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GENEOS, COMPACTIFICATIONS, AND GRAPHS

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GENEOS, COMPACTIFICATIONS, AND GRAPHS

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Abstract

Our objective in this thesis is to study the structure of the space of group equivariant non-expansive operators (GENEOs). In particular, we explore the pseudometric and topological properties of this space. We introduce the notions of compactification of a perception pair, collectionwise surjectivity of a space of GENEOs, and compactification of a space of GENEOs. We obtain some compactification results for perception pairs and the space of GENEOs. We show that when the data spaces are totally bounded and endow the common domains with metric structures, the perception pairs can be embedded isometrically in compact ones. Moreover, we show that when these conditions are satisfied, every collectionwise surjective space of GENEOs admits a compactification. We also show that the underlying embeddings are compatible.

An important part of the study of topology of the space of GENEOs is to populate it in a rich manner. We introduce the notion of a generalized permutant and show that this concept is useful in defining new GENEOs. The concept of a generalized permutant extends the applicability of the techniques based on the former idea of a permutant to the case when we might be working with distinct perception pairs.

We define the analogues of some of the aforementioned concepts in a graph theoretic setting, enabling us to use the power of the theory of GENEOs for the study of graphs in an efficient way. We define the notions of graph perception pair, graph permutant, and graph GENEO. We develop two models for the theory of graph GENEOs. The first model addresses the case of graphs having weights assigned to the vertices, while the second one addresses the case of so called weighted graphs, i.e., graphs with weights assigned to the edges. We prove some new results in the proposed theory of graph GENEOs and show the power of our models by describing their applications to the structural study of simple graphs.

We introduce the concept of a graph permutant and show that this concept

can be used to define new graph GENEOs between distinct graph perception pairs, thereby enabling us to populate the space of graph GENEOs in a rich manner. The richness of the techniques for defining GENEOs puts us in a better position to shed more light on the structure of the space of GENEOs.

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Chapter 1

Introduction

The importance of equivariance in machine learning is widely recognized. The use of equivariant operators allows one to incorporate domain knowledge into the learning process and introduce symmetries in data space, thereby paving the way not only to speeding up machine learning and reducing large dimensionality of data but also to the introduction of new abstract representations [1, 2, 3, 4, 5].

From the epistemological perspective, equivariant operators can be interpreted as observers that transform data into (usually simpler and more interpretable) data. In our mathematical framework, we are interested in data observers that are represented by functional operators transforming data in a regular and stable way, while respecting the compatibility with the action of an underlying group G of transformations, which describes the equivalence between data [6, 7]. The essence of group equivariant operators lies in their commutativity with respect to the action of G, and one of the most important regularity, *viz.* non-expansivity, enables one to avoid instability and divergent behavior.

Our research focuses on the study of topological properties of these *group equivariant non-expansive operators* (GENEOs, for short). Such operators can be seen as components of a new kind of neural networks as well as selected observers whose expertise is leveraged to improve data analysis. The use of GENEOs opens new possibilities in applications, making use of their ability to incorporate domain knowledge in the analysis hierarchy. For example, a shallow and interpretable neural network based on GENEOs, *viz.,* GENEOnet, has been recently proposed for the efficient detection of protein pockets that can host ligands. In this method, a set of GENEOs is allowed to fall on a collection of operators that process functions representing various properties of a protein. Families of operators are networked through a convex combination and a specific parameter is used to generate binary functions, where 1's represent promising binding sites [8].

In some sense, GENEOs constitute a bridge between geometric deep learning [9, 10] and topological data analysis. They make available a mathematical model for the concepts of agent and observer, seen from a geometrical perspective. The concept of a GENEO helps us affect a paradigm shift in data analysis from the geometry of data to that of the space of observers, thereby enriching our understanding of geometric deep learning.

Moreover, GENEOs present interesting links with persistent homology and allow one to get lower bounds for the natural pseudo-distance associated with the action of a group of homeomorphisms [6]. Furthermore, the concept of GENEO is useful in the architectural analysis of neural networks. Therefore, it is natural to study the metric and topological properties of the spaces of GENEOs. This study, coupled with our compactification results, could prove useful for the research in artificial intelligence.

Formally speaking, GENEOs are maps between so-called *perception pairs* (Φ, G), where Φ is a set of bounded real-valued maps defined on a non-empty set X and G is a group of Φ-preserving bijections of X. The space Φ represents the signals or measurements that the observer can interpret, while G is the equivariance group associated with the action of the observer. The space Φ is naturally endowed with a metric structure and endows X and G with suitable pseudo-metrics or metrics. This reflects the epistemological assumption that any information (and hence any quantitative structure) follows from physical measurements. It is interesting to observe that some of the pseudo-metric and topological properties of Φ are propagated to X and G, but not all. For example, if Φ is totally bounded, then so are X and G [11, Theorem 1, Theorem 4], while there are simple examples of perception pairs with compact Φ but incomplete X and G [6].

While the literature concerning equivariant neural networks is already extensive, the topological research about them is still quite limited. Until now, most of the attention has been devoted to what is called *topological machine learning*; i.e., the joint use of topology-based methods and machine learning algorithms [12], in general terms. In this field, some research focuses on the study of so-called *intrinsic topological features*, which concerns the employment of topological features to analyze or influence the machine learning model. In particular, some regularisation techniques have been considered, such as *topological autoencoders* [13, 14] (based on the idea of building networks that can simplify the data without changing their topology) or methods to simplify the topological complexity of the decision boundary [15]. More fundamental principles of regularisation using topological features have been investigated in [16]. The inclusion of topological features of graph neighborhoods into a standard graph neural network (GNN) has been proposed in [17], and the employment of GNNs to learn suitable filtrations have been examined in [18]. Furthermore, topological techniques have also been used for model analysis in machine learning. For example, topological analysis has been applied to evaluate generative adversarial networks (GANs) by the concept of *Geometry Score* [19], while *neural persistence* has been introduced as a complexity measure summarizing topological features that arise when filtrations of the neural network graphs are calculated [20]. The topological analysis of the decision boundary of a given classifier has been considered in [21], and the topological information encoded in the weights of convolutional neural networks (CNNs) has been studied in [22].

However, we stress that the development of the theory of GENEOs differs greatly from these lines of research, which are not focused on equivariance concerning arbitrary transformation groups and do not study the topology of suitable operator spaces, but most of them consider the properties of individual techniques and applications. In other words, the approach we are interested in is devoted to studying the topological properties of a space of equivariant operators as a whole. In this mathematical setting, the compactification problem can arise and admit resolution, and it seems natural to devise techniques for defining a rich variety of GENEOs.

In recent years, the need for an extension of Deep Learning to non-Euclidean domains has led to the development of Geometric Deep Learning (GDL) [9, 10]. This line of research focuses on applying neural networks on manifolds and graphs, so making available new geometric models for artificial intelligence. In doing that, GDL uses techniques coming from differential geometry, combinatorics, and algebra. In particular, it largely uses the concepts of group action and equivariant operator [1, 2, 4, 23, 24, 25, 26, 27], which allow for a strong reduction in the number of parameters involved in machine learning.

Topological Data Analysis (TDA) [28, 29, 30, 31, 32] is giving a contribution to the development of GDL, grounding on the use of *non-expansive* equivariant operators [7, 33]. The main idea is to benefit from classical and new results of TDA to study the "shape" of the spaces of equivariant operators by employing suitable topologies and metrics. The topological, geometric, and algebraic properties of these spaces have an important role to play in the identification of the operators that are more efficient for our application purposes. We stress that the assumption of non-expansivity is fundamental in this framework, since it guarantees that the space of group equivariant non-expansive operators (GENEOs) is compact (and hence finitely approximable), provided that the data space is compact for a suitable topology [6]. We also observe that, from a practical point of view, non-expansivity can be seen as the property of simplifying the metric structure of data. While particular applications may require locally violating this property, we remark that the usual long-term purpose of observers is the one of simplifying the available information by representing it in a much simpler and more meaningful way.

The approach based on TDA has allowed us to start shaping a compositional and topological theory for GENEOs. In particular, it has been proved that some operations are available to combine GENEOs and obtain other GENEOs, including composition, convex combination, minimization, maximization, and direct product. The compositional theory based on such operations leads us to think of GENEOs as elementary components that could be used to replace neurons in neural networks [6, 34, 35]. This new kind of network could be much more transparent in its behavior, because of the intrinsic interpretability of its components. Modularity is indeed a key tool for interpretability in machine learning, since it can make clear which processes control the behavior of artificial agents. The attention to this property corresponds to the rising interest in the so-called "explainable deep learning" [36, 37, 38].

To use GENEOs in applications, we need methods to build such operators for the transformation groups we are interested in. If we restrict our attention to linear operators, a constructive procedure is available for the case that the functions representing our data have a finite domain X. This procedure is based on the concept of "permutant", i.e., a set of permutations of X that is invariant under the conjugation action of the equivariance group G we are considering [39]. While the classical way of building equivariant operators requires integration on the (possibly large) group G [40], this construction method may be based on a simpler sum computed on a small permutant. We can prove that any linear GENEO can be obtained as a weighted arithmetic mean related to a suitable permutant, provided that the domains of the signals are finite and the equivariance groups transitively act on those domains [41]. By replacing the weighted arithmetic mean with other normalized symmetric functions, the method based on permutants can be easily extended to the construction of non-linear GENEOs [35].

We serve the purpose of studying the structure of the space of GENEOs on three fronts. First of all, we prove that the space of GENEOs can be embedded into a compact space of GENEOs, under some mild conditions. Secondly, we extend the definition of a permutant to that of a generalized permutant. This allows us to define new GENEOs even if we are working with distinct perception pairs. Finally, we define the concepts of graph GENEO and graph permutant which allow us to apply the theory of GENEOs to the structural study of simple graphs.

Compactness of the space of GENEOs provides us with fundamental guarantees in machine learning. It ensures, in addition to the existence of bounds, finite approximability which is useful in computations and paves the way to the search for efficient operators. In our model, everything stems from measurements, i.e., the data set Φ consisting of real-valued functions $\varphi : X \to \mathbb{R}$, which allows us not only to define the equivariance group G but also suitable pseudo-metrics for X and G. It is natural to expect the effect of metric and topological properties of Φ on those of X and G. It is known that the total boundedness of Φ implies the total boundedness of X and G. But not all the properties of Φ are propagated to X and G; for example, one can easily construct compact data sets Φ with incomplete X and G. Since compactness is very important in computations, one would like to embed all the non-compact spaces in question into the compact ones. We formalize the concepts of *compactification of a perception pair* and *compactification of a space of GENEOs*, and show that a wide variety of spaces of GENEOs, along with the underlying perception pairs, admit compactifications under some mild conditions.

We give our compactification results under the assumption that our data sets are totally bounded and endow the common domains with metric structures; moreover, we assume that the space of GENEOs under consideration is *collectionwise surjective*, a notion that we introduce in this thesis. These assumptions are already satisfied in many practical applications. For example, if we are working with a space Φ of grey-level images represented as matrices with values in the interval $[0, 1]$, then it is totally bounded.

Although the existence of a rich variety of GENEOs has implications for the structure of their space, yet in order to apply the theory of GENEOs to solve the practical problems, one needs to come up with techniques for defining new GENEOs. The method based on the concept of a permutant is very useful in this regard. But it has a serious limitation: it works only in the cases when we are working with a single perception pair, whereas in applications, we need also to use GENEOs between distinct perception pairs. We solve this problem by extending the concept of a permutant to that of a *generalized permutant*, and show that this concept too can be used to define GENEOs.

