_1$ and $\sigma(\mathcal{T}|_{\ker(\mathcal{T})}) = \sigma_2$.*

Proof. One can prove that the definition of $P$ does not depend on $\gamma$ repeating the analogous proof for analytic functions. Let $\bar\gamma$ be another curve containing $\gamma$ in its interior and satisfying the hypothesis. Then, we compute

$$\begin{aligned}
P^2 &= \frac{1}{(2\pi i)^2}\int_\gamma\int_{\bar\gamma}(z\mathrm{id} - \mathcal{T})^{-1}(\bar z\mathrm{id} - \mathcal{T})^{-1} d\bar z dz = \\
&= \frac{1}{(2\pi i)^2}\int_\gamma\int_{\bar\gamma}\frac{1}{z - \bar z}[(z\mathrm{id} - \mathcal{T})^{-1} - (\bar z\mathrm{id} - \mathcal{T})^{-1}] d\bar z dz = \\
&= \frac{1}{(2\pi i)^2}\int_\gamma(z\mathrm{id} - \mathcal{T})^{-1}\int_{\bar\gamma}\frac{1}{\bar z - z} d\bar z dz - \frac{1}{(2\pi i)^2}\int_{\bar\gamma}(\bar z\mathrm{id} - \mathcal{T})^{-1}\int_\gamma\frac{1}{\bar z - z} dz d\bar z,
\end{aligned}$$

where in the second equality we used

$$\begin{aligned}
&\frac{1}{\bar z - z}[(z\mathrm{id} - \mathcal{T})^{-1} - (\bar z\mathrm{id} - \mathcal{T})^{-1}](z\mathrm{id} - \mathcal{T})(\bar z\mathrm{id} - \mathcal{T}) = \\
=&\frac{1}{\bar z - z}[\bar z\mathrm{id} - \mathcal{T} - z\mathrm{id} + \mathcal{T}] = \mathrm{id}.
\end{aligned}$$

Since $\bar\gamma$ is outside $\gamma$, $(\bar z - z)^{-1}$ is analytic in the region bordered by $\gamma$, we obtain $\int_\gamma (\bar z - z)^{-1} d\bar z = 0$. On the other hand, the function $\phi(\bar z) = (\bar z - z)^{-1}$ has a simple pole inside the region bordered by $\bar\gamma$. Accordingly, by Cauchy's residue theorem $\int_{\bar\gamma}(\bar z - z)^{-1} d\bar z = 2\pi i$ and we conclude that

$$P^2 = \frac{1}{2\pi i}\int_\gamma (z\mathrm{id} - \mathcal{T})^{-1} dz = P.$$

Since $(z\mathrm{id}-\mathcal{T})\mathcal{T}^{-1} = z\mathcal{T}^{-1}-\mathrm{id} = \mathcal{T}^{-1}(z\mathrm{id}-\mathcal{T})$, $\mathcal{T}$ commutes with $(z\mathrm{id}-\mathcal{T})$, hence $\mathcal{T}$ commutes with $P$. As a consequence, $T(\ker(P)) \subseteq \ker(P)$ and $T(\mathrm{im}(P)) \subseteq \mathrm{im}(P)$, that is $T|_{\ker(P)}$ and $T|_{\mathrm{im}(P)}$ are well-defined linear endomorphisms. It remains to prove that $\sigma(\mathcal{T}|_{\mathrm{im}(\mathcal{T})}) = \sigma_1$ and $\sigma(\mathcal{T}|_{\ker(\mathcal{T})}) = \sigma_2$. Let $\lambda \in \sigma_2$, we must show that $(\lambda\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})$ is invertible. Let us define, given $\gamma$ as above,

$$Q = \frac{1}{2\pi i}\int_\gamma \frac{(z\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})^{-1}}{\lambda - z}dz.$$

Notice that

$$\begin{aligned}(\lambda\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})Q = \frac{1}{2\pi i}\int_\gamma \frac{(\lambda\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})(z\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})^{-1}}{\lambda - z}dz = \\ =\frac{1}{2\pi i}\int_\gamma \frac{(\lambda\mathrm{id} - z\mathrm{id} + z\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})(z\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})^{-1}}{\lambda - z}dz = \\ =\frac{1}{2\pi i}\int_\gamma (z\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})^{-1}dz + \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{id}}{\lambda - z}dz = P,\end{aligned} \tag{A.3}$$

where we used the fact that $(\lambda - z)^{-1}$ is analytic for $\lambda \in \sigma_2$, hence last integral equals 0. Therefore, given $y = Px \in \mathrm{im}(P)$, $(\lambda\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})Qy = Py = y$, that is $(\lambda\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})Q = \mathrm{id}|_{\mathrm{im}(P)}$. Accordingly, $\sigma(\mathcal{T}|_{\mathrm{im}(P)}) \subseteq \sigma_1$. Let us prove that $\sigma(\mathcal{T}|_{\ker(P)}) \subseteq \sigma_2$. Let $\lambda \in \sigma_1$, then we consider $Q$ as above. The computation (A.3) holds with $\mathcal{T}|_{\mathrm{im}(P)}$ replaced by $\mathcal{T}|_{\ker(P)}$, except for last integral. In fact, since $\lambda$ is inside $\gamma$, then it is a pole of $(\lambda - z)^{-1}$. Using again Cauchy's residue theorem, we get $(\lambda\mathrm{id} - \mathcal{T}|_{\ker(P)})Q = P + \mathrm{id}$. Consequently, if $x \in \ker(P)$, then $(\lambda\mathrm{id} - \mathcal{T}|_{\ker(P)})Qx = Px + x = x$, i.e., $(\lambda\mathrm{id} - \mathcal{T}|_{\mathrm{im}(P)})Q = \mathrm{id}$. By Lemma A.10, we conclude that $\sigma(\mathcal{T}|_{\mathrm{im}(P)}) = \sigma_1$ and $\sigma(\mathcal{T}|_{\ker(P)}) = \sigma_2$. Q.E.D.

**Definition A.12.** *Let $\mathcal{T} \in L(\mathcal{B}, \mathcal{B})$ be a bounded linear operator. We say that $\lambda \in \sigma(\mathcal{T})$ belongs to the discrete spectrum of $\mathcal{T}$ if*

1. *$\lambda$ is isolated inside $\sigma(\mathcal{T})$;*
2. *the projection*
$$P = \frac{1}{2\pi i}\int_\gamma (z\mathrm{id} - \mathcal{T})^{-1}dz,$$
*where $\gamma$ is a simple smooth closed curve in $\mathcal{R}(\mathcal{T})$ around $\lambda$, has finite rank;*
3. *the range of $\lambda\mathrm{id} - \mathcal{T}$ is closed.*

*We denote by $\sigma_{\text{dis}}(\mathcal{T}) \subseteq \sigma(\mathcal{T})$ the discrete spectrum, while we denote by $\sigma_{\text{ess}}(\mathcal{T}) \subseteq \sigma(\mathcal{T})$ the complementary part, that is the essential spectrum. In particular, $\lambda$ belongs to the essential spectrum if at least one of the following conditions is true:*

1. *the range of $(\lambda\text{id} - \mathcal{T})$ or the range of $(\lambda\text{id} - \mathcal{T})^{-1}$ is not closed;*
2. *$\cup_{i\geq 1} \ker((\lambda\text{id} - \mathcal{T})^i)$ is infinite dimensional;*
3. *$\lambda$ is a limit point in $\sigma(\mathcal{T}) \setminus \{\lambda\}$.*

*We finally denote by*

$$\rho_{\text{ess}}(\mathcal{T}) = \sup_{z \in \sigma_{\text{ess}}(\mathcal{T})} |z|$$

*the essential spectral radius.*

**Remark A.13.** There are many different definitions of essential spectrum. The one we use in this thesis is the most common in the field of dynamical systems and it is Browder's definition [18]. We point out that, although the essential spectrum depends on the definition, for bounded linear operator, the essential spectral radius is independent of it [30, Corollary 4.11].

**Definition A.14.** *Let $\mathcal{T} \in L(\mathcal{B}, \mathcal{C})$ be a bounded linear operator between Banach spaces. We say that $\mathcal{T}$ is quasi-compact if there exist a bounded linear operator $\mathcal{T}_{\text{ess}} \in L(\mathcal{B}, \mathcal{C})$ and a compact linear operator of finite rank $\mathcal{T}_{\text{dis}} \in L(\mathcal{B}, \mathcal{C})$, such that $\mathcal{T} = \mathcal{T}_{\text{ess}} + \mathcal{T}_{\text{dis}}$ and $\|\mathcal{T}_{\text{ess}}\| < \|\mathcal{T}_{\text{dis}}\|$. As a consequence of Proposition A.11, if $\mathcal{T} \in L(\mathcal{B}, \mathcal{B})$ is a linear endomorphism, then $\mathcal{T}$ is quasi-compact if and only if $\rho_{\text{ess}}(\mathcal{T}) < \rho(\mathcal{T})$.*

We conclude this section recalling a useful result [9, Lemma A.1.] about the spectrum of quasicompact operators acting on different Banach spaces.

**Lemma A.15.** *Let $\mathcal{B}$ be a separable topological vector space and let $(\mathcal{B}_1, \|\cdot\|_1)$ and $(\mathcal{B}_2, \|\cdot\|_2)$ be two Banach spaces that are continuously embedded in $\mathcal{B}$. Assume that there exists a linear subspace $\mathcal{B}_0 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$ which is dense in $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\mathcal{T} \colon \mathcal{B} \to \mathcal{B}$ be a continuous linear map preserving $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$. Assume that there exists $\bar{\rho} > 0$ such that*

$$\max\{\rho_{ess}(\mathcal{T}|_{\mathcal{B}_1}), \rho_{ess}(\mathcal{T}|_{\mathcal{B}_2})\} < \bar{\rho}.$$

*Then the eigenvalues of the operators $\mathcal{T}|_{\mathcal{B}_1}$ and $\mathcal{T}|_{\mathcal{B}_2}$ coincide out of the ball $\{z \in \mathbb{C} : |z| < \bar{\rho}\}$. In addition, the generalized eigenvectors corresponding to such eigenvalues coincide and belong to $\mathcal{B}_1 \cap \mathcal{B}_2$.*

Proof. We firstly prove that

$$\rho_{ess}(\mathcal{T}) = \inf\{\rho(\mathcal{T}|_W) : \ W \in \mathcal{W}\}, \tag{A.4}$$

where $\mathcal{W}$ is the family of closed $\mathcal{T}$-invariant linear subspaces $W \in \mathcal{W}$ of finite codimension. In effect, for any $\widetilde{\rho} > \rho_{ess}(\mathcal{T})$, let $W$ be the image of the projection corresponding to the spectrum contained in $\{z \in \mathbb{C} : \ |z| < \widetilde{\rho}\}$. As a consequence of Lemma A.10, Proposition A.11 and Definition A.12, $W$ is a closed $\mathcal{T}$-invariant linear subspace of finite codimension, i.e., $W \in \mathcal{W}$. Accordingly, $\widetilde{\rho} \geq \inf\{\rho(\mathcal{T}|_W) : W \in \mathcal{W}\}$ for any $\widetilde{\rho} > \rho_{ess}(\mathcal{T})$, hence $\rho_{ess}(\mathcal{T}) \geq \inf\{\rho(\mathcal{T}|_W) : \ W \in \mathcal{W}\}$. On the other hand, let $W \in \mathcal{W}$ and let $W'$ a complementary finite dimensional subspace in $\mathcal{B}$. Denoting by $\pi \colon \mathcal{B} \to W$ and $\pi' \colon \mathcal{B} \to W'$ the corresponding projections, one can decompose $\mathcal{T} = \mathcal{T} \circ \pi + \mathcal{T} \circ \pi'$, where $\mathcal{T} \circ \pi'$ is a finite rank operator. This implies that $\rho_{ess}(\mathcal{T}) \leq \rho(\mathcal{T}|_W)$ for any $W \in \mathcal{W}$, hence $\rho_{ess}(\mathcal{T}) \leq \inf\{\rho(\mathcal{T}|_W) : \ W \in \mathcal{W}\}$.

Let us consider $\mathcal{B}_1 \cap \mathcal{B}_2$. This is a Banach space once we endow it with $\|\cdot\|_1 + \|\cdot\|_2$. Using (A.4), we get $\rho_{ess}(\mathcal{T}|_{\mathcal{B}_1\cap\mathcal{B}_2}) \leq \max\{\rho_{ess}(\mathcal{T}|_{\mathcal{B}_1}), \rho_{ess}(\mathcal{T}|_{\mathcal{B}_2})\}$. In fact, let $\mathcal{W}_i$ be the family of closed $\mathcal{T}$-invariant linear subspaces of finite codimension in $\mathcal{B}_i$, for $i \in \{1, 2\}$. Similarly, let $\mathcal{W}_{1,2}$ be the analogous family for $\mathcal{B}_1 \cap \mathcal{B}_2$. The existence of the linear subspace $\mathcal{B}_0 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$, which is dense in $\mathcal{B}_1$ and $\mathcal{B}_2$, implies that $\mathcal{W}_{1,2} \subseteq \mathcal{W}_1 \cup \mathcal{W}_2$, i.e., the relation among essential spectral radii. As a consequence, we assume, without loss of generality, that $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\|\cdot\|_2 \leq \|\cdot\|_1$.