The recent success of the theory of GENEOs in some industrial applications [8] has motivated us to synthesize it with the theory of graphs. We develop two analogous models dealing with graphs weighted on vertices or edges and apply the aforementioned ideas to the structural study of simple graphs, focusing on the important role that these graphs have in GDL [9] and generally in TDA [42, 43, 44, 45]. To this end, we devise a way of using GENEOs with graphs, defining the concept of a *graph GENEO*. Moreover, we extend the definition of permutant to the one of a *graph permutant*, showing how this new concept allows us to build GENEOs between graphs. Our final purpose in developing the theory of graph GENEOs is to devise a new technique to build operators that can transform graphs according to our needs, and make these operators available as components for applications in GDL.

Our theory of graph GENEOs differs in many respects from the recent work on the use of *Hom*-complexes in the study of protein-protein interactions (PPIs). Xiang Liu et al. (2022) [46] build *Hom*-complexes to gain insights into the structure of graphs, while the series of nested *Hom*-complexes are constructed providing us with multiscale representations of PPIs. Moreover, they use the concepts of graph homomorphism and multihomomorphism. On the other hand, we apply our graph GENEOs to the graphs under consideration, and the outcomes of these applications are used to classify our graphs into various isomorphism classes. We hope that our work will prove instrumental in solving many industrial problems.

Chapter 2

Elements of the Theory of GENEOs

In this chapter, we give a brief introduction to the theory of GENEOs and speak about some of its recent applications. We will include the most important results describing the topology of the space of GENEOs that will be used frequently in the forthcoming chapters [6, 11], along with a short description of a method to build GENEOs based on the concept of a permutant [39].

Contents

2.1 The Mathematical Setting

The concept of a GENEO involves the auxiliary idea of a perception pair, which is nevertheless important in its own right too. It consists of a data set along with the corresponding equivariance group. The data set is made up of bounded real-valued functions, often called measurements or signals, defined on a non-empty set, and plays fundamental role in defining all the concepts used or introduced in this thesis. First of all, we use this data set to define an equivariance group consisting of bijections of the common domain of its members. This group acts on our data set through composition on the right. The common domain and the equivariance group both are endowed with pseudo-metric structures, which in turn furnish them with suitable topologies.

This preparation sets the stage for defining a GENEO between two not necessarily distinct perception pairs. Precisely, a GENEO transforms the domain data set into the codomain data set. Its non-expansivity is defined with respect to the pseudo-metric structures on the data sets under consideration, while the equivariance consists in its commutativity with respect to the corresponding group actions through the mediation of a homomorphism between the selected equivariance groups. Finally, the set of all GENEOs with respect to this homomorphism is endowed with a metric, and hence a topological, structure. It is precisely this space that we refer to as the space of GENEOs, the study of whose topological properties is the subject matter of our work. This study though will also be supplemented with the consideration of alternative pseudo-metrics on the space of GENEOs.

We will preserve, and elaborate upon, the notation used in [6] throughout this thesis.

Let X be a non-empty set and consider the normed vector space $(\mathbb{R}_b^X, \|\cdot\|_\infty)$, where

$$\mathbb{R}_b^X := \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is bounded}\},$$

and $\|\cdot\|_\infty$ denotes the usual uniform norm given by

$$\|\varphi\|_\infty := \sup_{x\in X} |\varphi(x)|, \text{ for every } \varphi \in \Phi.$$

Any metric subspace (Φ, D_Φ) of $(\mathbb{R}_b^X, \|\cdot\|_\infty)$, where

$$D_\Phi(\varphi_1, \varphi_2) := \|\varphi_1 - \varphi_2\|_\infty = \sup_{x\in X} |\varphi_1(x) - \varphi_2(x)|, \text{ for every } \varphi_1, \varphi_2 \in \Phi,$$

endows X with the topology induced by the extended pseudo-metric

$$D_X(x_1, x_2) := \sup_{\varphi\in\Phi} |\varphi(x_1) - \varphi(x_2)|.$$

The space X is interpreted as the space where one makes measurements, and the elements φ of Φ are called *admissible measurements* or *signals.* The function spaces Φ are sometimes called *data sets.* Moreover, we set $\mathrm{dom}(\Phi) := X$.

We stress here that any pseudo-metric structure that we will be considering in our model stems from the specification of a space of measurements. This encapsulates our assumption that data cannot be approached directly and are known only through suitable measurements made by an observer endowed with specific goals [6].

In our model, self-maps of the space X have an important role to play.

Definition 2.1.1. *A map $g : X \to X$ is said to be a $\Phi-$operation if the composite function φg is an element of Φ for every $\varphi \in \Phi$. A bijective $\Phi-$operation is called an invertible $\Phi-$operation if g^{-1} is also a $\Phi-$operation.*

The set of all invertible $\Phi-$operations is denoted by $\mathrm{Aut}_\Phi(X)$; i.e.,

$$\mathrm{Aut}_\Phi(X) := \{g : X \to X \mid g \text{ is a bijection, and } \varphi g,\ \varphi g^{-1} \in \Phi, \text{ for all } \varphi \in \Phi\},$$

and forms a group under the function composition. It acts on the space Φ through the right action

$$\rho : \Phi \times \mathrm{Aut}_\Phi(X) \to \Phi,\ (\varphi, g) \mapsto \varphi g.$$

We say that a bijection $f : X \to X$ is an *isometry* of X if $D_X(f(x), f(y)) = D_X(x, y)$, for every $x, y \in X$, and denote the set of all isometries of X by $\mathrm{Iso}(X)$.

Let $C(X, X) \supseteq \mathrm{Iso}(X)$ denote the set of all continuous functions $f : X \to X$. The following pseudo-metric will be used frequently in the sequel.

$$d_\infty(f, g) := \sup_{x \in X} D_X(f(x), g(x)), \text{ for every } f, g \in C(X, X).$$

If Φ is rich enough to endow X with a metric structure, instead of a pseudo-metric one, then d_∞ is an extended metric, and is called the *metric of uniform convergence* on $C(X, X)$.

Definition 2.1.2. *If G is a subgroup of $\mathrm{Aut}_\Phi(X)$, then (Φ, G) is called a perception pair.*

Definition 2.1.3. *We say that a perception pair (Φ, G) with $\mathrm{dom}(\Phi) = X$ is compact if Φ, G, and X are all compact.*

The data set Φ endows $\mathrm{Aut}_\Phi(X)$ with a pseudo-metric structure where the (extended) pseudo-distance D_{Aut} is given by

$$D_{\mathrm{Aut}}(f, g) := \sup_{\varphi \in \Phi} D_\Phi(\varphi f, \varphi g), \text{ for every } f, g \in \mathrm{Aut}_\Phi(X).$$

Conversely, each group $G \subseteq \mathrm{Aut}_\Phi(X)$ induces on the space Φ a pseudo-metric $d_G : \Phi \times \Phi \to \mathbb{R}$:

$$d_G(\varphi_1, \varphi_2) := \inf_{g \in G} D_\Phi(\varphi_1, \varphi_2 g), \text{ for every } \varphi_1, \varphi_2 \in \Phi.$$

We call d_G the *natural pseudo-distance* associated with the group G. This pseudo-metric represents the ground truth in our model and allows us to compare functions in the sense that it vanishes for the pairs of functions that are equivalent with respect to the action of the group G representing the data similarities useful for the observer [47, 48, 49].

It is known that each invertible $\Phi-$operation is an isometry with respect to D_X; that is, $\mathrm{Aut}_\Phi(X) \subseteq \mathrm{Iso}(X)$ [6, Proposition 2], though the reverse inclusion does not hold in general [6, Remark 2.4]. But d_∞ does not endow the space $(\mathrm{Aut}_\Phi(X), D_{\mathrm{Aut}})$ with any additional pseudo-metric structure:

$$\begin{aligned} D_{\mathrm{Aut}}(f,g) &:= \sup_{\varphi \in \Phi} D_\Phi(\varphi f, \varphi g) \\ &= \sup_{\varphi \in \Phi} \sup_{x \in X} |\varphi f(x) - \varphi g(x)| \\ &= \sup_{x \in X} D_X(f(x), g(x)) \\ &=: d_\infty(f,g), \end{aligned}$$

for all $f, g \in \mathrm{Aut}_\Phi(X)$. So, d_∞ coincides with the pseudo-distance D_{Aut} on $\mathrm{Aut}_\Phi(X)$; that is

$$d_\infty|_{\mathrm{Aut}_\Phi(X)} = D_{\mathrm{Aut}}.$$

In general, D_{Aut} is an extended pseudo-metric. But when (X, D_X) is a metric space, then so is (G, D_{Aut}): If $g, h \in G$ are distinct functions, then there is an $x_0 \in X$ such that $g(x_0) \neq h(x_0)$. Since D_X is a metric,

$$0 < D_X(g(x_0), h(x_0)) \leq \sup_{x \in X} D_X(g(x), h(x)) = d_\infty(g,h) = D_{\mathrm{Aut}}(g,h),$$

whence D_{Aut} is a metric as well.

Definition 2.1.4. *Let* (Φ, G) *and* (Ψ, H) *be perception pairs with* $\mathrm{dom}(\Phi) = X$ *and* $\mathrm{dom}(\Psi) = Y$*, and* $T : G \to H$ *be a group homomorphism. A map* $F : \Phi \to \Psi$ *is said to be a group equivariant non-expansive operator (GENEO) with respect to*

T if

$$F(\varphi \circ g) = F(\varphi) \circ T(g), \text{ for every } \varphi \in \Phi, g \in G,$$

and

$$\|F(\varphi_1) - F(\varphi_2)\|_\infty \leq \|\varphi_1 - \varphi_2\|_\infty, \text{ for every } \varphi_1, \varphi_2 \in \Phi.$$

A map $F : \Phi \to \Psi$ satisfying the first condition is called *$T-$equivariant* or a *group equivariant operator* (GEO), and it is called *non-expansive* if it satisfies the second condition. For the sake of conciseness, we often write a GENEO as $(F, T) : (\Phi, G) \to (\Psi, H)$.

The set $\mathcal{F}_T^{\text{all}}$ of all GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$ corresponding to a group homomorphism $T : G \to H$ is a metric space with the distance function given by

$$D_{\text{GENEO}}(F_1, F_2) = \sup_{\varphi \in \Phi} D_\Psi(F_1(\varphi), F_2(\varphi)), \text{ for every } F_1, F_2 \in \mathcal{F}_T^{\text{all}}.$$

The natural pseudo-distance allows us to define another pseudo-metric on this space:

$$D_{\text{GENEO},H}(F_1, F_2) := \sup_{\varphi \in \Phi} d_H(F_1(\varphi), F_2(\varphi)), \text{ for every } F_1, F_2 \in \mathcal{F}_T^{\text{all}}.$$

The spaces $\mathcal{F} \subseteq \mathcal{F}_T^{\text{all}}$ of GENEOs prove instrumental in comparing data. For example, one can consider the following pseudo-metric:

$$D_{\mathcal{F},\Phi}(\varphi_1, \varphi_2) := \sup_{F \in \mathcal{F}} D_\Psi(F(\varphi_1), F(\varphi_2)), \text{ for every } \varphi_1, \varphi_2 \in \Phi.$$

Conti et al. (2022) [35] give examples demonstrating how the use of GENEOs increases our ability to distinguish between data.

Our objective is to obtain isometric embeddings of perception pairs and of the spaces of GENEOs into compact ones while retaining the metric properties of the original spaces. The reader is referred to [6, 11] for further details about the concepts we have so far introduced in this section.

We will assume in Section 3.1 that the data set Φ is rich enough to endow the common domain X with a metric structure. The first step towards constructing our compactifications, under this assumption, is to consider the metric completion of X. It is well known that every metric space (M, D_M) admits a unique metric completion $(\hat{M}, \hat{D}_{\hat{M}})$ up to homeomorphisms. We can assume that the completion

$\hat{M}$ contains M; i.e., we have the inclusion

$$j : M \to \hat{M},$$

and the metric $\hat{D}_{\hat{M}}$ is given by

$$\hat{D}_{\hat{M}}(\hat{x}, \hat{y}) = \lim_{n\to\infty} D_M(x_n, y_n),$$

where $\hat{x}, \hat{y} \in \hat{M}$, and $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are arbitrary sequences in M converging to $\hat{x}$ and $\hat{y}$ respectively.

2.2 Topology of the Space of GENEOs

The geometric and topological properties of the space of GENEOs enable one choose better the operators that are more useful in practical applications. For example, compactness ensures the existence of a finite set of operators that can represent the whole space in a reliable manner [6], while the introduction of Riemannian structure paves the way for minimization of cost functions by means of gradient descent methods [50]. Bergomi et al. [6] laid the foundations of a topological theory of the space of GENEOs. Retaining the notation of Section 2.1, we recall the following results from [6, 11, 51] which will be used frequently in the sequel. The proofs of the results that appear only in [11] will be given in Section 2.4 for the sake of completeness.

Proposition 2.2.1. *[11, Proposition 1.2.10] Each function $\varphi \in \Phi$ is non-expansive, and hence uniformly continuous with respect to D_X.*

Therefore, the topology τ_{D_X} induced by D_X is finer than the so called *initial topology* τ_{in} on X, which by definition is the coarsest topology on X with respect to which all the signals $\varphi \in \Phi$ are continuous.

Theorem 2.2.1. *[6, Supplementary Methods: Theorem 2.1] If Φ is totally bounded, then τ_{D_X} coincides with τ_{in}.*

Theorem 2.2.2. *[11, Theorem 1] If Φ is totally bounded, then so is (X, D_X).*

Proposition 2.2.2. *[6, Proposition 2]* $\mathrm{Aut}_\Phi(X) \subseteq \mathrm{Iso}(X)$.

That is, each $g \in G \subseteq \mathrm{Aut}_\Phi(X)$ is an isometry of X.

Recall that a subgroup of a topological group is topological, and

Proposition 2.2.3. *[51] If A is a subgroup of a topological group G, then $cl_G(A)$ is also a subgroup, and hence a topological subgroup of G.*

This proposition will be used in conjunction with

Theorem 2.2.3. *[6, Supplementary Methods: Theorem 2.7] $\mathrm{Aut}_\Phi(X)$ is a topological group and the action $\rho : \Phi \times \mathrm{Aut}_\Phi(X) \to \Phi$ is continuous.*

Theorem 2.2.4. *[11, Theorem 4] If Φ is totally bounded, then so is (G, D_{Aut}).*

Proposition 2.2.4. *[11, Proposition 1.2.20] If (X, D_X) is a compact metric space, then $(\mathrm{Iso}(X), d_\infty)$ is also compact.*

Theorem 2.2.5. *[11, Theorem 5] If $\Phi \subseteq \mathbb{R}_b^X$ and (X, D_X) are both compact metric spaces, then $\mathrm{Aut}_\Phi(X)$ is closed in $\mathrm{Iso}(X)$, and hence compact.*

Theorem 2.2.6. *[6, Theorem 7] The space $(\mathcal{F}_T^{\mathrm{all}}, D_{\mathrm{GENEO}})$ of GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$ is compact whenever the spaces Φ and Ψ are compact.*

2.3 Permutants

Conti et al. [35] give a method to build GENEOs by means of the concept of a permutant. Roughly speaking, a permutant is a finite set of invertible data-preserving operations that remains invariant under the conjugation action of the equivariance group. The group action is then used to define new GENEOs on the given perception pair. If G is a subgroup of $\mathrm{Aut}_\Phi(X)$, then the conjugation map

$$\alpha_g \colon \mathrm{Aut}_\Phi(X) \to \mathrm{Aut}_\Phi(X),$$

given by $f \mapsto g \circ f \circ g^{-1}$, $g \in G$, plays a key role in this technique.

Definition 2.3.1. *Let H be a finite subset of $\mathrm{Aut}_\Phi(X)$. We say that H is a permutant for G if $H = \emptyset$ or $\alpha_g(H) \subseteq H$ for every $g \in G$; i.e., $\alpha_g(f) = g \circ f \circ g^{-1} \in H$ for every $f \in H$ and $g \in G$.*

Example 2.3.1. *Let Φ be the set of all functions $\varphi : X = S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\} \to [0, 1]$ that are non-expansive with respect to the Euclidean distances on S^1 and $[0, 1]$. Let us consider the group G of all isometries of $\mathbb{R}^2$, restricted to S^1. If h is the clockwise rotation of ℓ radians for a fixed $\ell \in \mathbb{R}$, then the set $H = \{h, h^{-1}\}$ is a permutant for G.*

Other examples of permutants will be given in Example 4.2.6, Example 4.2.7 and Proposition 4.2.1.

We recall the following result. As usual, in the following we will denote the set of all functions from the set A to the set B by the symbol B^A.

Proposition 2.3.1. *Let (Φ, G) be a perception pair with $dom(\Phi) = X$. If H is a nonempty permutant for $G \subseteq \mathrm{Aut}_\Phi(X)$, then the restriction to Φ of the operator $F \colon \mathbb{R}^X \to \mathbb{R}^X$ defined by*

$$F(\varphi) := \frac{1}{|H|} \sum_{h \in H} \varphi \circ h$$

is a GENEO from (Φ, G) to (Φ, G) with respect to T, provided that $F(\Phi) \subseteq \Phi$.

The reader is referred to [11, 41, 35] for further details.

2.4 Supplementary Proofs

For the sake of completeness, we recall here the proofs of some results reported in Section 2.2 that have been given only in [11].

Proof of Proposition 2.2.1. Let $\varphi \in \Phi$ and $x_1, x_2 \in X$. Then

$$|\varphi(x_1) - \varphi(x_2)| \leq \sup_{\varphi' \in \Phi} |\varphi'(x_1) - \varphi'(x_2)| = D_X(x_1, x_2).$$

So $\varphi : X \to \mathbb{R}$ is non-expansive. □

Proof of Theorem 2.2.2. It suffices to show that every sequence $(x_i)_{i \in \mathbb{N}}$ in X admits a Cauchy subsequence [52]. Let us consider an arbitrary sequence $(x_i)_{i \in \mathbb{N}}$ in X and an arbitrarily small $\varepsilon > 0$. Since Φ is totally bounded, we can find a finite subset $\Phi_\varepsilon = \{\varphi_1, \ldots, \varphi_n\}$ such that $\Phi = \bigcup_{i=1}^n B_\Phi(\varphi_i, \varepsilon)$, where $B_\Phi(\varphi, \varepsilon) = \{\varphi' \in \Phi : D_\Phi(\varphi', \varphi) < \varepsilon\}$. In particular, we can say that for any $\varphi \in \Phi$ there exists $\varphi_{\bar{k}} \in \Phi_\varepsilon$ such that $\|\varphi - \varphi_{\bar{k}}\|_\infty < \varepsilon$. Now, we consider the real sequence $(\varphi_1(x_i))_{i \in \mathbb{N}}$ that is bounded because all the functions in Φ are bounded. From Bolzano-Weierstrass Theorem it follows that we can extract a convergent subsequence $(\varphi_1(x_{i_h}))_{h \in \mathbb{N}}$. Then we consider the sequence $(\varphi_2(x_{i_h}))_{h \in \mathbb{N}}$. Since φ_2 is bounded, we can extract a convergent subsequence $(\varphi_2(x_{i_{h_t}}))_{t \in \mathbb{N}}$. We can repeat the same argument for any $\varphi_k \in \Phi_\varepsilon$. Thus, we obtain a subsequence $(x_{p_j})_{j \in \mathbb{N}}$ of $(x_i)_{i \in \mathbb{N}}$, such that $(\varphi_k(x_{p_j}))_{j \in \mathbb{N}}$ is a real convergent sequence for any $k \in \{1, \ldots, n\}$, and hence a Cauchy sequence in $\mathbb{R}$. Moreover, since Φ_ε is a finite set, there exists an index $\bar{\jmath}$ such that for any $k \in \{1, \ldots, n\}$ we have that

$$|\varphi_k(x_{p_r}) - \varphi_k(x_{p_s})| < \varepsilon, \quad \text{for all } r, s \geq \bar{\jmath}.$$

We observe that $\bar{\jmath}$ does not depend on k, but only on ε and Φ_ε.

In order to prove that $(x_{p_j})_{j\in\mathbb{N}}$ is a Cauchy sequence in X, we observe that for any $r, s \in \mathbb{N}$ and any $\varphi \in \Phi$, by choosing a k such that $\|\varphi - \varphi_k\|_\infty < \varepsilon$ we have:

$$\begin{aligned}
|\varphi(x_{p_r}) - \varphi(x_{p_s})| &= |\varphi(x_{p_r}) - \varphi_k(x_{p_r}) + \varphi_k(x_{p_r}) - \varphi_k(x_{p_s}) + \varphi_k(x_{p_s}) - \varphi(x_{p_s})| \\
&\leq |\varphi(x_{p_r}) - \varphi_k(x_{p_r})| + |\varphi_k(x_{p_r}) - \varphi_k(x_{p_s})| + |\varphi_k(x_{p_s}) - \varphi(x_{p_s})| \\
&\leq \|\varphi - \varphi_k\|_\infty + |\varphi_k(x_{p_r}) - \varphi_k(x_{p_s})| + \|\varphi_k - \varphi\|_\infty.
\end{aligned}$$

It follows that $|\varphi(x_{p_r}) - \varphi(x_{p_s})| < 3\varepsilon$ for every $\varphi \in \Phi$ and every $r, s \geq \bar{\jmath}$. Thus, $\sup_{\varphi\in\Phi} |\varphi(x_{p_r}) - \varphi(x_{p_s})| = D_X(x_{p_r}, x_{p_s}) \leq 3\varepsilon$. Hence, the subsequence $(x_{p_j})_{j\in\mathbb{N}}$ is a Cauchy sequence in X, and the theorem is proved. □

Proof of Theorem 2.2.4. Let $(g_i)_{i\in\mathbb{N}}$ be a sequence in G and take a real number $\varepsilon > 0$. Given that Φ is totally bounded, we can find a finite subset $\Phi_\varepsilon = \{\varphi_1, \ldots, \varphi_n\}$ such that for every $\varphi \in \Phi$ there exists $\varphi_h \in \Phi_\varepsilon$ for which $D_\Phi(\varphi_h, \varphi) < \varepsilon$.