Fix $\widetilde{\rho}$ as in the statement of the lemma. Denote by $E \subseteq \mathcal{B}_1$ the finite dimensional subspace obtained as the direct sum of generalized eigenspaces corresponding to eigenvalues $\lambda$ with $|\lambda| \geq \widetilde{\rho}$. Up to considering the quotient spaces $\mathcal{B}_i/E$, for $i = 1, 2$, we can assume that $E = \{0\}$, so that $\rho(\mathcal{T}|_{\mathcal{B}_1}) < \widetilde{\rho}$. We finally show that there are no eigenvalues $\lambda$ for $\mathcal{T} \colon \mathcal{B}_2 \to \mathcal{B}_2$ such that $|\lambda| \geq \widetilde{\rho}$. Assume by contradiction that such a $\lambda$ exists, then there exists a nonzero $v \in \mathcal{B}_2$ such that $\mathcal{T}v = \lambda v$. By the density of $\mathcal{B}_0 \subseteq \mathcal{B}_2$, there exists $v' \in \mathcal{B}_0$, close enough to $v$ in both the $\mathcal{B}_1$ and $\mathcal{B}_2$ topologies, such that, for all $n \geq 0$, $\|\mathcal{T}^n v'\|_1 \geq \|\mathcal{T}^n v'\|_2 \geq |\lambda|^n \|v'\|_2$. Accordingly $\|\mathcal{T}\|_{\mathcal{B}_1\to\mathcal{B}_1} \geq |\lambda|$, and this contradicts the assumption $\rho(\mathcal{T}|_{\mathcal{B}_1}) < \widetilde{\rho}$.

Q.E.D.

## A.4 Proof of Hennion's theorem

In this section we prove Theorem 2.9. We essentially follow [27], but we give the original proof of Hennion [41], which is a consequence of Nussbaum formula [59]. In effect, the authors of [27] prove a generalization of Hennion's original result (see also [10]).

We firstly recall some prerequisites.

**Definition A.16.** *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space and let $W \subseteq \mathcal{B}$ be a linear subspace. The distance between a point $x \in \mathcal{B}$ and the subspace $W$ is*

$$d(x, W) = \inf_{y \in W} \|x - y\|.$$

The following result, which is intuitive for finite-dimensional Banach spaces, also holds for infinite-dimension spaces.

**Lemma A.17.** *Let $W \subset \mathcal{B}$ be a proper subspace of the Banach space $\mathcal{B}$, that is $0 \neq W \neq \mathcal{B}$. Then for each $\epsilon > 0$ there exists $x \in \mathcal{B}$ such that $\|x\| = 1$ and $d(x, W) \geq 1 - \epsilon$.*

Proof. Let $y \in \mathcal{B} \setminus W$, then by definition, for all $\delta > 0$ there exists $z \in W$ such that $d(y, W) \leq \|y - z\| \leq d(y, W) + \delta$. Let consider $x = \frac{y-z}{\|y-z\|}$ and $\delta = \frac{\epsilon d(y,W)}{1-\epsilon}$. For every $q \in W$ we have

$$\|x - q\| = \frac{\|y \overbrace{-z - q\|y - z\|}^{\in W} \|}{\|y - z\|} \geq \frac{d(y, W)}{d(y, W) + \delta} = 1 - \epsilon.$$

Q.E.D.

**Definition A.18.** *Let $\mathcal{B}$ be a Banach space. $\mathcal{B}$ is locally compact if every bounded sequence in $\mathcal{B}$ has a convergent subsequence.*

**Lemma A.19.** *Let $\mathcal{B}$ be a locally compact Banach space. Then $\mathcal{B}$ is finite-dimensional.*

Proof. Assume by contradiction that $\mathcal{B}$ is infinite-dimensional. Let $W_1 = \mathrm{span}\{w_1\}$ be a 1-dimensional subspace with $\|w_1\| = 1$. Then, by Lemma A.17, we can define by induction $W_n = \mathrm{span}\{w_1, \ldots, w_n\}$, $n \in \mathbb{Z}^+$, such that $\|w_i\| = 1$, for $i = 1, \ldots, n$, and $\|w_i - w_j\| \geq \frac{1}{2}$, for all $i, j = 1, \ldots, n$, $i \neq j$. Therefore, we have built a bounded sequence $(x_n)_{n \in \mathbb{N}}$ with no convergent subsequences. Q.E.D.

**Definition A.20.** *Let $T\colon X \to Y$ be a continuous function between topological spaces. We say that $T$ is proper if, for every compact set $A \subseteq Y$, $T^{-1}(M)$ is compact in $X$.*

**Lemma A.21.** *Let $\mathcal{T} \in L(\mathcal{B}, \mathcal{C})$ be a bounded linear operator, such that $\mathcal{T}$ restricted to closed bounded sets is proper. Then* $\ker(\mathcal{T})$ *is finite-dimensional and* $\mathrm{im}(\mathcal{T})$ *is closed.*

Proof. By assumption, $\mathcal{T}$ is proper when restricted to bounded closed sets. Therefore, $\ker(\mathcal{T})$ is locally compact. As a consequence of Lemma A.19, it is finite-dimensional. To prove the second part, let $(\mathcal{T}(x_n))_{n\in\mathbb{N}}$ be a sequence in $\mathrm{im}(\mathcal{T})$, such that $(\mathcal{T}(x_n)) \xrightarrow[n\to+\infty]{} y \in \mathcal{C}$. The set $A = \{y, \mathcal{T}(x_n)|\ n \in \mathbb{N}\}$ is compact, hence $\mathcal{T}^{-1}A$ is compact and we can extract a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$, such that $x_{n_k} \xrightarrow[k\to+\infty]{} x \in \mathcal{B}$. By continuity, $\mathcal{T}(x_{n_k}) \xrightarrow[k\to+\infty]{} \mathcal{T}(x) = y \in \mathrm{im}(\mathcal{T})$. We conclude that $\mathrm{im}(\mathcal{T})$ is closed. Q.E.D.

**Definition A.22** (Measure of noncompactness)**.** *Let $\mathcal{B}$ be a Banach space and let $B \subseteq \mathcal{B}$ be a bounded subset. The measure on noncompactness of $B$, denoted by $\gamma(B) \geq 0$, is the infimum over $d > 0$, such that there exists $B_1, \dots, B_n$, subsets of $\mathcal{B}$ of diameter* $\mathrm{diam} B_i \leq d$*, for which $B \subseteq \cup_{i=1}^n B_i$. When $B_1, \dots, B_n$ are balls of radius smaller that $d$, the same definition gives the ball-measure of noncompactness $\widetilde{\gamma}(B)$. Given a bounded linear operator $\mathcal{T} \in L(\mathcal{B}, \mathcal{C})$, we define the measure of noncompactness of $\mathcal{T}$*

$$\gamma(\mathcal{T}) = \inf\{d > 0 |\ \gamma_{\mathcal{C}}(\mathcal{T}(B)) \leq d\gamma_{\mathcal{B}}(B),\ \textit{for each bounded } B \subseteq \mathcal{B}\}$$

*and the ball-measure of noncompactness of $\mathcal{T}$*

$$\widetilde{\gamma}(\mathcal{T}) = \inf\{d > 0 |\ \widetilde{\gamma}_{\mathcal{C}}(\mathcal{T}(B)) \leq d\widetilde{\gamma}_{\mathcal{B}}(B),\ \textit{for each bounded } B \subseteq \mathcal{B}\}$$

The following lemma collects some properties of the (ball-)measure of noncompactness.

**Lemma A.23.** *Let $A, B \subseteq \mathcal{B}$ be bounded subsets of the Banach space $\mathcal{B}$ and let $\mathcal{T} \in L(\mathcal{B}, \mathcal{C})$ be a bounded linear operator. The following properties hold:*

1) *The closure of $A$, $\bar{A}$, is compact if and only if $\gamma(A) = 0$. Moreover, $\bar{A}$ is compact if and only if $\widetilde{\gamma}(A) = 0$;*

*2)* $\mathcal{T}$ *is compact if and only if* $\gamma(\mathcal{T}) = 0$*. Moreover,* $\mathcal{T}$ *is compact if and only if* $\widetilde{\gamma}(\mathcal{T}) = 0$*;*

*3)* $\gamma(\mathcal{T}) \leq \|\mathcal{T}\|$*;*

*4)* $\gamma(A + B) \leq \gamma(A) + \gamma(B)$ *and* $\widetilde{\gamma}(A + B) \leq \widetilde{\gamma}(A) + \widetilde{\gamma}(B)$

Proof. If $\bar{A}$ is compact, then for all $\epsilon > 0$ there is a finite open covering of $\bar{A}$ made of bounded subsets of $\mathcal{B}$ of radius $\epsilon$. Thus, $\gamma(A) = 0$. Similarly, $\widetilde{\gamma}(A) = 0$. On the other hand, if $\bar{A}$ is not compact, then there is a sequence $(x_n)_{n\in\mathbb{N}}$ with no accumulation points. Let us consider the family of balls $(\mathcal{B}(x_n, \epsilon))_{n\in\mathbb{N}}$ of radius $\epsilon$. We claim that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, such that $\mathcal{B}(x_{n_{k_1}}, \epsilon) \cap \mathcal{B}(x_{n_{k_2}}, \epsilon) = \varnothing$, for some $\epsilon > 0$ and for each $k_1, k_2 \in \mathbb{N}$. In effect, if this is not true, then for all $\epsilon > 0$ and for any subsequence $(x_{n_k})_{k\in\mathbb{N}}$, there are $k_1, k_2 \in \mathbb{N}$ such that $d(x_{n_{k_1}}, x_{n_{k_2}}) < 2\epsilon$. Accordingly, there is Cauchy subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to some point $x \in \bar{A}$, against the assumption. We conclude that $\widetilde{\gamma}(\mathcal{T}) \geq \gamma(\mathcal{T}) > \epsilon > 0$.

Let us prove 2). If $\mathcal{T}$ is compact, then, by Remark A.2, $\overline{\mathcal{T}(B)}$ is compact, whenever $B \subset \mathcal{B}$ is bounded. By 1) $\widetilde{\gamma}(\mathcal{T}(B)) = \gamma(\mathcal{T}(B)) = 0$, i.e., $\widetilde{\gamma}(\mathcal{T}) = \gamma(\mathcal{T}) = 0$. If $\gamma(T) = 0$, then let us take $B \subset \mathcal{B}$ bounded and contained in a ball of radius $R$. For any $\epsilon > 0$, we have $\gamma(\mathcal{T}(B)) < \frac{\epsilon}{R}\gamma(B) < \epsilon$, i.e., $\gamma(\mathcal{T}(B)) = 0$, which implies that $\mathcal{T}(B)$ is compact by 1).

For any $B \subseteq \mathcal{B}$ bounded, let $l > \gamma(B)$, so that $B$ is covered by bounded sets $B_1, \ldots, B_n$ of diameter smaller than $l$. Consequently, $\mathcal{T}(B)$ is covered by $\mathcal{T}(B_1), \ldots, \mathcal{T}(B_n)$ and $\operatorname{diam}(\mathcal{T}(B_i)) = \sup_{x,y\in B_i} \|\mathcal{T}(x) - \mathcal{T}(y)\| \leq \|T\|\operatorname{diam}(B_i) \leq l\|\mathcal{T}\|$. This proves 3).

We now show 4). Let $l > \gamma(A)$ and let $t > \gamma(B)$. By definition there are coverings $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^m$ of $A$ and $B$, respectively, such that $\operatorname{diam}(A_i) \leq l$ and $\operatorname{diam}(B_i) \leq t$. It is clear that $A + B$ is covered by sets $\{A_i + B_j\}_{i,j}$ whose diameter is bounded by $l + t$. We conclude that $\gamma(A+B) \leq \gamma(A)+\gamma(B)$. The proof of second inequality is essentially the same noticing that the sum of two balls is again a ball.

Q.E.D.

**Lemma A.24.** *Let* $\mathcal{T} \in L(\mathcal{B}, \mathcal{B})$ *be a bounded linear operator. Suppose that there exists* $n \in \mathbb{Z}^+$ *such that* $\widetilde{\gamma}(\mathcal{T}) \leq k < 1$*. Then* $(\mathrm{id} - \mathcal{T})^r$ *restricted to bounded closed sets is proper, for any* $r \geq 1$*.*

Proof. Let $B \subset \mathcal{B}$ be a bounded closed set. By definition, we need to show that, for any compact set $C \in \mathcal{B}$, $D = (\mathrm{id} - \mathcal{T})^{-1}(C)$ is compact, which is equivalent to $\widetilde{\gamma}(D) = 0$. For any $x \in D$ $x = \mathcal{T}x + c$, with $c \in C$. Iterating this equality we get $x = \mathcal{T}^n(x) + \sum_{i=0}^{n-1} \mathcal{T}^i(c)$. Writing $\widetilde{C} = \sum_{i=0}^{n-1} \mathcal{T}^i C$, we get that $\widetilde{C}$ is compact, since it is the image of a compact set under a continuous map, and $D \subseteq \mathcal{T}^n(D) + \widetilde{C}$. Accordingly, $\widetilde{\gamma}(D) \leq \widetilde{\gamma}(\mathcal{T}^n D) \leq k\widetilde{\gamma}(D)$, i.e., $\widetilde{\gamma}(D) = 0$. For $r > 1$ we proceed by induction. Assume the statement is true up to $r - 1$, then, for any compact set $C \subseteq \mathcal{B}$, $[(\mathrm{id} - \mathcal{T})^r]^{-1}(C) = (\mathrm{id} - \mathcal{T})^{-1}[(\mathrm{id} - \mathcal{T})^{r-1}]^{-1}(C)$ is compact and $(\mathrm{id} - \mathcal{T})^r$ is proper. Q.E.D.