Let us consider the sequence $(\varphi_1 g_i)_{i\in\mathbb{N}}$ in Φ. Since Φ is totally bounded, we can extract a Cauchy subsequence $(\varphi_1 g_{i_h})_{h\in\mathbb{N}}$ [52]. Then we consider the sequence $(\varphi_2 g_{i_h})_{h\in\mathbb{N}}$. Again, we can extract a Cauchy subsequence $(\varphi_2 g_{i_{h_t}})_{t\in\mathbb{N}}$. We can repeat the same argument for any $\varphi_k \in \Phi_\varepsilon$. Thus, we are able to extract a subsequence $(g_{i_j})_{j\in\mathbb{N}}$ of $(g_i)_{i\in\mathbb{N}}$ such that $(\varphi_k g_{i_j})_{j\in\mathbb{N}}$ is a Cauchy sequence for any $k \in \{1, \ldots, n\}$. For the finiteness of set Φ_ε, we can find an index $\bar{\jmath}$ such that for any $k \in \{1, \ldots, n\}$

$$D_\Phi(\varphi_k g_{i_r}, \varphi_k g_{i_s}) < \varepsilon, \text{ for every } s, r \geq \bar{\jmath}.$$

In order to prove that $(g_{i_j})_{j\in\mathbb{N}}$ is a Cauchy sequence, we observe that for any $\varphi \in \Phi$, any $\varphi_k \in \Phi_\varepsilon$, and any $r, s \in \mathbb{N}$ we have

$$\begin{aligned}
D_\Phi(\varphi g_{i_r}, \varphi g_{i_s}) &\leq D_\Phi(\varphi g_{i_r}, \varphi_k g_{i_r}) + D_\Phi(\varphi_k g_{i_r}, \varphi_k g_{i_s}) + D_\Phi(\varphi_k g_{i_s}, \varphi g_{i_s}) \\
&= D_\Phi(\varphi, \varphi_k) + D_\Phi(\varphi_k g_{i_r}, \varphi_k g_{i_s}) + D_\Phi(\varphi_k, \varphi).
\end{aligned}$$

We observe that $\bar{\jmath}$ does not depend on φ, but only on ε and Φ_ε. By choosing a

$\varphi_k \in \Phi_\varepsilon$ such that $D_\Phi(\varphi_k, \varphi) < \varepsilon$, we get $D_\Phi(\varphi g_{i_r}, \varphi g_{i_s}) < 3\varepsilon$ for every $\varphi \in \Phi$ and every $r, s \geq \bar{\jmath}$. Thus, $D_{\mathrm{Aut}}(g_{i_r}, g_{i_s}) \leq 3\varepsilon$. Hence, the sequence $(g_{i_j})_{j\in\mathbb{N}}$ is a Cauchy sequence. Therefore, G is totally bounded. $\square$

Proof of Proposition 2.2.4. Let $C(X, X)$ denote the metric space of all continuous self-maps of X with respect to the metric d_∞ given by

$$d_\infty(f, g) := \sup_{x\in X} D_X\left(f(x), g(x)\right), \text{ for every } f, g \in C(X, X).$$

It suffices to show that $\mathrm{Iso}(X)$ is closed in $C(X, X)$, as it is relatively compact by Arzelà-Ascoli theorem [53]. Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{Iso}(X)$ that converges to some $f \in C(X, X)$; we show that $f \in \mathrm{Iso}(X)$. Note that $f(x) = \lim_{i\to\infty} f_i(x)$ with respect to D_X, for each $x \in X$; indeed,

$$0 \leq \lim_{i\to\infty} D_X(f(x), f_i(x)) \leq \lim_{i\to\infty} d_\infty(f, f_i) = 0.$$

So,

$$\begin{aligned} D_X(f(x), f(y)) &= D_X(\lim_{i\to\infty} f_i(x), \lim_{i\to\infty} f_i(y)) \\ &= \lim_{i\to\infty} D_X(f_i(x), f_i(y)) \\ &= \lim_{i\to\infty} D_X(x, y) \\ &= D_X(x, y), \end{aligned}$$

whence f preserves D_X.

It immediately follows that f is injective. As for surjectivity, let x_0 be an arbitrary point of X; we show that $x_0 \in f(X)$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by setting $x_{n+1} := f(x_n)$. Since X is compact, $(x_n)_{n\in\mathbb{N}}$ admits a converging subsequence $(x_{n_i})_{i\in\mathbb{N}}$. Let $\varepsilon > 0$ be an arbitrary real number. Then there is an $n_0 \in \mathbb{N}$ such that $D_X(x_{n_i}, x_{n_j}) < \varepsilon$ for every $i, j \geq n_0$. If $n_j \geq n_i$, then $D_X(x_{n_i}, x_{n_j}) = D_X(x_0, x_{n_j - n_i})$, as f preserves D_X. Hence, $D_X(x_0, f(X)) := \inf_{x\in f(X)} D_X(x_0, x) \leq D_X(x_0, x_{n_j-n_i}) = D_X(x_{n_i}, x_{n_j}) < \varepsilon$. From the arbitrariness of ε, it follows that $D_X(x_0, f(X)) = 0$. As f preserves D_X, it is continuous, and $f(X)$ then is compact. In particular, $f(X)$ is closed, and hence $x_0 \in f(X)$. $\square$

Proof of Theorem 2.2.5. For the sake of conciseness, we will rephrase the proof given in [11]. Consider the collection $\mathcal{H}$ of all non-empty compact subsets of the space $\mathbf{NE}(X,\mathbb{R})$ of all real-valued non-expansive functions on (X, D_X), endowed with the distance induced by the uniform norm. Of course, by Proposition 2.2.1, $\mathbf{NE}(X,\mathbb{R}) \supseteq \Phi$. We know that $(\mathcal{H}, d_{\mathcal{H}})$ is a metric space, where $d_{\mathcal{H}}$ is the usual Hausdorff distance [54]. If $g \in \mathrm{Iso}(X)$, then the map $R_g : \Phi \to \mathbb{R}^X_b$ that takes φ to φg is continuous (it indeed preserves the max-morm distance), and hence $\Phi g := \{\varphi g, \varphi \in \Phi\} \in \mathcal{H}$.

We now observe that if $g, h \in \mathrm{Iso}(X)$, then

$$
\begin{aligned}
d_{\mathcal{H}}(\Phi g, \Phi h) := \max &\left\{ \sup_{\varphi \in \Phi g} \inf_{\psi \in \Phi h} \|\varphi - \psi\|_\infty, \sup_{\psi \in \Phi h} \inf_{\varphi \in \Phi g} \|\varphi - \psi\|_\infty \right\} \\
&= \max \left\{ \sup_{\varphi \in \Phi} \inf_{\psi \in \Phi} \|\varphi g - \psi h\|_\infty, \sup_{\psi \in \Phi} \inf_{\varphi \in \Phi} \|\varphi g - \psi h\|_\infty \right\} \\
&\le \max \left\{ \sup_{\varphi \in \Phi} \|\varphi g - \varphi h\|_\infty, \sup_{\psi \in \Phi} \|\psi g - \psi h\|_\infty \right\} \\
&= \sup_{\varphi \in \Phi} \|\varphi g - \varphi h\|_\infty \\
&= \sup_{\varphi \in \Phi} \sup_{x \in X} |\varphi g(x) - \varphi h(x)| \\
&= \sup_{x \in X} \sup_{\varphi \in \Phi} |\varphi g(x) - \varphi h(x)| \\
&= \sup_{x \in X} D_X(g(x), h(x)) \\
&= d_\infty(g, h).
\end{aligned}
$$

Therefore, the map $\chi : \mathrm{Iso}(X) \to \mathcal{H}$ that takes g to Φg is non-expansive and hence continuous. Since $\mathrm{Aut}_\Phi(X) = \chi^{-1}(\Phi)$, such a group is the preimage of a closed set under a continuous function. It follows that it is closed in $\mathrm{Iso}(X)$, and hence compact.

□

Chapter 3

Compactification of Perception Pairs and Spaces of GENEOs

In this chapter, we define the notions of compactification of a perception pair, collectionwise surjectivity, and compactification of a space of GENEOs, and give our compactification results. Under the hypothesis that our data spaces are totally bounded and endow the common domains with metric structures, we show that our perception pairs and every collectionwise surjective space of GENEOs admit compactifications. We require the underlying embeddings to be compatible, and show that this requirement too is satisfied by our constructions [55].

Contents

3.1 The Compactification Problem

Compactness results provide us with fundamental guarantees in machine learning. It is known that the space of all group equivariant non-expansive operators associated with a given group homomorphism is compact whenever the spaces of signals are compact [6, Theorem 7]. In some sense, this result states that if the spaces of data are compact, then the space of observers is compact too, provided that suitable topologies are used. This ensures that, for any specified tolerance, there is always a finite set of GENEOs in the space that can approximate the behavior of each GENEO within any acceptable proximity.

Therefore, it is natural to seek embeddings of mathematical structures of interest into compact ones. This process is called *compactification* in general topology. Formally, a compact Hausdorff space K is a compactification of a given space A if it contains a dense subspace D homeomorphic to A. In the case of metric spaces, we require the underlying homeomorphism $e : A \to D \subseteq K$ to be an isometry.

In view of the widely recognized importance of compactifications, we seek conditions under which a given space $\mathcal{F} \subseteq \mathcal{F}_T^{\text{all}}$ of GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$, and the respective perception pairs (Φ, G), $\text{dom}(\Phi) = X$ and (Ψ, H), $\text{dom}(\Psi) = Y$, can be embedded isometrically into compact ones, where $\mathcal{F}_T^{\text{all}}$ denotes the topological space of all GENEOs between the perception pairs (Φ, G), (Ψ, H), with respect to the homomorphism $T : G \to H$. In this article, we ascertain which spaces of GENEOs allow us to construct the surrounding compact spaces of GENEOs isometrically containing the original ones. We prove that, in many practical applications, every perception pair and an important class of spaces of GENEOs can be viewed as parts of compact perception pairs and compact spaces of GENEOs.

We will be assuming that our data sets Φ and Ψ are totally bounded and are rich enough to endow X and Y, and therefore G and H respectively, with a metric structure. Moreover, we will also assume that the collection $\{F(\Phi) \mid F \in \mathcal{F} \subseteq \mathcal{F}_T^{\text{all}}\}$ covers the data set Ψ. These hypotheses will be recalled several times in order to facilitate the reading.

Our approach, in brief, is as follows. The total boundedness of Φ ensures that X is totally bounded (Theorem 2.2.2), and therefore, its metric completion $\hat{X}$ is compact. We extend the functions $\varphi \in \Phi$ to functions $\hat{\varphi} : \hat{X} \to \mathbb{R}$ on the metric completion $\hat{X}$ (Section 3.2), and use the isometries $g \in G$ to define the isometries $\hat{g} : \hat{X} \to \hat{X}$ (Subsection 3.3.1). The set $\hat{\Phi}$, being isometric to the totally bounded space Φ is likewise totally bounded (Corollary 3.2.2), while the set $\hat{G}$ of all $\hat{g}$ may or may not be compact despite Φ being totally bounded [6]. We, therefore, consider their closures $\overline{\hat{\Phi}}$ and $\overline{\hat{G}}$ in the complete space $C(\hat{X}, \mathbb{R})$ and the compact space $\mathrm{Iso}(\hat{X})$ of isometries of $\hat{X}$ respectively, and, constructing successively the perception pairs $(\hat{\Phi}, \hat{G}), (\overline{\hat{\Phi}}, \hat{G})$, and $(\overline{\hat{\Phi}}, \overline{\hat{G}})$, we obtain the compatible embedding of the original perception pair (Φ, G) into the compact perception pair $(\overline{\hat{\Phi}}, \overline{\hat{G}})$ (Subsection 3.3.2). If $\mathcal{F}$ is a space of GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$, then these perception pairs allow us to define two suitable spaces $\mathcal{F}_1 \subseteq \mathcal{F}^{\mathrm{all},1}_{\hat{T}}$ and $\mathcal{F}_2 \subseteq \mathcal{F}^{\mathrm{all},2}_{\hat{T}}$ of GENEOs $(\hat{F}, \hat{T}) : (\hat{\Phi}, \hat{G}) \to (\hat{\Psi}, \hat{H})$ and $(\overline{\hat{F}}, \hat{T}) : (\overline{\hat{\Phi}}, \hat{G}) \to (\overline{\hat{\Psi}}, \hat{H})$ respectively (Subsection 3.4.1 and Section 3.5). Under the covering assumption stated above, we can define a suitable space $\mathcal{F}_3 \subseteq \mathcal{F}^{\mathrm{all}}_{\overline{\hat{T}}}$ of GENEOs $(\overline{\hat{F}}, \overline{\hat{T}}) : (\overline{\hat{\Phi}}, \overline{\hat{G}}) \to (\overline{\hat{\Psi}}, \overline{\hat{H}})$, while the closure of $\mathcal{F}_3 = \{\overline{\hat{F}} : \overline{\hat{\Phi}} \to \overline{\hat{\Psi}} \mid F \in \mathcal{F}\}$ in the compact space $\mathcal{F}^{\mathrm{all}}_{\overline{\hat{T}}}$ serves as the requisite compactification of the space $\mathcal{F} \subseteq \mathcal{F}^{\mathrm{all}}_{T}$ (Section 3.5).

Let (Φ, G), $\mathrm{dom}(\Phi) = X$ and (Ψ, H), $\mathrm{dom}(\Psi) = Y$ be perception pairs and $\mathcal{F} \subseteq \mathcal{F}^{\mathrm{all}}_{T}$, where $\mathcal{F}^{\mathrm{all}}_{T}$ denotes, as usual, the space of all GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$ with respect to a fixed homomorphism $T : G \to H$.

In this chapter, we will be assuming that

i) Φ and Ψ are totally bounded, and are rich enough to endow each of X and Y with metric structures;

ii) the collection $\{F(\Phi) \mid F \in \mathcal{F}\}$ covers Ψ.

We know that even if Φ and Ψ are compact, let alone being totally bounded, X, G, Y, and H need not be compact [6], though $\mathcal{F}^{\mathrm{all}}_{T}$ is indeed compact in that case. Moreover, an arbitrary subspace $\mathcal{F}$ of $\mathcal{F}^{\mathrm{all}}_{T}$ need not necessarily be compact either. Since compactness is an important property, as it provides us with essential guarantees in machine learning context, it is natural to prefer compact spaces in practical applications. We therefore ask: If compactness of X, G, Y, and H is not guaranteed even by the compactness of data sets Φ and Ψ, let alone their total boundedness, can we at least prove that these spaces can be isometrically and densely embedded in compact ones while the corresponding sought after compact spaces preserve the former mutual relations between the original spaces? That is, can we find compactifications of perception pairs? Furthermore, can we find compactifications of the spaces of GENEOs? These notions need being made precise, which we do in the sequel, and prove that our assumptions are sufficient to grant the answer to this question in the affirmative.

Somewhat formally, *given* the perception pairs (Φ, G), $\mathrm{dom}(\Phi) = X$ and (Ψ, H), $\mathrm{dom}(\Psi) = Y$ and a space $\mathcal{F} \subseteq \mathcal{F}_T^{\mathrm{all}}$ of GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$ with respect to a fixed homomorphism $T : G \to H$, we assume that the data sets Φ and Ψ are totally bounded and rich enough to endow X and Y with metric structures, and the collection $\{F(\Phi) \mid F \in \mathcal{F}\}$ covers the space Ψ. Under these assumptions, we *find* perception pairs (Φ^*, G^*), $\mathrm{dom}(\Phi^*) = X^*$ and (Ψ^*, H^*), $\mathrm{dom}(\Psi^*) = Y^*$, a space $\mathcal{F}^* \subseteq \mathcal{F}_{T^*}^{\mathrm{all}}$ of GENEOs $(F^*, T^*) : (\Phi^*, G^*) \to (\Psi^*, H^*)$ with respect to a fixed homomorphism $T^* : G^* \to H^*$, and isometric embeddings $j_1 : X \to X^*$, $j_2 : Y \to Y^*$, $i_1 : \Phi \to \Phi^*$, $i_2 : \Psi \to \Psi^*$, $k_1 : G \to G^*$, $k_2 : H \to H^*$, and $f : \mathcal{F} \to \mathcal{F}^*$. We *require* that the spaces $\Phi^*, G^*, X^*, \Psi^*, H^*, Y^*$ and $\mathcal{F}^*$ are all compact, and the following commutativity conditions are satisfied: $i_1(\varphi) \circ j_1 = \varphi$ for every $\varphi \in \Phi$, $i_2(\psi) \circ j_2 = \psi$ for every $\psi \in \Psi$; $k_1(g) \circ j_1 = j_1 \circ g$ for every $g \in G$, $k_2(h) \circ j_2 = j_2 \circ h$ for every $h \in H$; $i_2 \circ F = f(F) \circ i_1$ for every $F \in \mathcal{F}$; and $k_2 \circ T = T^* \circ k_1$.

These compatibility conditions formalize the requirement that the spaces Φ, G, X, Ψ, H, Y and $\mathcal{F}$ do not lose any of their metric or topological properties while being viewed as subspaces of Φ^*, G^*, X^*, Ψ^*, H^*, Y^* and $\mathcal{F}^*$ respectively. In this case, we say that (Φ^*, G^*), $\mathrm{dom}(\Phi^*) = X^*$ is a compactification of the perception pair (Φ, G), $\mathrm{dom}(\Phi) = X$, and $\mathcal{F}^*$ is a compactification of the space $\mathcal{F}$ of GENEOs. We will give formal definitions in the forthcoming sections. Our assumptions here are mild; in many practical applications, they are already satisfied.

Precisely, the intermediary results and constructions in Subsections 4.1 and 4.2 are aimed at proving that every perception pair (Φ, G), $\mathrm{dom}(\Phi) = X$, with totally bounded Φ endowing X with a metric structure, admits a compactification (Φ^*, G^*), $\mathrm{dom}(\Phi^*) = X^*$. Similarly, the Subsections 4.3 and 4.4 are devoted to proving that every space $\mathcal{F} \subseteq \mathcal{F}_T^{\mathrm{all}}$ of GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$ with $\mathrm{dom}(\Phi) = X$ and $\mathrm{dom}(\Psi) = Y$ such that the collection $\{F(\Phi) \mid F \in \mathcal{F}\}$ covers Ψ admits a compactification $\mathcal{F}^*$, provided the data sets Φ and Ψ are totally bounded and endow X and Y with metric structures. Again, this proof will require several auxiliary constructions and corresponding results.

In order to set the stage for the requisite compactification of the perception pair (Φ, G), $\mathrm{dom}(\Phi) = X$, we consider the unique metric completion $\hat{X}$ of X, and assume that $X \subseteq \hat{X}$. Since X is totally bounded by Theorem 2.2.2, $\hat{X}$ is totally bounded by virtue of the isometric embedding $j : X \to \hat{X}$, and hence compact. This serves as the sought after X^* in our construction. Then we use the measurements $\varphi \in \Phi$ and isometries $g \in G$ to define the measurements $\hat{\varphi} : \Phi \to \mathbb{R}$

on the compact space $\hat{X}$ and its corresponding isometries $\hat{g} : \hat{X} \to \hat{X}$.

3.2 The Extension of Signals

Let us consider a perception pair (Φ, G), $\mathrm{dom}(\Phi) = X$, with totally bounded Φ endowing X with a metric structure, and denote the unique metric completion of X by $\hat{X}$. We can assume without loss of generality that $X \subseteq \hat{X}$. There is a wealth of results in analysis allowing one to extend specific real-valued functions from X to $\hat{X}$. With suitable modifications, they can be adapted to our mathematical setting and one can easily prove

Proposition 3.2.1. *Let (M, d_M) be a metric space, and S a subset of M. Then every non-expansive map $f\colon S \to \mathbb{R}$ admits a unique non-expansive extension $\bar{f}\colon \overline{S} \to \mathbb{R}$.*

Since each $\varphi \in \Phi$ is non-expansive by Proposition 2.2.1, we have

Corollary 3.2.1. *Each signal $\varphi \in \Phi$ can be uniquely extended to a non-expansive signal $\hat{\varphi}\colon \hat{X} \to \mathbb{R}$, where $\hat{X}$ is the completion of $X = \mathrm{dom}(\Phi)$, by setting*

$$\hat{\varphi}(\hat{x}) = \lim_{n\to\infty} \varphi(x_n),$$

for any arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to $\hat{x} \in \hat{X}$.

Let us put

$$\hat{\Phi} := \{\hat{\varphi} : \hat{X} \to \mathbb{R} \mid \varphi \in \Phi\}.$$

Since the extensions $\hat{\varphi}\colon \hat{X} \to \mathbb{R}$ of signals $\varphi \in \Phi$ are unique, we get a one-to-one correspondence $i\colon \Phi \to \hat{\Phi}$ between signals in Φ and signals in $\hat{\Phi}$ given by

$$\varphi \mapsto \hat{\varphi}.$$

The notations $\mathbb{R}_b^{\hat{X}}$, $\mathrm{Iso}(\hat{X})$, $\hat{d}_\infty$ and $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ are self-explanatory. Clearly, $\hat{\Phi} \subseteq \mathbb{R}_b^{\hat{X}}$.

The set $\hat{\Phi}$ of extended signals induces the pseudo-metric $D_{\hat{X}}$ on the completion $\hat{X}$ given by

$$D_{\hat{X}}(x, y) := \sup_{\varphi\in\hat{\Phi}} |\varphi(x) - \varphi(y)|\,, \quad x, y \in \hat{X}.$$

At this point, the completion $\hat{X}$ appears to be equipped with the previously defined metric $\hat{D}_{\hat{X}}$ associated with the completion, and the pseudo-metric $D_{\hat{X}}$. It is worth investigating their relationship. We will prove later in this section that these seemingly distinct functions are in fact numerically equal on $\hat{X}$, thereby establishing in addition that $D_{\hat{X}}$ is in fact a metric.