**Lemma A.25.** *Let $\mathcal{T} \in L(\mathcal{B}, \mathcal{C})$ be a bounded linear operator. Denote by $\mathcal{B}^*$ the dual space of the Banach space $\mathcal{B}$, that is the space of bounded linear forms $l \colon \mathcal{B} \to \mathbb{C}$. Let $\mathcal{T}^* \in L(\mathcal{C}^*, \mathcal{B}^*)$ be the dual map of $\mathcal{T}$, i.e.,$\mathcal{T}^*(l)(x) = l(\mathcal{T}(x))$ for any $l \in \mathcal{C}^*$ and for any $x \in \mathcal{B}$. Then $\gamma(\mathcal{T}^*) \leq \widetilde{\gamma}(\mathcal{T})$.*

Proof. We have to prove that for any bounded set $S \subseteq \mathcal{C}^*$, $\mathcal{T}^*(S)$ can be covered by finitely many bounded set of diameter less or equal than $\widetilde{\gamma}(\mathcal{T})\mathrm{diam}(S) + \epsilon$, for any $\epsilon > 0$. By definition, $\widetilde{\gamma}(\overline{\mathcal{B}(0,1)}) \leq 1$, hence $\mathcal{T}(\overline{\mathcal{B}(0,1)})$ can be covered by finitely many balls $B_1, \dots, B_n$ in $\mathcal{C}$ of radius $\widetilde{\gamma}(\mathcal{T}) + \frac{\epsilon}{2\mathrm{diam}(S)}$. Let $c_i \in \mathcal{C}$ be the center of $B_i$. We fix $M \in \mathbb{R}^+$ such that $\|c_i\| \leq M$ for any $i = 1, \dots, n$, and $\|c^*\| \leq M$ for any $c^* \in S$. Consequently, $|c^*(c_i)| < M^2$ for any $c^* \in S$ and any $i = 1, \dots, n$. Now, we subdivide the interval $[-M^2, M^2]$ into subintervals $\Delta_j$, $j = 1, \dots, p$, of length smaller than $\frac{\epsilon}{2}$ and we define the following equivalence relation: $c^* \sim d^*$ if and only if $c^*(c_i)$ and $d^*(c_i)$ belong to the same interval $\Delta_j(i)$. Let $\widetilde{S} = S/\sim$ be the quotient w.r.t. the relation $\sim$. We show that, for any $s \in S$, $\mathrm{diam}([s]) \leq \widetilde{\gamma}(\mathcal{T})\mathrm{diam}(S) + \epsilon$, where $[s] \in \widetilde{S}$ represents the equivalence class of $s \in S$. Consider $c_1^*, c_2^* \in [s]$. Then

$$\|\mathcal{T}^*(c_1^*) - \mathcal{T}^*(c_2^*)\| = \sup_{x \in \overline{\mathcal{B}(0,1)}} |c_1^*(\mathcal{T}(x)) - c_2^*(\mathcal{T}(x))| = \sup_{y \in \mathcal{T}(\overline{\mathcal{B}(0,1)})} |c_1^*(y) - c_2^*(y)|.$$

Assume that $y \in \overline{\mathcal{B}\left(c_i, \widetilde{\gamma}(\mathcal{T}) + \frac{\epsilon}{2\mathrm{diam}(S)}\right)}$, then

$$\begin{aligned}
&|c_1^*(y) - c_2^*(y)| \leq |c_1^*(y - c_i) - c_2^*(y - c_i)| + |c_1^*(c_i) - c_2^*(c_i)| = \\
=&|(c_1^* - c_2^*)||y - c_i| + |c_1^*(c_i) - c_2^*(c_i)| \leq \mathrm{diam}(S)\left(\widetilde{\gamma}(\mathcal{T}) + \frac{\epsilon}{2\mathrm{diam}(S)}\right) + \frac{\epsilon}{2} = \\
=&\mathrm{diam}(S)\widetilde{\gamma}(\mathcal{T}) + \epsilon.
\end{aligned}$$

This gives $\operatorname{diam}(\mathcal{T}^*(S_i)) \leq \operatorname{diam}(S)\widetilde{\gamma}(\mathcal{T}) + \epsilon$ and, since these sets cover $\mathcal{T}^*(S)$, we conclude that $\gamma(\mathcal{T}^*) < \widetilde{\gamma}(\mathcal{T}) + \frac{\operatorname{diam}(S)}{\epsilon}$ for any $\epsilon > 0$, that is $\gamma(\mathcal{T}^*) \leq \widetilde{\gamma}(\mathcal{T})$.

Q.E.D.

The following theorem gives an important formula to estimate the essential spectral radius.

**Theorem A.26** (Nussbaum formula [59])**.** *Let $\mathcal{T} \in L(\mathcal{B}, \mathcal{B})$ be a bounded linear operator. Then it holds*

$$\lim_{n\to+\infty} \gamma(\mathcal{T}^n)^{\frac{1}{n}} = \inf_n \gamma(\mathcal{T}^n)^{\frac{1}{n}} \leq \inf_n \widetilde{\gamma}(\mathcal{T}^n)^{\frac{1}{n}} = \lim_{n\to+\infty} \widetilde{\gamma}(\mathcal{T}^n)^{\frac{1}{n}}.$$

*Moreover, denoting by $\widetilde{\rho}_{\text{ess}} = \inf_{n\in\mathbb{N}} \widetilde{\gamma}(\mathcal{T}^n)^{\frac{1}{n}}$, if $|\lambda| > \widetilde{\rho}_{\text{ess}}$, then $\ker((\lambda\mathrm{id} - \mathcal{T})^r)$ is finite-dimensional, for any $r \in \mathbb{Z}^+$, $\operatorname{im}(\lambda\mathrm{id} - \mathcal{T})$ is closed and $\lambda$ is not a limit point of $\sigma(\mathcal{T}) \setminus \{\lambda\}$. By Definition A.12, we get $\rho_{\text{ess}} \leq \widetilde{\rho}_{\text{ess}}$.*

Proof. Let $\mathcal{T}, \mathcal{S} \in L(\mathcal{B}, \mathcal{B})$ be bounded operators. We firstly show that $\widetilde{\gamma}(\mathcal{T} \circ \mathcal{S}) \leq \widetilde{\gamma}(\mathcal{T})\widetilde{\gamma}(\mathcal{S})$, so that $\ln \widetilde{\gamma}(\cdot)$ is a subadditive function and the last limit exists, and equals $\widetilde{\rho}_{\text{ess}}$ by the same reason of the limit in Lemma A.9. By definition, for any bounded $B \subseteq \mathcal{B}$, $\widetilde{\gamma}(\mathcal{T}(B)) \leq \widetilde{\gamma}(\mathcal{T})\widetilde{\gamma}(B)$. Therefore $\widetilde{\gamma}(\mathcal{T} \circ \mathcal{S}(B)) \leq \widetilde{\gamma}(\mathcal{T})\widetilde{\gamma}(\mathcal{S}(B)) \leq \widetilde{\gamma}(\mathcal{T})\widetilde{\gamma}(\mathcal{S})\widetilde{\gamma}(B)$, hence $\widetilde{\gamma}(\cdot)$ is submultiplicative. The proof for $\gamma(\cdot)$ is exactly the same, while the inequality is a consequence of $\gamma(\cdot) \leq \widetilde{\gamma}(\cdot)$.

Let $|\lambda| > \widetilde{\rho}_{\text{ess}}$, then there exists $n$ such that $\widetilde{\gamma}(\mathcal{T}^n)^{\frac{1}{n}} < |\lambda|$. Accordingly $\widetilde{\gamma}(\mathcal{T}^n\lambda^{-n}) < 1$ and, by Lemma A.24, $(\mathrm{id} - \mathcal{T}\lambda^{-1})$ is proper on closed and bounded sets. By Lemma A.21, $\ker((\lambda\mathrm{id} - \mathcal{T})^r) = \ker((\mathrm{id} - \mathcal{T}\lambda^{-1})^r)$ is finite-dimensional and $\operatorname{im}(\lambda\mathrm{id} - \mathcal{T}) = \operatorname{im}(\mathrm{id} - \mathcal{T}\lambda^{-1})$ is closed.

Finally, we prove that $\lambda$ is an isolated point of $\sigma(\mathcal{T})$. In particular, we show that there is a neighborhood $U$ containing $\lambda$, such that $U \setminus \{\lambda\} \in \mathcal{R}(\mathcal{T})$. Since the resolvent is open, the result is trivial if $\lambda \in \mathcal{R}(\mathcal{T})$. Assume $\lambda \in \sigma(\mathcal{T})$. We claim that $\ker(\mathrm{id} - \mathcal{T}) \neq 0$ or $\ker(\mathrm{id} - \mathcal{T}^*) \neq 0$. Otherwise, the inverse function $(\lambda\mathrm{id} - \mathcal{T})^{-1} \colon \operatorname{im}(\lambda\mathrm{id} - \mathcal{T}) \to \mathcal{B}$ exists and the domain is closed. If we prove that $\operatorname{im}(\lambda\mathrm{id} - \mathcal{T}) = \mathcal{B}$, then $\lambda \in \mathcal{R}(\mathcal{T})$ and we gain a contradiction. On the other hand, if $\operatorname{im}(\lambda\mathrm{id} - \mathcal{T}) \neq \mathcal{B}$, then, by Lemma A.17, there exists $x \in \mathcal{B}$, $\|x\| = 1$ and $d(x, \operatorname{im}(\lambda\mathrm{id} - \mathcal{T})) > \frac{1}{2}$. Let $y \in \operatorname{span}\{x, \operatorname{im}(\lambda\mathrm{id} - \mathcal{T})\}$, hence $y = ax + q$, with $q \in \operatorname{im}(\lambda\mathrm{id} - \mathcal{T})$ and $a \in \mathbb{C}$. We define $l(y) = a$, so that

$$\|y\| = \|ax + q\| = |a|\|x - (-a^{-1}q)\| > \frac{|a|}{2} = \frac{l(y)}{2}.$$

Since, $|l(y)| \leq 2\|y\|$, by Hahn-Banach theorem we can extend $l$ to a linear form on $\mathcal{B}$. Consequently, $(\lambda\mathrm{id} - \mathcal{T}^*)l(z) = l((\lambda\mathrm{id} - \mathcal{T})(z)) = 0$, for any $z \in \mathcal{B}$. Thus, $l \in \ker(\lambda\mathrm{id} - \mathcal{T}^*)$ against the assumption. So, $\mathcal{B} = \mathrm{im}(\lambda\mathrm{id} - \mathcal{T})$ and $\lambda \in \mathcal{R}((T))$, which is a contradiction. To conclude we need to show that there is no sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\sigma(\mathcal{T}) \setminus \{\lambda\}$ which converges to $\lambda$. Assume that such a sequence exists. Then, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|\lambda_n - \lambda| < \epsilon$, for each $n > n_0$. By the claim, there are infinitely many nonzero $x_n \in \ker(\lambda_n\mathrm{id} - \mathcal{T})$ or infinitely many nonzero $l_n \in \ker(\lambda_n\mathrm{id} - \mathcal{T}^*)$. In the first case, we consider $X_k = \mathrm{span}\{x_{n_0+1}, \dots, x_{n_0+k}\}$ for any $k \in \mathbb{Z}^+$. $X_{k-1}$ is proper inside $X_k$, hence, by Lemma A.17, there exists $y_k \in X_k$ such that $d(y_k, X_{k-1}) \geq 1 - \epsilon$. In addition, $y_k = a_k x_{n_0+k} + w_{k-1}$, with $a_k \in \mathbb{C}$ and $w_{k-1} \in X_{k-1}$. For any $k, s, r \in \mathbb{Z}^+$, with $s > k$

$$
\begin{aligned}
&\|\mathcal{T}^r(y_s) - \mathcal{T}^r(y_k)\| = \|\mathcal{T}^r(a_s x_{n_0+s}) + \mathcal{T}^r(w_{s-1}) - \mathcal{T}^r(y_k)\| = \\
=&\|\lambda_s^n a_s x_{n_0+s} + \mathcal{T}^r(w_{s-1}) - \mathcal{T}^r(y_k)\| = |\lambda_s^n| \|y_s - (w_{s-1} - \lambda_s^{-n}(\mathcal{T}^r(w_{s-1} - y_k)))\| \geq \\
\geq&|\lambda_s^r|(1-\epsilon) = |\lambda_s^r - \lambda^r + \lambda^r|(1-\epsilon) = |\lambda^r| \left|1 + \frac{\lambda_s^r - \lambda^r}{\lambda^r}\right| (1-\epsilon) \geq \\
\geq&|\lambda^r| \left(1 - \left|\frac{\lambda_s - \lambda}{\lambda}\right|\right)^r (1-\epsilon) \geq |\lambda^r|(1-\epsilon)^{r+1}.
\end{aligned}
$$

The computation shows that we cannot cover $\mathcal{T}^r(\mathcal{B}(0,1))$ with finitely many sets of diameter $\frac{1}{4}|\lambda|^r(1-\epsilon)^r$, for any $r \in \mathbb{Z}^+$ and any $\epsilon > 0$. We have prove that $\widetilde{\gamma}(\mathcal{T}^r) \geq \gamma(\mathcal{T}^r) > \frac{1}{4}|\lambda|^r$. In the other case, the same proof gives $\gamma((\mathcal{T}^*)^r) > \frac{1}{4}|\lambda|^r$. Lemma A.25 implies that $\widetilde{\gamma}(\mathcal{T}^r) \geq \gamma((\mathcal{T}^*)^r) > \frac{1}{4}|\lambda|^r$, hence $\widetilde{\rho}_{\mathrm{ess}} > |\lambda|$ against the assumption. This proves that $\lambda$ is an isolated point in $\sigma(\mathcal{T})$. Q.E.D.

We now have all the ingredients to prove Hennion's theorem (see Theorem 2.9 for the statement).

Proof of Theorem 2.9. We want to apply Nussbaum formula (Theorem A.26), hence we need to compute $\widetilde{\gamma}(\mathcal{T}^n)$. Since $i$ is compact and $\mathcal{B}$ is bounded, we get that $\overline{i \circ \mathcal{T}(\mathcal{B}_{\|\cdot\|}(0,1))}$ is compact. Consequently, for any $\epsilon > 0$, there exists $x_1, \dots, x_m \in \mathcal{B}_{\|\cdot\|}(0,1)$, such that

$$
i \circ \mathcal{T}(\mathcal{B}_{\|\cdot\|}(0,1)) \subseteq \bigcup_{i=1}^{m} \mathcal{B}_{|\cdot|}(\mathcal{T}(x_i), \epsilon) \cap \mathcal{B}_{|\cdot|}(0, \|\mathcal{T}\|).
$$

For any $y \in \mathcal{B}_{|\cdot|}(\mathcal{T}(x_i), \epsilon) \cap \mathcal{B}_{|\cdot|}(0, \|T\|)$, then, by second assumption,

$$
\|\mathcal{T}^n(x_i) - \mathcal{T}^n(y)\| \leq r_n \|\mathcal{T}(x_i) - y\| + R_n |\mathcal{T}(x_i) - y| < 2r_n \|\mathcal{T}\| + 2R_n \epsilon.
$$

Accordingly, $\widetilde{\gamma}(\mathcal{T}^{n+1}) \leq 2r_n\|\mathcal{T}\|$, hence $\rho_{\text{ess}} \leq \liminf_n (r_n)^{\frac{1}{n}}$.

Q.E.D.

# Appendix B

# A toolbox of Hodge theory

In this appendix we recall some basic facts about Hodge $\star$ operator. References we suggest to the reader are [46, Chapter 2] and [22], where these concepts are deepened.

Let $M$ be an orientable, closed, Riemannian manifold equipped with the metric $g$, which induces a volume form $\omega_0$, as in Chapter 1. We assume to work with an oriented atlas $\{U_i, \psi_i\}$, with an orthonormal, positively oriented basis of vector fields $\{\partial_{x_1}, \ldots, \partial_{x_{\dim(M)}}\}$, and with the corresponding basis of differential forms (see Section 2.3).

**Definition B.1.** *The Hodge $\star$ operator is the unique linear map*

$$\star\colon \Omega^l(M) \to \Omega^{\dim(M)-l}(M)$$

*such that*

$$\langle \omega_1, \omega_2 \rangle \omega_0 = \omega_1 \wedge \star\omega_2, \ \textit{for } \omega_1, \omega_2 \in \Omega^l(M).$$

The existence of $\star\omega$ is a consequence of the nondegeneracy of the scalar product $\langle \cdot, \cdot \rangle$. In addition, $\star$ is unique, because if $*$ is another Hodge operator, than

$$\omega_1 \wedge (\star\omega_2 - *\omega_2) = \langle \omega_1, \omega_2 - \omega_2 \rangle \omega_0 = 0,$$

for any $\omega_1, \omega_2 \in \Omega^l(M)$. Thus, $\star\omega_2 = *\omega_2$.

**Remark B.2.** Notice that, using the orthonormal basis $\{dx_1, \ldots, dx_{\dim(M)}\}$ of $TM$, the Hodge operator can be written as

$$\star dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l} = dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{\dim(M)-l}}$$

where $dx_{i_1}, dx_{i_2}, \dots, dx_{i_l}, dx_{j_1} dx_{j_2}, \dots, dx_{j_{\dim(M)-l}}$ is a positively oriented. Accordingly, if $h = \sum_{\bar{j} \in \mathcal{J}_l} h_{\bar{j}} dx_{\bar{j}}$ on $\psi_i(U_i)$, then

$$\star h = \sum_{\bar{j} \in \mathcal{J}_l} h_{\bar{j}} \star dx_{\bar{j}}.$$

**Definition B.3.** *We define the linear operator* $\delta \colon \Omega^l(M) \to \Omega^{l-1}(M)$

$$\delta = (-1)^{\dim(M)(l+1)+1} \star d \star .$$

The following proposition put together some properties of the Hodge operator, the exterior derivative $d$ and the linear operator $\delta$.

**Lemma B.4.** *The Hodge* $\star$ *operator satisfies*

1. $\star \star |_{\Omega^l(M)} = (-1)^{l(\dim(M)-l)} \mathrm{id}|_{\Omega^l(M)}$;
2. $\omega_2 \wedge \star\omega_1 = (-1)^{l(d-l)} \star \omega_1 \wedge \omega_2 = \omega_1 \wedge \star\omega_2$;
3. $\delta$ *is the adjoint of* $d$ *w.r.t. the scalar product* $(\cdot, \cdot)$;
4. $\delta \circ \delta = 0$.

*Moreover, let* $f \colon M \to N$ *be a diffeomorphism of Riemannian manifolds. Let* $\omega_1 \in \omega^l(M)$ *and let* $\omega_2 \in \Omega^l(N)$, *then*

5. $\langle f_* \omega_1, \omega_2 \rangle = (-1)^l (\dim(M) - l) \langle \omega_1, \star f^* \star \omega_2 \rangle \circ f^{-1} \det(df^{-1})$

Proof. Let us prove *1.* Notice that $\star\star$ maps $\Omega^l(M)$ onto itself and suppose

$$\star dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l} = dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{\dim(M)-l}}.$$

Consequently,

$$\star\star\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l} = \star dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{\dim(M)-l}} = \epsilon dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l},$$

where $\epsilon$ is the sign. Since,

$$\begin{aligned} &dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l} \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{\dim(M)-l}} = \\ =&(-1)^{l(\dim(M)-l)} dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{\dim(M)-l}} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}, \end{aligned}$$

we conclude that $\epsilon = l(\dim(M) - l)$.

As a consequence of the definition, we obtain

$$\int_M \omega_1 \wedge \star\omega_2 = (\omega_1, \omega_2) = (\omega_2, \omega_1) = \int_M \omega_2 \wedge \star\omega_1 = (-1)^{l(d-l)} \int_M \star\omega_1 \wedge \omega_2.$$

Let us show that $\delta$ is the adjoint operator of $d$. For $\omega_1 \in \Omega^{l-1}(M)$ and $\omega_2 \in \Omega^l(M)$,

$$\begin{aligned} d(\omega_1 \wedge \star\omega_2) &= d\omega_1 \wedge \star\omega_2 + (-1)^{l-1}\omega_1 \wedge d \star \omega_2 = \\ =&d\omega_1 \wedge \star\omega_2 + (-1)^{l-1}(-1)^{-(l-1)(\dim(M)-l+1)}\omega_1 \wedge \star \star d \star \omega_2 = \\ =&d\omega_1 \wedge \star\omega_2 - (-1)^{d(l+1)+1}\omega_1 \wedge \star \star d \star \omega_2 \end{aligned}$$

By Stokes' theorem,

$$0 = \int_M d(\omega_1 \wedge \star\omega_2) = \int_M d\omega_1 \wedge \star\omega_2 - \int_M \omega_1 \wedge \star\delta\omega_2 = (d\omega_1, \omega_2) - (\omega_1, \delta\omega_2).$$

Trivially, $\delta \circ \delta = \star d \star \star d \star = \pm \star d \circ d \star = 0$ proves *4.*

Finally, we check point *5.* In fact, denoting by $\omega_M$, resp. $\omega_N$, the volume form induced by the metric on $M$, resp. $N$, we compute

$$\begin{aligned} \langle f_*\omega_1, \omega_2\rangle\omega_N &= f_*\omega_1 \wedge \star\omega_2 = f_*(\omega_1 \wedge f^* \star \omega_2) = (-1)^{l(\dim(M)-l)} f_*(\omega_1 \wedge \star \star f^* \star \omega_2) = \\ &= (-1)^{l(\dim(M)-l)} f_*(\langle \omega_1, \star f^* \star \omega_2\rangle)\omega_0 = (-1)^{l(\dim(M)-l)} \langle \omega_1, \star f^* \star \omega_2\rangle \circ f^{-1} \det(df^{-1})\omega_0. \end{aligned}$$

Q.E.D.

# Appendix C

# Technical results

In this section we collect some technicalities that we used in some proofs of this thesis, as well as some technical results.

**Lemma C.1** (Distortion Lemma). *Let $W \in \Sigma$ be an admissible leaf. Let $W_i$ be an admissible leaf, as in Lemma 2.1, such that $W_i \subseteq f^{-n}(W)$. Define $\lambda_n^s(x) = |\det(d_x f^n|_{T_x W_i})|$, that is the contraction of $f^n$ along the stable manifold $W_i$. Let $\lambda_{n,i}^s = \min_{x \in W_i} \lambda_n^s(x)$. Then there exists a constant $C > 0$ such that, for any $n \in \mathbb{N}$ and for each $x \in W_i$,*

$$\lambda_{n,i}^s \leq \lambda_n^s(x) \leq C\lambda_{n,i}^s.$$

*In addition, $\|\lambda_n^s\|_{C^q(W_i)} \leq |f^n(W_i)|$.*

Proof. We claim that there exists a constant $D > 0$ such that for any $n \in \mathbb{N}$ and for any $x, y \in W_i$

$$D^{-1} \leq \frac{\lambda_n^s(x)}{\lambda_n^s(y)} \leq D.$$

In fact, assuming that the claim is true, we get $\lambda_n^s(x) \leq D\lambda_n^s(y)$, for all $x, y \in W_i$, hence $\lambda_n^s(x) \leq D \min_{y \in W_i} \lambda_n^s(y) = D\lambda_{n,i}^s$.

Let us prove the claim. Denoting by $W_i^t = f^t(W_i)$, we obtain

$$\lambda_n^s(x) = |\det(d_x f^n|_{T_x W_i})| = \left|\det\left(\prod_{t=0}^{n-1} d_{f^t(x)} f|_{T_{f^t(x)} W_i^t}\right)\right| = \prod_{t=0}^{n-1} \left|\det\left(d_{f^t(x)} f|_{T_{f^t(x)} W_i^t}\right)\right|.$$

Consequently,

$$\ln\frac{\lambda_n^s(x)}{\lambda_n^s(y)} = \sum_{t=0}^{n-1}\ln\frac{\left|\det(d_{f^t(x)}f|_{T_{f^t(x)}W_i^t})\right|}{\det\left|d_{f^t(y)}f|_{T_{f^t(y)}W_i^t}\right|} = \sum_{t=0}^{n-1}\langle\nabla\ln\left|\det(d_{f^t(z)}f|_{T_{f^t(z)}W_i^t})\right|, x-y\rangle =$$

$$=\sum_{t=0}^{n-1}\langle(d_zf^t)^T\nabla\ln\left|\det(d_{f^t(z)}f|_{T_{f^t(z)}W_i^t})\right|, x-y\rangle =$$

$$=\sum_{t=0}^{n-1}\langle\nabla\ln\left|\det(d_{f^t(z)}f|_{T_{f^t(z)}W_i^t})\right|, d_zf^t(x-y)\rangle \leq$$

$$\leq\sum_{t=0}^{n-1}\underbrace{\max_{l\in W_i^t}\|\nabla\ln\left|\det(d_lf|_{T_lW_i^t})\right|\|\max_{x,y\in W_i}\|x-y\|}_{\leq C}\underbrace{\max_{z\in W_i}\|d_zf^t|_{T_xW_i}\|}_{C\lambda^{-t}} \leq$$

$$\leq\sum_{t=0}^{n-1}C\lambda^{-t}\leq C\frac{\lambda^{-1}}{1-\lambda^{-1}}\leq D.$$

We have proved $\|C^0(W_i)\|\leq C\lambda_{n,i}^s$. Notice that

$$|f^n(W_i)| = \int_{f^n(W_i)}\omega_W = \int_{W_i}\lambda_n^s(x)\omega_{W_i}(x)\geq\lambda_{n,i}^s|W_i|\geq C\lambda_{n,i}^s,$$

hence $\|\lambda_n^s\|_{C^0(W_i)}\leq C|f^n(W_i)|$. Let us consider derivative of $\lambda_n^s$. We compute

$$\partial_{x_j}\lambda_n^s(x) = \partial_{x_j}\left(\prod_{t=0}^{n-1}\left|\det(d_{f^t(x)}f|_{T_{f^t(x)}W_i^t})\right|\right) =$$

$$=\sum_{h=0}^{n-1}\prod_{t=0}^{n-1}\left|\det(d_{f^t(x)}f|_{T_{f^t(x)}W_i^t})\right|\frac{\partial_{x_j}\left|\det(d_{f^h(x)}f|_{T_{f^h(x)}W_i^h})\right|}{\left|\det(d_{f^h(x)}f|_{T_{f^h(x)}W_i^h})\right|} =$$

$$=\lambda_n^s(x)\sum_{h=0}^{n-1}\partial_{x_j}\ln\left(\left|\det(d_{f^h(x)}f|_{T_{f^h(x)}W_i^h})\right|\right) =$$

$$=\lambda_n^s(x)\sum_{h=0}^{n-1}\left\langle\nabla\ln\left(\left|\det(d_{f^h(x)}f|_{T_{f^h(x)}W_i^h})\right|\right), \partial_{x_j}f^h(x)|_{T_xW_i}\right\rangle \leq$$

$$\leq\lambda_n^s(x)\sum_{h=0}^{n-1}\max_{y\in W_i^h}\|\nabla\ln|\det(d_yf|_{T_yW_i^h})|\|\|d_xf^h|_{T_xW_i}\|\leq C\lambda_n^s(x)\sum_{h=0}^{n-1}\lambda^{-h}\leq C\lambda_n^s(x).$$

Accordingly, $\|\lambda_n^s\|_{C^1(W_i)}\leq C\|\lambda_n^s\|_{C^0(W_i)}\leq C|f^n(W_i)|$ and, iterating the reasoning, $\|\lambda_n^s\|_{C^q(W_i)}\leq C|f^n(W_i)|$.