Before proceeding, we record another general proposition, omitting the easy proof, which will be used frequently in the sequel:

Proposition 3.2.2. *Let K be a compact topological space and A be dense in K. If f is a continuous real-valued function on K, then*

$$\sup f(K) = \sup f(A).$$

We are now ready to prove the following theorem:

Theorem 3.2.1. *The correspondence $i\colon \Phi \to \hat{\Phi}$ defined by $\varphi \mapsto \hat{\varphi}$, for every $\varphi \in \Phi$, is an isometry.*

Proof. The map i is surjective by construction; it will suffice to prove that it preserves distances, i.e.,

$$\|i(\varphi_1) - i(\varphi_2)\|_\infty = \|\varphi_1 - \varphi_2\|_\infty,$$

for any $\varphi_1, \varphi_2 \in \Phi$. Since X is dense in the compact topological space $\hat{X}$, and $i(\varphi) = \hat{\varphi}$ is an extension of $\varphi \in \Phi$, by Proposition 3.2.2 we have

$$\begin{aligned}
\|i(\varphi_1) - i(\varphi_2)\|_\infty &:= \sup_{x \in \hat{X}} |\widehat{\varphi_1}(x) - \widehat{\varphi_2}(x)| \\
&= \sup_{x \in X} |\widehat{\varphi_1}(x) - \widehat{\varphi_2}(x)| \\
&= \sup_{x \in X} |\varphi_1(x) - \varphi_2(x)| \\
&=: \|\varphi_1 - \varphi_2\|_\infty,
\end{aligned}$$

for any $\varphi_1, \varphi_2 \in \Phi$. Therefore, i is an isometry. $\square$

Corollary 3.2.2. *The space $\hat{\Phi} := \{\hat{\varphi} : \hat{X} \to \mathbb{R} \mid \varphi \in \Phi\}$ of extended signals is totally bounded.*

Remark 3.2.1. *Let us consider the isometry $i : \Phi \to \hat{\Phi}$ and the inclusion $j : X \hookrightarrow \hat{X}$. Since $\hat{\varphi}$ extends φ, the following natural commutativity condition holds* *for each $\varphi \in \Phi$:*

$$i(\varphi) \circ j = \varphi.$$

The total boundedness of $\hat{\Phi}$ allows us to prove the following crucial statement.

Proposition 3.2.3. *On $\hat{X}$, $D_{\hat{X}} = \hat{D}_{\hat{X}}$.*

Proof. Since $\hat{\Phi}$ is totally bounded, $D_{\hat{X}}$ induces on $\hat{X}$ the initial topology with respect to $\hat{\Phi}$ by Theorem 2.2.1. Moreover, by Corollary 3.2.1 the functions in $\hat{\Phi}$ are continuous with respect to $\hat{D}_{\hat{X}}$ as well. Hence, the topology induced by $\hat{D}_{\hat{X}}$ is finer than the topology induced by $D_{\hat{X}}$. This directly implies that $D_{\hat{X}}$ is a continuous function with respect to $\hat{D}_{\hat{X}}$. Then, we have

$$\hat{D}_{\hat{X}}(\hat{x}, \hat{y}) := \lim_{n\to\infty} D_X(x_n, y_n) = \lim_{n\to\infty} D_{\hat{X}}(x_n, y_n) = D_{\hat{X}}(\hat{x}, \hat{y})$$

where $\hat{x}, \hat{y} \in \hat{X}$, and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are sequences in X converging to $\hat{x}$ and $\hat{y}$ respectively, with reference to the topology induced by $\hat{D}_{\hat{X}}$. $\square$

So, $D_{\hat{X}}$ is a metric. As pointed out in Section 2.1, this directly implies that the pseudo-metric $\hat{D}_{\mathrm{Aut}}$ induced by $\hat{\Phi}$ on $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ is also a metric.

3.3 The Isometries of the Completions

Perception pairs are made up of a data set along with the corresponding equivariance groups. Given a perception pair (Φ, G), $\mathrm{dom}(\Phi) = X$ with totally bounded Φ endowing X with a metric structure, we seek to construct a compact perception pair (Φ^*, G^*), $\mathrm{dom}(\Phi^*) = X^*$. The metric completion $\hat{X}$ of X already provides us with the requisite compactification X^*. We have constructed an intermediary data set $\hat{\Phi}$ and now we turn our attention to the construction of an auxiliary topological group $\hat{G} \subseteq \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$, isometric to the given group $G \subseteq \mathrm{Aut}_{\Phi}(X)$, whose closure $G^* = \overline{\hat{G}}$ in the compact space $\mathrm{Iso}(\hat{X})$ finally serves our purposes.

3.3.1 The Induced Bijections

Let us consider as before a perception pair (Φ, G), $\mathrm{dom}(\Phi) = X$ with totally bounded Φ endowing X with a metric structure. Each $g \in G \subseteq \mathrm{Aut}_{\Phi}(X) \subseteq \mathrm{Iso}(X)$ induces a self-map $\hat{g} : \hat{X} \to \hat{X}$ on the metric completion $\hat{X}$ by the following association:

$$\hat{g}(\hat{x}) := \lim_{n\to\infty} g(x_n),$$

where $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to $\hat{x} \in \hat{X}$. The sequence $(g(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence since g is an isometry by Proposition 2.2.2. Also, if $(y_n)_{n\in\mathbb{N}}$ is another sequence in X converging to $\hat{x}$, then as g is an isometry, we have

$$0 = \lim_{n\to\infty} D_X(x_n, y_n) = \lim_{n\to\infty} D_X(g(x_n), g(y_n)),$$

whence

$$g(y_n) \to \lim_{n\to\infty} g(x_n)$$

as well. So, the function $\hat{x} \mapsto \lim_{n\to\infty} g(x_n)$ is well defined.

Moreover, note that $\hat{g}|_X = g$.

Proposition 3.3.1. *The map $\hat{g} : \hat{X} \to \hat{X}$ defined by setting*

$$\hat{g}(\hat{x}) := \lim_{n\to\infty} g(x_n),$$

where $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to $\hat{x} \in \hat{X}$, is bijective for every $g \in G$, and $\hat{g}^{-1} = \widehat{g^{-1}}$.

Proof. Let $\hat{x}_1, \hat{x}_2 \in \hat{X}$ with $\hat{x}_1 \neq \hat{x}_2$, and $(x_{1,n})_{n\in\mathbb{N}}$ and $(x_{2,n})_{n\in\mathbb{N}}$ be sequences in X converging respectively to $\hat{x}_1$ and $\hat{x}_2$. As g is an isometry,

$$\begin{aligned} 0 \neq \hat{D}_{\hat{X}}(\hat{x}_1, \hat{x}_2) &:= \lim_{n\to\infty} D_X(x_{1,n}, x_{2,n}) \\ &= \lim_{n\to\infty} D_X(g(x_{1,n}), g(x_{2,n})) \\ &=: \hat{D}_{\hat{X}}(\hat{g}(\hat{x}_1), \hat{g}(\hat{x}_2)), \end{aligned}$$

whence $\hat{g}(\hat{x}_1) \neq \hat{g}(\hat{x}_2)$ and $\hat{g}$ is injective.

As for surjectivity, let $g \in G$. If $\hat{y} \in \hat{X}$, then there is a sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $y_n \to \hat{y}$ in $\hat{X}$. As g^{-1} exists, we can put $x_n := g^{-1}(y_n)$, for each $n \in \mathbb{N}$. Since g^{-1} is an isometry and $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, $(x_n)_{n\in\mathbb{N}}$ too is a Cauchy sequence; so it converges to some $\hat{x} \in \hat{X}$. Of course, $\hat{g}(\hat{x}) := \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} y_n = \hat{y}$, and hence $\hat{g}$ is surjective.

Also, the equality $\hat{g}(\hat{x}) = \hat{y}$ just proved can be rewritten as

$$\hat{g}^{-1}(\hat{y}) = \hat{x}.$$

But at the same time, as $g^{-1} \in G$, by definition we have

$$\widehat{g^{-1}}(\hat{y}) = \widehat{g^{-1}}(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} g^{-1}(y_n) = \lim_{n\to\infty} x_n = \hat{x}.$$

By the arbitrariness of $\hat{y} \in \hat{X}$, we get

$$\hat{g}^{-1} = \widehat{g^{-1}}.$$

□

The following important property will be used frequently in the sequel.

Proposition 3.3.2. *For each $\varphi \in \Phi$ and each $g \in G$,*

$$\hat{\varphi}\hat{g} = \widehat{\varphi g}.$$

Proof. As $g \in \mathrm{Aut}_\Phi(X)$, $\varphi g \in \Phi$. So, if $\hat{x} \in \hat{X}$ and $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to $\hat{x}$, we compute:

$$\begin{aligned}
\hat{\varphi}\hat{g}(\hat{x}) &= \hat{\varphi}(\hat{g}(\hat{x})) \\
&= \hat{\varphi}(\lim_{n\to\infty} g(x_n)) \\
&= \lim_{n\to\infty} \varphi(g(x_n)) \\
&= \lim_{n\to\infty} \varphi g(x_n) \\
&= \widehat{\varphi g}(\lim_{n\to\infty} x_n) \\
&= \widehat{\varphi g}(\hat{x}).
\end{aligned}$$

By the arbitrariness of $\hat{x}$, we have the proposed equality. □

Corollary 3.3.1. *For each $g \in G$, $\hat{g} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X}) \subseteq \mathrm{Iso}(\hat{X})$.*

Proof. Let $\hat{\varphi} \in \hat{\Phi}$. As $g \in G \subseteq \mathrm{Aut}_\Phi(X)$, $\varphi g \in \Phi$; so, $\hat{\varphi}\hat{g} = \widehat{\varphi g} \in \hat{\Phi}$ by Proposition 3.3.2; whence $\hat{g}$ is a $\hat{\Phi}-$operation. Since $\hat{g}^{-1} = \widehat{g^{-1}}$ (Proposition 3.3.1), by applying Proposition 3.3.2 again to $g^{-1} \in G$, we infer that $\hat{g}^{-1}$ is a $\hat{\Phi}-$operation too; whence $\hat{g} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$.

The inclusion $\mathrm{Aut}_{\hat{\Phi}}(\hat{X}) \subseteq \mathrm{Iso}(\hat{X})$ is stated in Proposition 2.2.2. □

Proposition 3.3.3. *For each $g, h \in G$,*

$$\hat{g}\hat{h} = \widehat{gh}.$$

Proof. Let $\hat{x} \in \hat{X}$, and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to $\hat{x}$ in $\hat{X}$. Then, by recalling the definitions of $\hat{h}$ and $\hat{g}$, we have

$$\begin{aligned}\hat{g}\hat{h}(\hat{x}) &= \hat{g}(\hat{h}(\hat{x})) \\ &= \hat{g}(\lim_{n\to\infty} h(x_n)) \\ &= \lim_{n\to\infty} g(h(x_n)) \\ &= \lim_{n\to\infty} gh(x_n) \\ &= \widehat{gh}(\hat{x}),\end{aligned}$$

whence by the arbitrariness of $\hat{x}$, the proposition is proved. □

Let us put

$$\hat{G} := \{\hat{g} : \hat{X} \to \hat{X} \mid g \in G\}$$

.

Remark 3.3.1. *Clearly,* $\widehat{\mathrm{id}_X} = \mathrm{id}_{\hat{X}} \in \hat{G}$.

Corollary 3.3.2. *Let* (Φ, G), $\mathrm{dom}(\Phi) = X$ *be a perception pair with totally bounded* Φ *endowing* X *with a metric structure. Then the set* $\hat{G} := \{\hat{g} : \hat{X} \to \hat{X} \mid g \in G\}$ *is a subgroup of* $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$.