Q.E.D.

**Lemma C.2.** *Let $W \in \Sigma$ be an admissible leaf in $\psi_z(U_z)$. Let $W_i$ be an admissible leaf, as in Lemma 2.1, such that $W_i \subseteq f^{-n}(W)$ in $\psi_i(U_i)$. Define $\lambda_n^s(x) = |\det(d_x f^n|_{T_x W_i})|$, that is the contraction of $f^n$ along the stable manifold $W_i$. Let $dx_{\overline{k}} \in \Omega^l(\psi_i(U_i))$ and $dx_{\overline{j}} \in \Omega^l(\psi_z(U_z))$, as defined in Section 2.3. Then,*

$$\|\langle dx_{\overline{j}}, f_*^n dx_{\overline{k}}\rangle \circ f^n \lambda_n^s\|_{C^q(W_i)} \leq C\lambda^{-|d_s - l|n}.$$

Proof. We firstly consider the case $l \leq d_s$. Let $\mathcal{S}_x^l$ be the family of $l$-dimensional vector subspaces of $\mathcal{C}_x^s$. For any $V_x \in \mathcal{S}_x^l$ and for any $n \in \mathbb{N}$, let us denote by $V_{f^n(x)} = d_x f^n V_x$. Then

$$\begin{gathered}|\langle dx_{\overline{j}}, f_*^n dx_{\overline{k}}\rangle \circ f^n(x)\lambda_n^s(x)| = |\langle dx_{\overline{j}}, f_*^n dx_{\overline{k}}\rangle \circ f^n(x)\det(d_x f^n|_{T_x W_i})| \leq \\ \leq \max_{V_x \in \mathcal{S}_x^l} |\det(d_{f^n(x)} f^{-n}|_{V_{f^n(x)}})\det(d_x f^n|_{T_x W_i})| = \max_{V_x \in \mathcal{S}_x^l}\left|\frac{\det(d_{f^n(x)} f^{-n}|_{V_{f^n(x)}})}{\det(d_{f^n(x)} f^{-n}|_{T_{f^n(x)} W})}\right|\end{gathered}$$

Notice that

$$\max_{V_x \in \mathcal{S}_x^l} |\det(d_{f^n(x)} f^{-n}|_{V_{f^n(x)}})| \leq \max_{V_x \in \mathcal{S}_x^{d_s}} |\det(d_{f^n(x)} f^{-n}|_{V_{f^n(x)}})|\lambda^{-(d_s-l)n},$$

hence

$$\max_{V_x \in \mathcal{S}_x^l}\left|\frac{\det(d_{f^n(x)} f^{-n}|_{V_{f^n(x)}})}{\det(d_{f^n(x)} f^{-n}|_{T_{f^n(x))W}})}\right| \leq \max_{V_x \in \mathcal{S}_x^{d_s}}\left|\frac{\det(d_{f^n(x)} f^{-n}|_{V_{f^n(x)}})}{\det(d_{f^n(x)} f^{-n}|_{T_{f^n(x)W}})}\right|\lambda^{-(d_s-l)n}$$

Since both $d_{f^n(x)} f^{-n}(V_{f^n(x)})$ and $d_{f^n(x)} f^{-n}(T_{f^n(x)W})$ converge to the stable subbundle as $n \to +\infty$, by continuity of the differential, there exist $\nu \in (0,1)$ and $\bar{n} \in \mathbb{N}$ such that, for $m > \bar{n}$,

$$|\det(d_{f^m(x)} f^{-1}|_{V_{f^m(x)}})| - |\det(d_{f^m(x)} f^{-1}|_{T_{f^m(x)} f^m(W_i)})| < \nu < 1,$$

hence

$$|\det(d_{f^n(x)} f^{-n}|_{V_{f^n(x)}})| - |\det(d_{f^n(x)} f^{-n}|_{T_{f^n(x)} W})| < C\nu^{n-\bar{n}} \leq C$$

We obtain that

$$\max_{V_x \in \mathcal{S}_x^{d_s}}\left|\frac{\det(d_{f^n(x)} f^{-n}|_{V_{f^n(x)}})}{\det(d_{f^n(x)} f^{-n}|_{T_{f^n(x)W}})}\right| \leq 1 + \frac{1}{|\det(d_{f^n(x)} f^{-n}|_{T_{f^n(x)W}})|} \leq 1 + \frac{C}{\lambda^{nd_s}} \leq C$$

We conclude that

$$\|\langle dx_{\overline{j}}, f_*^n dx_{\overline{k}}\rangle \circ f^n(x)\lambda_n^s(x)\|_{C^0(W_i)} \leq C\lambda^{-(d_s-l)n}$$

Let us compute derivatives. W.l.o.g. we can assume that $\partial_{x_j} \in T_x W_i$. Then

$$\begin{aligned}\partial_{x_j}(|\det(d_{f^n(x)}f^{-n}|_{V_{f^n(x)}})\lambda_n^s(x)|) = &\,\partial_{x_j}(|\det(d_{f^n(x)}f^{-n}|_{V_{f^n(x)}})\|)\lambda_n^s(x)+ \\ &+|\det(d_{f^n(x)}f^{-n}|_{V_{f^n(x)}})\|\,\partial_{x_j}\,\lambda_n^s(x)\end{aligned}$$

Since $\partial_{x_j}\,\lambda_n^s(x)$ (see the proof of Lemma C.1), the second term of the sum can be estimated by the $C^0$-norm. Similarly, one can repeat the argument of the proof of Lemma C.1 to prove that

$$\partial_{x_j}\,|\det(d_{f^n(x)}f^{-n}|_{V_{f^n(x)}})| \leq C|\det(d_{f^n(x)f^{-n}|_{V_{f^n(x)}}})|$$

Thus, also the first term of the sum can be estimated by the $C^0$-norm. By iterating that procedure one obtains the estimate for the $C^q$-norm. It remains to prove the case $l > d_s$. Let $V_x$ be an $l$-dimensional subspace of $T_xM$. The worst estimate for $|\det(d_{f^n(x)}f^{-n}|_{V_{f^n(x)}})|$ is given by the case for which $V_x = S_x \oplus U_x$, where $S_x$ is a $d_s$-dimensional subspace in the stable cone and $U_x$ is a $(l-d_s)$-dimensional unstable subspace. Since $|\det(d_{f^n(x)}f^{-n}|_{U_{f^n(x)}})| \leq \lambda^{-(l-d_s)n}$, we obtain again

$$|\langle dx_{\overline{j}}, f_*^n dx_{\overline{k}}\rangle \circ f^n(x)\lambda_n^s(x)| \leq \max_{S_x \in \mathcal{S}_x^{d_s}} \left|\frac{\det(d_{f^n(x)}f^{-n}|_{S_{f^n(x)}})}{\det(d_{f^n(x)}f^{-n}|_{T_{f^n(x)}W})}\right| \lambda^{-(l-d_s)n}$$

and we can conclude as above. Q.E.D.

We now prove two generalizations of Poincaré's lemma that we used in Section 3.3 to get the isomorphism between the anisotropic de Rham cohomology and the Čech cohomology.

Proof of Lemma 3.23. The first part the lemma informally says that closed 0-currents are constant and it generalizes [20, Lemma 5.15] to higher-dimensional manifolds. Let $h \in \mathcal{C}^{p,q,0}$ be a closed current of degree 0 (and dimension $\dim(M)$). By Lemma 2.8, $\mathcal{B}^{p,q,0}$ can be identified with a subspace the dual space of $C^{p+q}$ function on $M$, i.e., a space of distribution, and the behavior of $h$, acting on $\phi \in C^{p+q}$ as a current, is obtained disintegrating the integral

$$i(h)(\phi) = \int_M h\phi\omega_0$$

along leaves of $\Sigma$. The same holds true for $\mathcal{C}^{p,q,0}$, which is a subspace of $\mathcal{B}^{p,q,0}$. Let $\phi \in C^{p+q}(M)$ such that $supp(\phi) \subseteq int(supp(\phi_k))$, where we recall that $\{\phi_k\}_{k=1}^m$ is the partition of unity subordinated to the contractible open covering $\{V_k = \psi_k(U_k)\}$. Then

$$i(\partial_{x_i} h)(\phi) = \int_M \phi\, \partial_{x_i} h\omega_0 = \int_{a_1}^{b_1} \dots \int_{a_{d_u}}^{b_{d_u}} \int_{W_{t_1,\dots t_{d_u}}} \phi\, \partial_{x_i} h f_{t_1,\dots,t_{d_u}} \omega_{t_1,\dots,t_{d_u}} dt_1 \dots dt_{d_u}$$

where we disintegrated the integral along $d_s$-dimensional "stable" leaves $W_{t_1,\dots t_{d_u}} \in \Sigma$ depending on $d_u$ parameters $t_1, \dots t_{d_u}$. Notice that, since leaves are smooth, the Jacobian $f_{t_1,\dots,t_{d_u}}$ is a smooth function. Consequently,

$$i(\partial_{x_i} h)(\phi) = \int_{a_1}^{b_1} \dots \int_{a_{d_u}}^{b_{d_u}} \int_{W_{t_1,\dots t_{d_u}}} \langle f_{t_1,\dots,t_{d_u}} \phi_k^{-1} \phi dx_i, \phi_k dh\rangle \omega_{t_1,\dots,t_{d_u}} dt_1 \dots dt_{d_u} = 0.$$

This implies that $\partial_{x_i} h = 0$ as a distribution on the interior of *supp*$(\phi_k)$. Since, by assumption, *supp*$(\phi_k)$ is simply connected, hence connected, there exists $c_k \in \mathbb{C}$ such that $\phi_k \cdot (h - c_k) = 0$ in $(C^{p+q})^*$. Thus, $\phi_k \cdot (h - c_k) = 0$ in $\mathcal{C}^{p,q,0}$, because the inclusion $i' \colon \mathcal{C}^{p,q,0} \to \mathcal{B}^{p,q,0}$ and the map $i \colon \mathcal{B}^{p,q,0} \to (C^{p+q})^*$ are injective (see Lemma 2.8).

Let us prove the second part of the lemma. We firstly introduce an enlarged partition of unity $\{\bar{\phi}_k\}_{k=1}^m$ such that *supp*$(\phi_k) \subseteq$ *supp*$(\phi_k^+) \subseteq V_k$ and $\phi_k^+ = 1$ on *supp*$(\phi_k)$. Next consider a differential form $\omega \in \Omega^l(M)$ such that $d\omega\phi_k = 0$ for some $k \in \{1, \dots, m\}$. We fix $x_k \in V_k \setminus$ *supp*$(\phi_k^+)$ and we define the linear operator $\alpha_k \colon \Omega^l(M) \to \Omega^{l-1}(M)$ such that

$$\alpha_k(\omega)_x = \psi_{k*}\left(\int_0^1 t^{l-1} (\iota_{\psi_k^{-1}(x) - \psi_k^{-1}(x_k)} \psi_k^* \omega)_{\psi^{-1}(x_k)(1-t) + \psi_k^{-1}(x)t} dt\right),$$

where $\iota_v \omega$ denotes the interior product of $\omega$ with a vector field $v$. Next, denoting by $y_k = (r_1^{(k)}, \dots, r_{\dim(M)}^{(k)}) = \psi_k^{-1}(x_k)$ and $y = (r_1, \dots, r_{\dim(M)}) = \psi_k^{-1}(x)$, we can compute

$$((d\alpha_k + \alpha_k d)\omega)_x = \psi_{k*}\left(\int_0^1 t^{l-1} ((d\iota_{y-y_k} + \iota_{y-y_k} d)\psi_k^* \omega)_{y_k(1-t)+yt} dt\right) =$$
$$= \psi_{k*}\left(\int_0^1 t^{l-1} (L_{y-y_k} \psi_k^* \omega)_{y_k(1-t)+yt} dt\right) = (*)$$

where, in the second line, we used Cartan's magic formula [57, Theorem 2.11] which states that, given a smooth vector field $X$, $L_X = d\iota_X + \iota_X d$. Let $\Phi_s(y) = y_k + e^s(y - y_k)$, so that $\Phi_0(y) = y$ and $\frac{d}{ds}|_{s=0} \Phi_s(y) = y - y_k$. Writing $v_k$ in place of $\psi_k^* \omega$ and assuming that $(v_k)_z = \sum_{\overline{j}\in\mathcal{J}^l} f_{\overline{j}}(z)dr_{\overline{j}}$ in coordinates, we get $(\Phi_s^* v_k)_z = e^{ls}(v_k)_{\Phi_s(z)}$ and

$$(L_{y-y_k}v_k)_z = \frac{d}{ds}\bigg|_{s=0}(\Phi_s^* v_k)_z = l(v_k)_z + \sum_{i=1}^{\dim(M)}(z-y_k)_i\frac{\partial f_{\overline{j}}}{\partial r_i}(z)dr_{\overline{j}},$$

where the first equality is the definition of the Lie derivative. Consequently,

$$\begin{aligned} t^{l-1}&(L_{y-y_k}v_k)_{y_k(1-t)+yt} = \\ &= t^{l-1}\left(l(v_k)_{y_k(1-t)+yt} + \sum_{\overline{j}\in\mathcal{J}^l}\sum_{i=1}^{\dim(M)} t(y-y_k)_i\frac{\partial f_{\overline{j}}}{\partial r_i}(y_k(1-t)+yt)dr_{\overline{j}}\right) = \\ &= \left(lt^{l-1}(v_k)_{y_k(1-t)+yt} + t^l\sum_{\overline{j}\in\mathcal{J}^l}\langle\nabla f_{\overline{j}}(y_k(1-t)+yt), y-y_k\rangle dr_{\overline{j}}\right) \\ &= \frac{d}{dt}\left(t^l(v_k)_{y_k(1-t)+yt}\right)\end{aligned}$$

hence

$$(*) = \psi_{k*}\int_0^1\frac{d}{dt}\left(t^l\psi_k^*\omega_{y_k(1-t)+yt}\right)dt = \omega_x.$$