Proof. It will suffice to show that $\hat{G}$ is closed under composition and computation of the inverse. The first property follows from Proposition 3.3.3, since if $\hat{g}, \hat{h} \in \hat{G}$, then $\hat{g}\hat{h} = \widehat{gh} \in \hat{G}$. The second property follows from Proposition 3.3.1, since if $\hat{g} \in \hat{G}$, then $\hat{g}^{-1} = \widehat{g^{-1}} \in \hat{G}$. □

Remark 3.3.2. *Corollary 3.3.2 implicitly states that* $(\hat{\Phi}, \hat{G})$ *is a perception pair.*

Note that $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$, and therefore $\hat{G} \subseteq \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$, are pseudo-metric spaces with the pseudo-metric $\hat{D}_{\mathrm{Aut}} : \mathrm{Aut}_{\hat{\Phi}}(\hat{X}) \times \mathrm{Aut}_{\hat{\Phi}}(\hat{X}) \to \mathbb{R}$ given by

$$\hat{D}_{\mathrm{Aut}}(\hat{g}, \hat{h}) := \sup_{\hat{\varphi}\in\hat{\Phi}} D_{\hat{\Phi}}(\hat{\varphi}\hat{g}, \hat{\varphi}\hat{h}), \text{ for every } \hat{g}, \hat{h} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X}).$$

Moreover,

$$\begin{aligned}
\hat{D}_{\mathrm{Aut}}(\hat{g},\hat{h}) &:= \sup_{\hat{\varphi}\in\hat{\Phi}} D_{\hat{\Phi}}(\hat{\varphi}\hat{g},\hat{\varphi}\hat{h}) \\
&= \sup_{\hat{\varphi}\in\hat{\Phi}} \sup_{\hat{x}\in\hat{X}} \left|\hat{\varphi}(\hat{g}(\hat{x})) - \hat{\varphi}(\hat{h}(\hat{x}))\right| \\
&= \sup_{\hat{x}\in\hat{X}} \sup_{\hat{\varphi}\in\hat{\Phi}} \left|\hat{\varphi}(\hat{g}(\hat{x})) - \hat{\varphi}(\hat{h}(\hat{x}))\right| \\
&= \sup_{\hat{x}\in\hat{X}} D_{\hat{X}}(\hat{g}(\hat{x}),\hat{h}(\hat{x})).
\end{aligned}$$

Therefore, if $\hat{g},\hat{h} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ and $\hat{D}_{\mathrm{Aut}}(\hat{g},\hat{h}) = 0$, then $D_{\hat{X}}(\hat{g}(\hat{x}),\hat{h}(\hat{x})) = 0$ for every $\hat{x} \in \hat{X}$. Since $\hat{\Phi}$ endows $\hat{X}$ with the metric structure induced by the coinciding metrics $D_{\hat{X}}$ and $\hat{D}_{\hat{X}}$ (Proposition 3.2.3), $\hat{g}(\hat{x}) = \hat{h}(\hat{x})$ for every $\hat{x} \in \hat{X}$, and hence $\hat{g} = \hat{h}$. It follows that $\hat{G}$, and therefore $\hat{G} \subseteq \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$, are metric spaces.

Proposition 3.3.4. *The correspondence $k : G \to \hat{G}$ given by $k(g) := \hat{g}$ is an isometry.*

Proof. The map k is injective: If $g,h \in G$ differ at some $x \in X$, $\hat{g}(x) \neq \hat{h}(x)$ as well, and $k(g) = \hat{g} \neq \hat{h} = k(h)$. Also, the definition of $\hat{G}$ immediately implies that k is surjective.

We show that k preserves distances. By Corollary 3.3.1, the real-valued function $f : \hat{X} \to \mathbb{R}$ defined by setting

$$f(\hat{x}) := D_{\hat{X}}(\hat{g}(\hat{x}),\hat{h}(\hat{x})), \text{ for every } \hat{x} \in \hat{X},$$

is continuous for every $\hat{g}$ and $\hat{h}$ in $\hat{G}$, since each isometry is by definition a continuous map.

Let $\hat{g},\hat{h} \in \hat{G}$; then by Propositions 3.2.2, we have

$$\begin{aligned}
\hat{D}_{\mathrm{Aut}}(\hat{g},\hat{h}) &= \sup_{\hat{x}\in\hat{X}} D_{\hat{X}}(\hat{g}(\hat{x}),\hat{h}(\hat{x})) \\
&= \sup_{x\in X} D_{\hat{X}}(\hat{g}(x),\hat{h}(x)) \\
&= \sup_{x\in X} D_{X}(g(x),h(x)) \\
&=: D_{\mathrm{Aut}}(g,h),
\end{aligned}$$

as required. □

Corollary 3.3.3. *If G is complete, then $\hat{G}$ is compact.*

Proof. Recall that the space Φ of admissible signals was assumed to be totally bounded; whence by Theorem 2.2.4, G is totally bounded, and being complete by hypothesis, it is compact. As $k : G \to \hat{G}$ is an isometry, $\hat{G}$ is compact as well. □

Remark 3.3.3. *The assumption here that G is complete cannot be removed. It is easy to give an example of a perception pair (Φ, G) where Φ is compact but G is not complete [6]. For example, if Φ is the compact space of all 1-Lipschitz functions from $X = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ to $[0, 1]$, and G is the group of all rotations $\rho_{2\pi q}$ of X of $2\pi q$ radians with q a rational number, then the topological group G is not complete. Moreover, in this case $\hat{X} = X$, and the topological group $\hat{G} = G$ is not compact either.*

It is easy to see that the embeddings $j : X \to \hat{X}$ and $k : G \to \hat{G}$ satisfy the following natural commutativity condition.

Proposition 3.3.5. *For each $g \in G$,*

$$k(g) \circ j = j \circ g.$$

That is, for each $g \in G$ and each $x \in X$,

$$\hat{g}(x) = \widehat{g(x)}.$$

Remark 3.3.4. *We observe that $\hat{g}$ is the only map in $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ with $\hat{g}|_X = g$. It is indeed easy to show that for any $\overline{g} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ such that $\overline{g}|_X = g$, the equality $\overline{g}(\hat{x}) = \hat{g}(\hat{x})$ holds for every $\hat{x} \in \hat{X}$.*

Before proceeding, we stress that while $\hat{X}$, by definition, is a complete topological space, the topological spaces $\hat{\Phi}$ and $\hat{G}$, in general, are not complete.

3.3.2 The Embedding of Perception Pairs

We are now ready to show that every perception pair (Φ, G), $\mathrm{dom}(\Phi) = X$ can be embedded in a compact perception pair (Φ^*, G^*), $\mathrm{dom}(\Phi^*) = X^*$, provided that the space Φ of signals is totally bounded and (X, D_X) is a metric space.

We have so far obtained only an isometric image $\hat{G}$ of G. The group G is chosen arbitrarily; so G and $\hat{G}$ may or may not be closed in $\mathrm{Aut}_\Phi(X)$ and $\widehat{\mathrm{Aut}_\Phi(X)}$ respectively. Similarly, the space $\hat{\Phi}$, being isometric to Φ, need not necessarily be compact. However, the space $C(\hat{X},\mathbb{R})$ is complete; so, $\overline{\hat{\Phi}}$, the closure of $\hat{\Phi}$ in $C(\hat{X},\mathbb{R})$, being closed, is complete as well. Also, $\hat{\Phi}$, being isometric to the totally bounded space Φ (Theorem 3.2.1), is totally bounded, and so is its closure. Consequently,

Proposition 3.3.6. *Let* (Φ, G), $\mathrm{dom}(\Phi) = X$ *be a perception pair with totally bounded* Φ *endowing* X *with a metric structure. Then the metric space* $\overline{\hat{\Phi}} \subseteq \mathbb{R}_b^{\hat{X}}$ *is compact.*

The data set $\overline{\hat{\Phi}}$ endows $\hat{X}$ with a pseudo-metric structure where the pseudo-distance is given by

$$\overline{D}_{\hat{X}}(\hat{x}_1,\hat{x}_2) := \sup_{\overline{\varphi}\in\overline{\hat{\Phi}}} \left|\overline{\varphi}(\hat{x}_1) - \overline{\varphi}(\hat{x}_2)\right|, \text{ for every } \hat{x}_1,\hat{x}_2 \in \hat{X}.$$

Proposition 3.3.7. *On* $\hat{X}$, $\overline{D}_{\hat{X}} = D_{\hat{X}}$*; so* $\overline{D}_{\hat{X}}$ *is a metric.*

Proof. Let $\hat{x}_1,\hat{x}_2 \in \hat{X}$. By applying Proposition 3.2.2 to the continuous function $f(\overline{\varphi}) := |\overline{\varphi}(\hat{x}_1) - \overline{\varphi}(\hat{x}_2)|$, we get:

$$\begin{aligned}\overline{D}_{\hat{X}}(\hat{x}_1,\hat{x}_2) &:= \sup_{\overline{\varphi}\in\overline{\hat{\Phi}}} |\overline{\varphi}(\hat{x}_1) - \overline{\varphi}(\hat{x}_2)| \\ &= \sup_{\hat{\varphi}\in\hat{\Phi}} |\hat{\varphi}(\hat{x}_1) - \hat{\varphi}(\hat{x}_2)| \\ &=: D_{\hat{X}}(\hat{x}_1,\hat{x}_2).\end{aligned}$$

As $\hat{x}_1,\hat{x}_2 \in \hat{X}$ are arbitrary, we have the proposed equality. □

In view of Proposition 3.3.7, the symbol $\mathrm{Iso}(\hat{X})$ can be used without any ambiguity about the underlying metric.

By Theorem 2.2.3, the set $\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X}) \subseteq \mathrm{Iso}(\hat{X})$ of all invertible $\overline{\hat{\Phi}}$-operations is a topological group with respect to the topology induced by the pseudo-distance $\overline{\hat{D}}_{\mathrm{Aut}} : \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X}) \times \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X}) \to \mathbb{R}$:

$$\overline{\hat{D}}_{\mathrm{Aut}}(\overline{g},\overline{h}) := \sup_{\overline{\varphi}\in\overline{\hat{\Phi}}} D_{\overline{\hat{\Phi}}}(\overline{\varphi}\circ\overline{g}, \overline{\varphi}\circ\overline{h}), \text{ for every } \overline{g},\overline{h} \in \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X}).$$

If $\overline{g}, \overline{h} \in \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$, then

$$
\begin{aligned}
\overline{\hat{D}}_{\mathrm{Aut}}(\overline{g}, \overline{h}) &= \sup_{\overline{\varphi} \in \overline{\hat{\Phi}}} D_{\overline{\hat{\Phi}}}(\overline{\varphi} \circ \overline{g}, \overline{\varphi} \circ \overline{h}) \\
&= \sup_{\overline{\varphi} \in \overline{\hat{\Phi}}} \sup_{\hat{x} \in \hat{X}} \left| \overline{\varphi}(\overline{g}(\hat{x})) - \overline{\varphi}(\overline{h}(\hat{x})) \right| \\
&= \sup_{\hat{x} \in \hat{X}} \overline{D}_{\hat{X}}(\overline{g}(\hat{x}), \overline{h}(\hat{x})) \\
&= \sup_{\hat{x} \in \hat{X}} D_{\hat{X}}(\overline{g}(\hat{x}), \overline{h}(\hat{x})).
\end{aligned}
$$

Therefore, if $\overline{\hat{D}}_{\mathrm{Aut}}(\overline{g}, \overline{h}) = 0$, then $D_{\hat{X}}(\overline{g}(\hat{x}), \overline{h}(\hat{x})) = 0$ for every $\hat{x} \in \hat{X}$. Since $D_{\hat{X}}$ is a metric, $\overline{g}(\hat{x}) = \overline{h}(\hat{x})$ for every $\hat{x} \in \hat{X}$, whence $\overline{g} = \overline{h}$. So, $(\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X}), \overline{\hat{D}}_{\mathrm{Aut}})$ is a metric space.