We have thus proved that $(d\alpha_k + \alpha_k d)\omega = \omega$. Accordingly, if $\omega$ is a $l$-form such that $d\omega\phi_k = 0$, then we get $d\alpha_k(\omega)\phi_k = \omega\phi_k - \alpha_k(d\omega)\phi_k = \omega\phi_k$, that is we have just proved Poincaré's lemma for differential forms. Once we have the result for forms, we could try to extend $\alpha_k$ to a bounded linear operator from $\mathcal{C}^{p,q,l}$ to $\mathcal{C}^{p,q,l-1}$. Unfortunately, $\alpha_k(\omega)$, for a differential form $\omega$, is not defined on every admissible stable leaf of $W \in \Sigma$. In fact, we can just consider leaves inside $\psi_k(U_k)$. On the other hand, the product $\alpha_k(\omega)\phi_k^+$ is well defined on the full manifold, because it is null out of $supp(\phi_k^+)$. Accordingly, we consider the operator $\beta_k\colon \Omega^l(M)\to\Omega^{l-1}(M)$, such that $\beta_k(\omega) = \alpha_k(\omega)\phi_k^+$, and we want to prove that it can be extended to a bounded linear operator from $\mathcal{C}^{p,q,l}$ to $\mathcal{C}^{p,q,l-1}$. Let $\omega \in \Omega^l(M)$, and $\psi_k \circ G_{p,F}(\mathcal{B}_{d_s}(0,\rho)) = W \in \Sigma$, with $p \in \mathcal{B}(0,\rho)$ and $F \in \mathcal{F}$, as defined in Section 2.2. Let $\phi \in \Gamma_0^{q,l-1}(W)$ be a test form. Then, denoting again by $y = \psi_k^{-1}(x)$, $y_k = \psi_k^{-1}(x_k)$, $v_k = \psi_k^*\omega$, and $\chi_k = \psi_k^*\phi$

$$\begin{aligned}\int_W\langle\phi_x, \alpha_k(\omega)_x\rangle\phi_k^+(x)\omega_W(x) &= \int_W\langle\phi_x, \psi_{k*}\left(\int_0^1 t^{l-1}(\iota_{y-y_k}v_k)_{y_k(1-t)+yt}dt\right)_x\rangle\phi_k^+(x)\omega_W(x) = \\ =\psi_{k*}&\left(\int_{\psi_k^{-1}W}\langle(\chi_k)_y, \int_0^1 t^{l-1}(\iota_{y-y_k}v_k)_{y_k(1-t)+yt}dt\rangle\phi_k^+(\psi_k(y))\psi_k^*\omega_W(y)\right) = (\star)\end{aligned}$$

Setting $l_t(y) = y_k(1-t)+yt$, we have $t^{l-1}(i_{y-y_k}v_k)_{l_t(y)} = (l_{t*}i_{y-y_k}v_k)_y$ and $(\chi_k)_y =$

$(l_t^* l_{t*}\chi_k)_y = t^{l-1}(l_{t*}\chi_k)_{l_t(y)}$. Thus,

$$
\begin{aligned}
(\star) =&\psi_{k*}\left(\int_{\psi_k^{-1}W}\int_0^1 \langle(\chi_k)_y, (l_t^*\iota_{y-y_k}v_k)_y\rangle \phi_k^+(\psi_k(y))dt\psi_k^*\omega_W(y)\right) = \\
=&\psi_{k*}\left(\int_{\psi_k^{-1}W}\int_0^1 t^{l-1}\langle l_{t*}(\chi_k)_{l_t(y)}, (\iota_{y-y_k}v_k)_{l_t(y)}\rangle \phi_k^+(\psi_k(y))dt\psi_k^*\omega_W(y)\right) = \\
=&\sum_{i=1}^{\dim(M)}\psi_{k*}\left(\int_{\psi_k^{-1}W}\int_0^1 t^{l-1}\langle l_{t*}\chi_k\wedge dr_i, v_k\rangle_{l_t(y)}(y-y_k)_i\phi_k^+(\psi_k(y))dt\psi_k^*\omega_W(y)\right) = (\diamond)
\end{aligned}
$$

Next, we compute

$$
\begin{aligned}
&\int_{\psi_k^{-1}W}\int_0^1 t^{l-1}\langle l_{t*}\chi_k\wedge dr_i, v_k\rangle_{l_t(y)}(y-y_k)_i\phi_k^+(\psi_k(y))dt\psi_k^*\omega_W(y) = \\
=&\int_{\mathcal{B}_{d_s}(0,\rho)}\int_0^1 t^{l-1}\langle l_{t*}\chi_k\wedge dr_i, v_k\rangle_{l_t(G_{p,F}(s))}(G_{p,F}(s)-y_k)_i G_{p,F}^*\psi_k^*\phi_k^+(s)dtG_{p,F}^*\psi_k^*\omega_W(s) = \\
=&\int_0^1\int_{\mathcal{B}_{d_s}(0,t\rho)} t^{l-1-d_s}\langle l_{t*}\chi_k\wedge dr_i, v_k\rangle_{l_t((G_{p,F}(t^{-1}z)))}(G_{p,F}(t^{-1}z)-y_k)_i G_{p,F}^*\psi_k^*\phi_k^+(t^{-1}z)G_{p,F}^*\psi_k^*\omega_W(z)dt
\end{aligned}
$$

Let $q_t = y_k(1-t)+tp$ and $F_t\colon \mathcal{B}_{d_s}(0,t\rho)\to\mathcal{B}_{d_u}(0,\rho)$ such that $F_t(z) = F(t^{-1}z)$. Then $q_t\in\mathcal{B}(0,\rho)$ and $F_t$ can be extended to a function $\bar{F}_t\in\mathcal{F}$. As a consequence, setting $W_t = \psi_k\circ G_{q_t,F_t}(\mathcal{B}_{d_s}(0,t\rho))\subseteq \widehat{W}_t = \psi_k\circ G_{q_t,F_t}(\mathcal{B}_{d_s}(0,\rho))\in\Sigma$ and

$$
(\bar{\phi}_t)_x = \sum_{i=1}^{\dim(M)}(\psi_{k*}l_{t*}\chi_k\wedge dx_i)_x(\psi_k^{-1}(x)-y_k)_i\phi_k^+(x), \tag{C.1}
$$

we obtain

$$
(\diamond) = \int_0^1 t^{l-1-d_s}\int_{W_t}\langle(\bar{\phi}_t)_x,\omega_x\rangle\omega_{W_t}(x)dt.
$$

In summary, we have shown that

$$
\left|\int_W\langle\phi_x,\alpha_k(\omega)_x\rangle\phi_k^+(x)\omega_W(x)\right| \le C\int_0^1 t^{l-1-d_s}\|\bar{\phi}_t\|_{\Gamma_0^{q,l}}\|\omega\|_{0,q,l}|W_t|dt
$$

One can easily check, using (C.1), that $\|\bar{\phi}_t\|_{\Gamma_0^{q,l}}\le C\|\phi\|_{\Gamma_0^{q,l-1}}$. In addition, $|W_t|\le Ct^{d_s}|\widehat{W}_t|\le Ct^{d_s}$, hence we conclude that

$$
\begin{aligned}
\left|\int_W\langle\phi_x,\alpha_k(\omega)_x\rangle\phi_k^+(x)\omega_W(x)\right| &\le C\int_0^1 t^{l-1}\|\phi\|_{\Gamma_0^{q,l-1}}\|\omega\|_{0,q,l}dt \\
&\le C\|\phi\|_{\Gamma_0^{q,l-1}}\|\omega\|_{0,q,l},
\end{aligned} \tag{C.2}
$$

i.e., $\|\beta_k(\omega)\|_{0,q,l-1} \le C\|\omega\|_{0,q,l}$ and $\beta_k\colon \mathcal{B}^{0,q,l} \to \mathcal{B}^{0,q,l-1}$ is a bounded linear operator. Similarly, let $\gamma_k\colon \Omega^l(M) \to \Omega^l(M)$, such that $\gamma_k(\omega) = \alpha_k(\omega) \wedge d\phi_k^+$. Then, by the same proof, one can easily check that it can be extended to a bounded linear operator $\gamma_k\colon \mathcal{B}^{0,q,l} \to \mathcal{B}^{0,q,l}$. Next, we compute

$$\begin{aligned}\int_W \langle \phi, d\beta_k(\omega)\rangle \omega_W &= \int_W \langle \phi, d\alpha_k(\omega)\rangle \phi_k^+ \omega_W + (-1)^l \int_W \langle \phi, \alpha_k(\omega) \wedge d\phi_k^+\rangle \omega_W = \\ &= \int_W \langle \phi, \omega\rangle \phi_k^+ \omega_W - \int_W \langle \phi, \alpha_k(d\omega)\rangle \phi_k^+ \omega_W + (-1)^l \int_W \langle \phi, \alpha_k(\omega) \wedge d\phi_k^+\rangle \omega_W,\end{aligned} \tag{C.3}$$

hence

$$\begin{aligned}\left|\int_W \langle \phi, d\beta_k(\omega)\rangle \omega_W\right| &\le C\|\phi\|_{\Gamma_0^{q,l}}\|\omega\|_{0,q,l} + C\|\phi\|_{\Gamma_0^{q,l}}\|\beta_k(d\omega)\|_{0,q,l} + C\|\phi\|_{\Gamma_0^{q,l}}\|\gamma_k(d\omega)\|_{0,q,l+1} \\ &\le C\|\phi\|_{\Gamma_0^{q,l}}\|\omega\|_{0,q,l} + C\|\phi\|_{\Gamma_0^{q,l}}\|d\omega\|_{0,q,l+1},\end{aligned} \tag{C.4}$$

where, in the last inequality, we used the continuity of $\beta_k\colon \mathcal{B}^{0,q,l+1} \to \mathcal{B}^{0,q,l}$ and $\gamma_k\colon \mathcal{B}^{0,q,l+1} \to \mathcal{B}^{0,q,l+1}$. From (C.2) and (C.4) we get

$$|\beta_k(\omega)|_{0,q,l-1} = \|\beta_k(\omega)\|_{0,q,l-1} + \|d\beta_k(\omega)\|_{0,q,l} \le C\|\omega\|_{0,q,l} + C\|d\omega\|_{0,q,l+1} \le C|\omega|_{0,q,l},$$

that is $\beta_k\colon \mathcal{C}^{0,q,l} \to \mathcal{C}^{0,q,l-1}$ is continuous. Since $\omega$ is closed, $d\gamma_k(\omega) = \omega \wedge d\phi_k^+$ and $\|\gamma_k(\omega)\|_{0,q,l} \le C\|\omega\|_{0,q,l}$. Therefore, $|\gamma_k(\omega)|_{0,q,l} \le C|\omega|_{0,q,l}$ and also $\gamma_k\colon \mathcal{C}^{0,q,l} \to \mathcal{C}^{0,q,l}$ is bounded.

We now want to prove that $\beta_k\colon \mathcal{C}^{1,q,l} \to \mathcal{C}^{1,q,l-1}$ is still a bounded linear operator. Therefore, let $\omega \in \Omega^l(M)$, $W \in \Sigma$, $\phi \in \Gamma_0^{q+1,l-1}(W)$ and $v \in \mathcal{V}^{q+1}(U(W))$. We firstly show that $\iota_v\alpha_k(\omega) = -\alpha_k(\iota_v\omega)$. In fact, once we have this preliminary result, we get $L_v\alpha_k(\omega) = (d\iota_v + \iota_v d)\alpha_k(\omega) = -d\alpha_k(\iota_v\omega) + \iota_v\omega - \iota_v\alpha_k(d\omega) = -\iota_v\omega + \alpha_k(d\iota_v\omega) + \iota_v\omega + \alpha_k(\iota_v d\omega) = \alpha_k(L_v\omega)$. With a slight abuse of notation, in order to simplify notations, we confuse $\omega$, $x$, $x_k$ and $v$, with $\psi_{k*}\omega$, $\psi_k(x)$, $\psi_k(x_k)$ and $\psi_{k*}v$, respectively. Accordingly, we compute

$$\begin{aligned}\iota_v\alpha_k(\omega) =& \iota_v\left(\int_0^1 t^{l-1}(\iota_{x-x_k}\omega)_{x_k(1-t)+xt}dt\right) = \int_0^1 \iota_v(l_t^*\iota_{x-x_k}\omega)_x dt \\ &= \int_0^1 (l_t^*\iota_{l_{t*}v}\iota_{x-x_k}\omega)_x dt = \int_0^1 t^{-1}(l_t^*\iota_v\iota_{x-x_k}\omega)_x dt = \\ &= -\int_0^1 t^{-1}(l_t^*\iota_{x-x_k}\iota_v\omega)_x dt = -\int_0^1 t^{l-2}\iota_{x-x_k}(\iota_v\omega)_{x_k(1-t)+tx}dt = -\alpha_k(\iota_v\omega).\end{aligned}$$

As a consequence,

$$\begin{aligned}\int_W \langle \phi, L_v \beta_k(\omega)\rangle \omega_W &= \int_W \langle \phi, L_v \alpha_k(\omega)\rangle \phi_k^+ \omega_W + \int_W \langle \phi, \alpha_k(\omega) v \phi_k^+ \rangle \omega_W = \\ &= \int_W \langle \phi, \alpha_k(L_v\omega)\rangle \phi_k^+ \omega_W + \int_W \langle \phi, \alpha_k(\omega) v \phi_k^+ \rangle \omega_W \\ &= \int_W \langle \phi, \beta_k(L_v\omega)\rangle \omega_W + \int_W \langle \phi, \alpha_k(\omega) v \phi_k^+ \rangle \omega_W, ,\end{aligned}$$

hence

$$\begin{aligned}\left| \int_W \langle \phi, L_v \beta_k(\omega)\rangle \omega_W \right| \leq & C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|\beta_k(L_v\omega)\|_{0,q+1,l-1} + C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|v\|_{C^{q+1}} \|\omega\|_{0,q+1,l} \leq \\ \leq & C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|v\|_{C^{q+1}} \|\omega\|_{1,q,l} + C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|v\|_{C^{q+1}} \|\omega\|_{0,q+1,l},\end{aligned}$$

which proves that $\|\beta_k(\omega)\|_{1,q,l-1} \leq C\|\omega\|_{1,q,l}$. By the same procedure of (C.3) we obtain

$$\begin{aligned}\int_W \langle \phi, L_v d\beta_k(\omega)\rangle \omega_W = & \int_W \langle \phi, L_v \omega\rangle \phi_k^+ \omega_W - \int_W \langle \phi, \alpha_k(L_v d\omega)\rangle \phi_k^+ \omega_W + \\ & +(-1)^l \int_W \langle \phi, \alpha_k(L_v\omega) \wedge d\phi_k^+ \rangle \omega_W,\end{aligned}$$

hence

$$\begin{aligned}\left| \int_W \langle \phi, L_v d\beta_k(\omega)\rangle \omega_W \right| \leq & C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|L_v\omega\|_{0,q+1,l} + C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|\beta_k(L_v d\omega)\|_{0,q+1,l} + \\ & + C\|\phi\|_{\Gamma_0^{q+1,,l-1}} \|\gamma_k(L_v\omega)\|_{0,q+1,l} \leq C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|v\|_{C^{q+1}} \|\omega\|_{1,q,l} + \\ & + C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|v\|_{C^{q+1}} \|d\omega\|_{1,q,l+1} + C\|\phi\|_{\Gamma_0^{q+1,l-1}} \|v\|_{C^{q+1}} \|\omega\|_{1,q,l},\end{aligned}$$

i.e $\|d\beta_k(\omega)\|_{1,q,l} \leq C\|\omega\|_{1,q,l} + C\|d\omega\|_{1,q,l+1}$. We conclude that

$$|\beta_k(\omega)|_{1,q,l-1} = \|\beta_k(\omega)\|_{1,q,l-1} + \|d\beta_k(\omega)\|_{1,q,l} \leq C\|\omega\|_{1,q,l} + C\|d\omega\|_{1,q,l+1} \leq C|\omega|_{1,q,l},$$

so that $\beta_k \colon \mathcal{C}^{1,q,l} \to \mathcal{C}^{1,q,l-1}$ is a bounded linear operator. A similar computation show that $\gamma_k \colon \mathcal{C}^{1,q,l} \to \mathcal{C}^{1,q,l}$ is a continuous operator. By the same argument $\beta_k$ and $\gamma_k$ extend to bounded linear operators $\beta_k \colon \mathcal{C}^{p,q,l} \to \mathcal{C}^{p,q,l-1}$ and $\gamma_k \colon \mathcal{C}^{p,q,l} \to \mathcal{C}^{p,q,l}$. Finally, since $\phi_k^+ \phi_k = \phi_k$, we obtain

$$\begin{aligned}d(\beta_k(\omega)\phi_k) = & d\beta_k(\omega)\phi_k + (-1)^{l-1}\beta_k(\omega) \wedge d\phi_k = \\ = & \omega\phi_k^+\phi_k + (-1)^{l-1}\alpha_k(\omega) \wedge d\phi_k^+ \phi_k + (-1)^{l-1}\beta_k(\omega) \wedge d\phi_k = \\ = & \omega\phi_k + (-1)^{l-1}\beta_k(\omega) \wedge d\phi_k,\end{aligned}$$

hence, setting $u_k = \beta_k(\omega)$ we get the lemma.

Q.E.D.

PROOF OF LEMMA 3.25. Let $h \in \mathcal{C}^{p,q,l,0}$ such that $\bar{\delta}h = 0$. Then $0 = \bar{\delta}h(a_0, a_1) = h(a_1) - h(a_0)$, i.e., $h(a_0) = h(a_1)$, for all $a_0, a_1 \in \{1, \dots, m\}$. Accordingly, defining $\omega = h(a_0) \in \mathcal{C}^{p,q,l}$, we get $i(\omega) = h$.

Next, we consider $\omega \in \mathcal{C}^{p,q,l,k}$, for some $k > 0$, such that $\bar{\delta}\omega = 0$. We define the operator $\Xi\colon \mathcal{C}^{p,q,l,k} \to \mathcal{C}^{p,q,l,k-1}$ such that

$$\Xi(\omega)(a_0, \dots, a_{k-1}) = \sum_{j=1}^{m} \phi_j \omega(j, a_0, \dots, a_{k-1}),$$

where $\{\phi_j\}_{j=1}^m$ is the partition of unity subordinated to the cover $\{V_j = \psi_j(U_j)\}_{j=1}^m$ that we used to define Čech cohomology. We show that $\bar{\delta} \circ \Xi + \Xi \circ \bar{\delta} = \mathrm{id}$. In effect,

$$\begin{aligned}\bar{\delta} \circ \Xi\omega(a_0, \dots, a_k) =& \sum_{i=0}^{k} (-1)^i \Xi\omega(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_k) = \\ =& \sum_{i=0}^{k} (-1)^i \sum_{j=1}^{m} \phi_j \omega(j, a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_k),\end{aligned}$$

while

$$\begin{aligned}\Xi \circ \bar{\delta}\omega(a_0, \dots, a_k) =& \sum_{j=1}^{m} \phi_j \bar{\delta}\omega(j, a_0, \dots, a_k) = \sum_{j=1}^{m} \phi_j \omega(a_0, \dots, a_k) + \\ &+ \sum_{i=0}^{k} (-1)^{i+1} \sum_{j=1}^{m} \phi_j \omega(j, a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_k) = \\ =& (\omega - \bar{\delta} \circ \Xi\omega)(a_0, \dots, a_k).\end{aligned}$$

Accordingly, if $\bar{\delta}(\omega) = 0$, defining $u = \Xi\omega$, we get

$$\bar{\delta}u = \bar{\delta} \circ \Xi\omega = \omega - \Xi \circ \bar{\delta}\omega = \omega.$$

Q.E.D.

# Appendix D

# A minimal introduction to entropy

For the sake of completeness, in this appendix we recall the key aspects of entropy of dynamical systems used in this thesis. We underline that this short survey is far to be a complete treatment of the topic, which can be found for instance in [47, 17, 71].

## D.1 Measure-theoretic entropy

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{P} = \{P_i\}_{i\in I}$ be finite or countable measurable partition. This means that every $P_i \in \mathcal{B}$, $\mu(X \setminus \bigcup_{i\in I} P_i) = 0$ and $\mu(P_i \cap P_j) = 0$, for any couple of distinct indexes $i, j \in I$. The entropy of this partition is

$$H(\mathcal{P}) = -\sum_{i\in I} \mu(P_i) \log \mu(P_i) \in [0, +\infty]$$

with the convention that $0 \log 0 = 0$. A finite or countable measurable partition $\mathcal{R}$ is finer than $\mathcal{P}$, or equivalently $\mathcal{P}$ is coarser than $\mathcal{R}$, if for any $R \in \mathcal{R}$ there exists $P \in \mathcal{P}$ such that $R \subseteq P \mod (\mu)$. Given another finite or countable measurable partition $\mathcal{Q} = \{Q_j\}_{j\in J}$, the joint partition is

$$\mathcal{P} \vee \mathcal{Q} = \{P_i \cap Q_j |\ i \in I,\ j \in J\}$$

Let $T\colon X \to X$ be a measure preserving transformations, i.e., a map which satisfies $\mu(T^{-1}(B)) = \mu(B)$, for any $B \in \mathcal{B}$. Given a finite or countable measurable partition $\mathcal{P}$ of $X$, we define

$$\mathcal{P}_n^T = \mathcal{P} \vee T^{-1}(\mathcal{P}) \vee \cdots \vee T^{-n}(\mathcal{P}),$$

where $T^{-m}(\mathcal{P}) = \{T^{-m}P_j|\ j \in J\}$. Notice that, since $\mu$ is $T$-invariant, then $T^{-m}(\mathcal{P})$ is again a measurable partition and $H(T^{-m}(\mathcal{P})) = H(\mathcal{P})$.

**Proposition D.1.** *The limit*

$$h_\mu(T, \mathcal{P}) = \lim_{n\to+\infty} \frac{1}{n} H(\mathcal{P}_n^T)$$

*exists and $h_\mu(T, \mathcal{P}) \in [0, +\infty]$ is called the metric entropy of $T$ relative to the partition $\mathcal{P}$.*

PROOF. Let us recall the definition of conditional entropy. In $\mathcal{P}$ and $\mathcal{Q}$ are finite or countable measurable partition, then the conditional entropy of $\mathcal{P}$ w.r.t. $\mathcal{Q}$ is

$$H(\mathcal{P}|\mathcal{Q}) = -\sum_{j\in J} \mu(Q_j) \sum_{i\in I} \mu(P_i|Q_j) \log \mu(P_i|Q_j),$$

where $\mu(P_i|Q_j) = \mu(P_j \cap Q_j)/\mu(Q_j)$ when $\mu(Q_j) \neq 0$, and it is zero otherwise. It is not difficult to prove that (see for instance [47, Proposition 4.3.3. - (4)])

$$H(\mathcal{P} \vee \mathcal{Q}) \leq H(\mathcal{P}) + H(\mathcal{Q}).$$

Thus,

$$H(\mathcal{P}_{n+m}^T) \leq H(\mathcal{P}_n^T) + H(\mathcal{P}_m^T),$$

hence $(H(\mathcal{P}_n^T))_{n\in\mathbb{N}}$ is subadditive and we conclude that this limit exists (see the proof of Lemma A.9). Q.E.D.

**Definition D.2.** *The entropy of $(T, \mu)$ is*

$$h_\mu(T) = \sup\{h_\mu(T, \mathcal{P})|\ h_\mu(T, \mathcal{P}) < +\infty\}$$

## D.2 Topological entropy and the Variational Principle

In this section we recall the topological counterpart to the metric entropy $h_\mu(T)$, i.e., the topological entropy, introduced by Dinaburg and Bowen. Roughly speaking, the topological entropy $h_{top}$ is a number which measures the exponential growth rate of orbits segments that can be distinguished with arbitrarily fine, but finite precision.

**Definition D.3.** *Let $(X, d)$ be a metric space and let $T$ be a continuous map on $X$. We define the dynamic distance of length $n$ of $X$*

$$d_n(x, y) = \max_{0 \leq i < n} d(T^i(x), T^i(y)).$$

*Let us denote by $B_n(x, r)$ the open dynamic ball of length $n$ and radius $r$, that is*

$$B_n(x, r) = \{y \in X | \; d_n(x, y) < r\}.$$

*Given $\epsilon > 0$, a set $E \subseteq X$ is said to be an $(n, \epsilon)$-spanning set, if*

$$X = \bigcup_{x \in E} B_n(x, \epsilon).$$

*Let $S_d(n, \epsilon) = \min\{\#(E) | \; E$ is a $(n, \epsilon)$-spanning set$\}$. Then we define*

$$h_d(T) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log(S_d(n, \epsilon))$$

**Proposition D.4.** *Let $d$ $d'$ be two metrics which induce the same topology on $X$. Then $h_d(T) = h_{d'}(T)$ and we can write $h_{top} = h_{top}(T) = h_d(T) = h_{d'(T)}$. We call $h_{top}$ the topological entropy of $T$. In particular, $h_{top}$ is invariant under topological conjugation.*

Proof. Let $d'$ be another metric on $X$. For any $\epsilon > 0$, let $D_\epsilon$ be the subset of points $(x, y) \in X \times X$ such that $d(x, y) \geq \epsilon$. $D_\epsilon$ is compact in $X \times X$ and the metric $d'$ is continuous on $D_\epsilon$. Accordingly, there exists the minimum $m_\epsilon = \min d'|_{D_\epsilon}$. Notice that $m_\epsilon > 0$, otherwise, there would exists $(x, y) \in D_\epsilon$, such that $d'(x, y) = 0$, but then $x = y$ and $d(x, y) = 0$, i.e., $(x, y) \notin D_\epsilon$. As a consequence, if $d'(x, y) < m_\epsilon$, then $d(x, y) < \epsilon$, hence $B_{n,d'}(x, m_\epsilon) \subseteq B_{n,d}(x, \epsilon)$ and $S_{d'}(n, m_\epsilon) \geq S_d(n, \epsilon)$. We conclude that $h_{d'}(T) \geq h_d(T)$. By reversing the roles of $d$ and $d'$ one obtains $h_d(T) = h_{d'}(T)$, hence $h_{top}$ does not depend on $d$.

Finally, assume that $\widetilde{T} \colon \widetilde{X} \to \widetilde{X}$ is topologically conjugated to $T$, with a homeomorphism $\Phi \colon X \to \widetilde{X}$. Let $d$ be a metric on $X$ and let $\widetilde{d}$ be the pushforward metric $\widetilde{d}(\widetilde{x}, \widetilde{y}) = d(\Phi^{-1}(\widetilde{x}), \Phi^{-1}(\widetilde{y}))$. Then $h_d(T) = h_{\widetilde{d}}(\widetilde{T})$ and we conclude that $h_{top}(T) = h_{top}(\widetilde{T})$.

Q.E.D.

**Remark D.5.** An equivalent definition of topological entropy is obtained through the $(n,\epsilon)$-separated sets. In fact, $E \subset X$ is $(n,\epsilon)$-separated if $d_n(x,y) > \epsilon$ for any $x, y \in E$. Setting$Z_d(n,\epsilon) = \max\{\#(E)|\ E \text{ is a } (n,\epsilon)\text{-separated set}\}$, we define

$$h_d(T) = \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log Z_d(n,\epsilon)$$

Notice that $S_d(n,\epsilon) \le Z_d(n,\epsilon) \le S_d(n,\epsilon/2)$. In effect, If $E$ is an $(n,\epsilon)$-separated set with maximal cardinality, then it is also an $(n,\epsilon)$-spanning set. Thus, $S_d(n,\epsilon) \le Z_d(n,\epsilon)$. On the other hand, if $E$ is an $(n,\epsilon)$-separated set and $F$ is an $(n,\epsilon/2)$-spanning set, then to every $x \in E$ we can assign injectively a point $\phi(x) \in F$ such that $d_n(x,\phi(x)) < \epsilon/2$. Accordingly, $\#(E) \le \#(F)$ and $Z_d(n,\epsilon) \le S_d(n,\epsilon/2)$. We conclude that

$$h_d(T) = \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log Z_d(n,\epsilon) = \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log S_d(n,\epsilon),$$

i.e., the two definitions of topological entropy are equivalent.

Next theorem gives the connection between metric entropy and topological entropy. We do not rewrite the proof, but we refer the interested reader to [47, Theorem 4.5.3] or [71, Theorem 8.6]

**Theorem D.6** (Variational Principle). *Let $\mathcal{M}(T)$ be the space of Borel $T$-invariant probability measures , where $T$ is a homeomorphism of a compact metric space $(X,d)$. Then*

$$h_{top}(T) = \sup\{h_\mu(T)|\ \mu \in \mathcal{M}(T)\}.$$

*If the r.h.s. is a maximum, then $\mu$ is said to be a measure of maximal entropy.*

## D.3 Volume growth of invariant manifolds for Anosov diffeomorphisms

We conclude this chapter showing that the volume of stable leaves of $\Sigma$ (see Section 2.2) of Anosov diffeomorphisms asymptotically grow as $e^{nh_{top}}$ under the action of $f^{-n}$. Notice that we used this estimate in the proof of Lasota-Yorke inequalities (Theorem 2.11). Consequently, in this section we assume the setup of the problem of this thesis (see Section 1.2). The proof is taken by [36, Appendix C]

We point out that there is a small mistake in their result, because $W$ are close to stable leaves, hence they grow under the action of $f^{-n}$, and not $f^n$, as specified in [36, Appendix C].

Let $\rho_n^+ = \sup_{W\in\Sigma} |f^{-n}(W)|$ and let $\rho_n^- = \inf_{W\in\Sigma} |f^{-n}(W)|$, where $|f^{-n}W|$ represents the volume of $f^{-n}(W)$. We prove the following theorem.

**Theorem D.7.** *If $f$ is topologically transitive, then, for every $n \in \mathbb{N}$,*

$$e^{nh_{top}} \leq \rho_n^+ \leq Ce^{nh_{top}}.$$

Before proving Theorem D.7, we need a couple of lemmas.

**Lemma D.8.** *Given $n, m \in \mathbb{N}$, $\rho_{n+m}^+ \leq C\rho_n^+\rho_m^+$ and $\rho_{n+m}^- \geq C\rho_n^-\rho_m^-$. Moreover, if $f$ is topologically transitive, then $\rho_n^+ \leq C\rho_n^-$.*

PROOF. Let $W \in \Sigma$ and let $\mathcal{W}_n = \{W_i\}_{i\in I} \subseteq \Sigma$, of maximal cardinality, such that $f^{-n}(W) \supseteq \cup_{i\in I} W_i$ and $W_i \cap W_j = \varnothing$ for $i \neq j$. Thus, $\#\mathcal{W}_n \geq C|f^{-n}(W)|$ and

$$|f^{-n-m}(W)| \geq \sum_{i\in I} |f^{-m}(W_i)| \geq C\rho_n^-\rho_m^-.$$

Taking the inf on the l.h.s., we obtain that $\rho_{n+m}^- \geq C\rho_n^-\rho_m^-$. The other inequality can be proved similarly by considering a minimal disjoint covering $\mathcal{W}_n$ of $f^{-n}(W)$. Let $\delta$ be the sup on the volume of leaves of $\Sigma$. By topological transitivity, for any $\epsilon > 0$ there exists $n_\epsilon$, such that, given $B_1, B_2$, balls of radius $\epsilon$, then $f^{-n}B_1 \cap B_2 \neq \varnothing$, for some $n \leq n_\epsilon$. We set $\bar{n} = n_\delta$. For any $n \geq \bar{n}$, let $W_n \in \Sigma$ such that $|f^{-n+\bar{n}}(W_n)| \geq \frac{1}{2}\rho_{n-\bar{n}}^+$. Let $x \in W_n$ and consider a ball $B_1$ of center $x$ and radius $c\delta$, for some $c \in (0,1)$, that we fix later. For any $W \in \Sigma$ let $B_2$ be a ball of radius $c\delta$ with center in a point $z \in W$. Since $f$ is topologically transitive, there exists $m \leq \bar{n}$ such that $f^{-m}(B_2) \cap B_1 \neq \varnothing$. By uniform transversality of stable and unstable manifolds, there exists a $c \in (0,1)$ such that, for any point $y \in W_n$, the local unstable manifold $W_\delta^u(y)$ intersect $f^{-m(W)}$ and consists of a single point $r \in f^{-m}(W)$. Since $r \in W_\delta^u(y)$, the distance between $y$ and $r$ is always smaller than $\delta$ iterating $f^{-1}$. Accordingly, $|f^{-k}(W)| \leq C|f^{-k-m}(W)|$. Hence,

$$|f^{-n}(W)| \geq C|f^{-n+m}(W_n)| \geq C\rho_{n-\bar{n}}^+ \geq \rho_n^+.$$

By taking the the inf on $W \in \Sigma$ we obtain the desired inequality.

Q.E.D.

**Lemma D.9.** *If $f$ is topologically transitive, then, for any $n \in \mathbb{N}$ and for each $\epsilon > 0$,*

$$C_\epsilon \rho_n^+ \leq Z_d(n, \epsilon) \leq C_\epsilon \rho_n^+,$$

*where $Z_d(n, \epsilon)$ is the maximal cardinality of a $(n, \epsilon)$-separated set (see Remark D.5).*

Proof. Let $W \in \Sigma$ and let $S_n$ be an $\epsilon$-separated set on $f^{-n}(W)$, in the sense that the distance between points of $S_n$ is at least $\epsilon$, with respect to the distance induced by the metric of $M$ on $f^{-n}(W)$. Since $W \in \Sigma$, this distance does not grow under iterations of $f$. Thus, $S_n$ is an $(n, \epsilon)$-separated set. Since $\# S_n \geq C_\epsilon |f^{-n}(W)|$, we obtain that $Z_d(n, \epsilon) \geq C_\epsilon \rho_n^+$.

Next, let $W \in \Sigma$ and let $E$ be an $(n, \epsilon)$-separated set of maximal cardinality. Let us consider a family of balls $\{B_i\}_{i=1}^k$ of radius $c\epsilon$. By topological transitivity, for any $i = 1, \ldots, k$, there is $n_i \leq \bar{n}$ such that $f^{-n_i} W \cap B_i \neq \varnothing$ ($\bar{n}$ is the same of the proof of Lemma D.8). Let $W_i \in \Sigma$, $W_i \subseteq f^{-n_i}(W)$, be a leaf which intersect the ball $B$. To every point of $x \in B \cap E$ we assign the unique point $y \in W_\delta^u(x) \cap W_i$. Two point $x_1, x_2 \in S \cap B$ are $(n, \epsilon)$-separated if and only if the corresponding $y_1, y_2$ are $(n, C\epsilon)$-separated. In particular, $y_1, y_2$ are $(n, C\epsilon)$-separated in $f^{-n}(W_i)$. We conclude that the number of these points $y_j$ is at most $C_\epsilon |f^{-n}(W_i)|$, hence

$$Z_d(n, \epsilon) \leq \sum_{i \in I} C_\epsilon |f^{-n-n_i}(W)| \leq C_\epsilon \rho_n^+.$$

Q.E.D.

Proof of Theorem D.7. Lemma D.8 and Lemma D.9 imply that

$$\frac{\log Z_d(n, \epsilon)}{kn} \leq \frac{\log(C_\epsilon \rho_{kn}^+)}{kn} \leq \frac{\log C_\epsilon}{kn} + \frac{\log \rho_n^+}{n}$$

By passing to the limit for $k \to +\infty$ and $\epsilon \to 0$, we get

$$e^{n h_{top}} \leq \rho_n^+$$

On the other hand, Lemma D.8 gives $\rho_{kn}^+ \geq C^k (\rho_n^+)^k$, hence

$$\log(C \rho_n^+) \leq \frac{\log(\rho_{kn}^+)}{k} \leq \lim_{n \to +\infty} \frac{\log(\rho_{kn}^+)}{k} \leq n \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{\log(Z_d(n\epsilon))}{kn} \leq n h_{top},$$

where we used again Lemma D.9. Q.E.D.

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# Ringraziamenti

Al termine di questo percorso di dottorato, concluso con la stesura di questa tesi, è giunto il momento di ringraziare alcune persone, senza le quali tutto questo non sarebbe stato possibile. Grazie innanzitutto al Professor Marco Lenci, che con il suo costante supporto mi ha guidato in questi tre anni, fornendomi preziosi suggerimenti per la ricerca scientifica e per i miei progetti futuri. Ringrazio il Professor Giovanni Forni, che mi ha accolto per sei mesi presso l'Università del Maryland, ha arricchito le mie conoscenze, mi ha fornito preziosi consigli e mi ha permesso di vivere un'esperienza indimenticabile. Ringrazio anche tutto il gruppo di Sistemi Dinamici della UMD, in particolare il Professor Rodrigo Treviño, per aver letto questo elaborato e segnalato alcune imprecisioni. Inoltre, ringrazio i referee di questa tesi, i Professori Carlangelo Liverani e Sébastien Gouëzel, che con i loro suggerimenti hanno permesso di valorizzare il lavoro svolto.

Non posso dimenticare di ringraziare i miei colleghi e i miei amici. In particolare, Matteo, Elisa, Gioia, Lorenzo, Valentina, Gabriele, Letizia e Fabio. La vostra presenza ha spesso consentito di dimenticare, per un po' di tempo, preoccupazioni e dubbi che questo lavoro comporta. Grazie di cuore ai miei genitori, ad Alessia, Francesco, nonna Ornella e i nonni che mi guardano da lassù. Il vostro sostegno non è mai mancato e tanti obiettivi che ho raggiunto durante la mia vita sono dovuti un po' anche a voi. Ringrazio con affetto Gianni, Lorella, Giulio e Anna, per avermi accolto e per essersi sempre interessati al mio percorso accademico.

Un grazie speciale lo dedico alla mia famiglia. Ad Emilia, per essere sempre stata al mio fianco, per avermi ascoltato, compreso, consolato e spronato. Sei stata e sei per me un sostegno costante, so di poter sempre fare affidamento su di te. A Sofia, che ha portato gioia nella nostra famiglia. Spero che un giorno possa essere orgogliosa di questo mio importante traguardo. A voi dedico questa mia tesi.