Proposition 3.3.8. *Every $\check{g} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ is a $\overline{\hat{\Phi}}-$operation.*

Proof. Let $\overline{\varphi} \in \overline{\hat{\Phi}}$ and $\check{g} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$. We show that $\overline{\varphi}\check{g} \in \overline{\hat{\Phi}}$.

There is a sequence $(\hat{\varphi}_n)_{n \in \mathbb{N}}$ in $\hat{\Phi}$ such that $\hat{\varphi}_n \to \overline{\varphi}$. As $\check{g}$ is a bijection of $\hat{X}$ by Proposition 2.2.2, we have

$$
\|\hat{\varphi}_n \check{g} - \overline{\varphi}\check{g}\|_\infty = \|\hat{\varphi}_n - \overline{\varphi}\|_\infty;
$$

whence $\hat{\varphi}_n \check{g} \to \overline{\varphi}\check{g}$ in the space $C(\hat{X}, \mathbb{R})$. As $\check{g}$ is a $\hat{\Phi}-$operation, $(\hat{\varphi}_n \check{g})_{n \in \mathbb{N}}$ is a sequence in the space $\hat{\Phi} \subseteq C(\hat{X}, \mathbb{R})$. Consequently, $\overline{\varphi}\check{g} \in \overline{\hat{\Phi}}$. □

We conclude

Corollary 3.3.4. *Given a perception pair* (Φ, G)*,* $\mathrm{dom}(\Phi) = X$ *with totally bounded* Φ *endowing* X *with a metric structure, we have* $\hat{G} \subseteq \mathrm{Aut}_{\hat{\Phi}}(\hat{X}) \subseteq \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$*.*

Proof. The first inclusion is given by Corollary 3.3.2. As for the second, let $\check{g} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$. By Proposition 3.3.8, $\check{g}$ is a $\overline{\hat{\Phi}}-$operation. As $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ is a group, $\check{g}^{-1} \in \mathrm{Aut}_{\hat{\Phi}}(\hat{X})$, and again by Proposition 3.3.8, is a $\overline{\hat{\Phi}}-$operation. Consequently, $\check{g} \in \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$, and by the arbitrariness of $\check{g}$, we have the second inclusion. □

Remark 3.3.5. *Corollary 3.3.4 implicitly states that* $(\overline{\hat{\Phi}}, \hat{G})$ *is a perception pair.*

Let $\overline{\hat{G}}$ and $\overline{\mathrm{Aut}_{\hat{\Phi}}(\hat{X})}$ respectively denote the closures of $\hat{G}$ and $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ in the space $\mathrm{Iso}(\hat{X})$ of all isometries of $(\hat{X}, D_{\hat{X}})$. Recall that the topology on $\mathrm{Iso}(\hat{X})$ is given by the restriction to $\mathrm{Iso}(\hat{X})$ of the metric $\hat{d}_\infty$ defined on $C(\hat{X}, \hat{X})$ by setting

$\hat{d}_\infty(f,g) := \sup_{\hat{x}\in\hat{X}} D_{\hat{X}}(f(\hat{x}), g(\hat{x}))$, for every $f, g \in C(\hat{X}, \hat{X})$. Note also that the isometries of $\hat{X}$ with respect to the metric $D_{\hat{X}}$ coincide with those induced by the metric $\overline{D}_{\hat{X}}$ (Proposition 3.3.7).

Corollary 3.3.5. $\overline{\hat{G}} \subseteq \overline{\mathrm{Aut}_{\hat{\Phi}}(\hat{X})} \subseteq \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X}) \subseteq \mathrm{Iso}(\hat{X})$.

Proof. The last inclusion is given by Proposition 2.2.2. As we have seen at the beginning of Section 3.1, $\hat{X}$ is compact, and $\overline{\hat{\Phi}}$ is compact by Proposition 3.3.6. It follows from Theorem 2.2.5 that $\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$ is a closed subspace of $\mathrm{Iso}(\hat{X})$. Since $\hat{G} \subseteq \mathrm{Aut}_{\hat{\Phi}}(\hat{X}) \subseteq \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$ (Corollary 3.3.4), we have the first two inclusions.

□

Remark 3.3.6. *Corollary 3.3.5 directly implies that the closure of $\hat{G}$ in $\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$ coincides with $\overline{\hat{G}}$, i.e., with the closure of $\hat{G}$ in the space $\mathrm{Iso}(\hat{X})$. Similarly the closure of $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ in $\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$ coincides with $\overline{\mathrm{Aut}_{\hat{\Phi}}(\hat{X})}$.*

Incidentally, Proposition 2.2.4 and Corollary 3.3.5 also give that the spaces $\overline{\hat{G}}$ and $\overline{\mathrm{Aut}_{\hat{\Phi}}(\hat{X})}$ are compact.

Proposition 3.3.9. *The groups $\hat{G}$ and $\overline{\hat{G}}$ are both topological subgroups of the compact group $\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$.*

Proof. By Corollary 3.3.5, we have $\hat{G} \subseteq \overline{\hat{G}} \subseteq \mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$. By Theorem 2.2.3, $\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$ is a topological group. So, $\hat{G}$ and $\overline{\hat{G}}$, being subgroups of a topological group are likewise topological. The compactness of $\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$ is given by Theorem 2.2.5. □

We can now state

Theorem 3.3.1. *Given any perception pair (Φ, G), $\mathrm{dom}(\Phi) = X$ with totally bounded Φ endowing X with a metric structure, the perception pair $(\overline{\hat{\Phi}}, \overline{\hat{G}})$, $\mathrm{dom}(\overline{\hat{\Phi}}) = \hat{X}$ is compact.*

Proof. Propositions 3.3.6 and 3.3.9 together give the assertion. □

Definition 3.3.1. *We say that the perception pair (Φ, G) with $\mathrm{dom}(\Phi) = X$ is isometrically embedded into the perception pair (Φ^*, G^*) with $\mathrm{dom}(\Phi^*) = X^*$ if there are isometric embeddings $j^* : X \to X^*$, $i^* : \Phi \to \Phi^*$, and $k^* : G \to G^*$ such that the images $j^*(X)$, $i^*(\Phi)$, and $k^*(G)$ are all dense in X^*, Φ^*, and G^* respectively, and the following commutativity conditions are satisfied: $i^*(\varphi) \circ j^* = \varphi$*

for every $\varphi \in \Phi$ *and* $k^*(g) \circ j^* = j^* \circ g$ *for every* $g \in G$*. If* (Φ^*, G^*) *is compact, it is said to be a compactification of* (Φ, G)*.*

With this definition at our disposal, we summarize

Theorem 3.3.2. *Every perception pair* (Φ, G)*,* $\mathrm{dom}(\Phi) = X$*, with totally bounded* Φ *endowing* X *with a metric structure, admits a compactification* (Φ^*, G^*)*, with* $\mathrm{dom}(\Phi^*) = X^*$*.*

Proof. Put $\Phi^* := \overline{\hat{\Phi}}$ and $G^* := \overline{\hat{G}}$ in Theorem 3.3.1. □

3.4 GENEOs and Completions

Our next goal is to construct compactifications of the spaces $\mathcal{F} \subseteq \mathcal{F}_T^{\mathrm{all}}$ of GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$ with the property that the images $F(\Phi), F \in \mathcal{F}$ form a cover for the data set Ψ, while maintaining the assumptions that Φ and Ψ are totally bounded and endow X and Y with metric structures. We have shown, under these assumptions, that the perception pairs (Φ, G), $\mathrm{dom}(\Phi) = X$ and (Ψ, H), $\mathrm{dom}(\Psi) = Y$ can be embedded nicely into the perception pairs $(\hat{\Phi}, \hat{G})$, $\mathrm{dom}(\hat{\Phi}) = \hat{X}$ and $(\hat{\Psi}, \hat{H})$, $\mathrm{dom}(\hat{\Psi}) = \hat{Y}$, respectively, through the compatible isometries $(j_1, j_2) : (X, Y) \to (\hat{X}, \hat{Y})$, $(i_1, i_2) : (\Phi, \Psi) \to (\hat{\Phi}, \hat{\Psi})$, and $(k_1, k_2) : (G, H) \to (\hat{G}, \hat{H})$. We are now in a position to use the GENEOs $(F, T) : (\Phi, G) \to (\Psi, H)$ to define new GENEOs $(\hat{F}, \hat{T}) : (\hat{\Phi}, \hat{G}) \to (\hat{\Psi}, \hat{H})$. Our construction will be further extended to the GENEOs $(\overline{\hat{F}}, \hat{T}) : (\overline{\hat{\Phi}}, \hat{G}) \to (\overline{\hat{\Psi}}, \hat{H})$ and $(\overline{\hat{F}}, \overline{\hat{T}}) : (\overline{\hat{\Phi}}, \overline{\hat{G}}) \to (\overline{\hat{\Psi}}, \overline{\hat{H}})$ later.

3.4.1 The Induced GENEOs

Given the perception pairs (Φ, G), $\mathrm{dom}(\Phi) = X$ and (Ψ, H), $\mathrm{dom}(\Psi) = Y$ and a homomorphism $T : G \to H$ with totally bounded Φ and Ψ endowing X and Y with metric structures respectively, let $(F, T) : (\Phi, G) \to (\Psi, H)$ be a GENEO in $\mathcal{F} \subseteq \mathcal{F}_T^{\mathrm{all}}$. We put

$$\hat{F}(\hat{\varphi}) := \widehat{F(\varphi)},$$

and

$$\hat{T}(\hat{g}) := \widehat{T(g)},$$

where $\varphi \in \Phi$, $g \in G$, and $F \in \mathcal{F} \subseteq \mathcal{F}_T^{all}$.

The maps $\hat{F} : \hat{\Phi} \to \hat{\Psi}$ and $\hat{T} : \hat{G} \to \hat{H}$ are clearly well defined, since the maps $i_1 : \Phi \to \hat{\Phi}$ (taking φ to $\hat{\varphi}$) and $k_1 : G \to \hat{G}$ (taking g to $\hat{g}$) are injective. Moreover,

Remark 3.4.1. *The map $\hat{F}$ is injective if and only if $F \in \mathcal{F}$ is an injection. As $i_2 : \Psi \to \hat{\Psi}$ is injective, we have*

$$\hat{F}(\hat{\varphi}_1) = \hat{F}(\hat{\varphi}_2) \underset{\text{def}}{\iff} \widehat{F(\varphi_1)} = \widehat{F(\varphi_2)} \iff F(\varphi_1) = F(\varphi_2),$$

for every $\varphi_1, \varphi_2 \in \Phi$. The injectivity of $i_1 : \Phi \to \hat{\Phi}$, together with these equivalences, gives the assertion.

Recalling Propositions 3.3.2 and 3.3.3, we prove

Proposition 3.4.1. *The map $\hat{F} : \hat{\Phi} \to \hat{\Psi}$ is a GENEO with respect to $\hat{T} : \hat{G} \to \hat{H}$.*

Proof. It is easy to see that $\hat{T} : \hat{G} \to \hat{H}$ is a group homomorphism: If $a, b \in G$, then

$$\begin{aligned} \hat{T}(\hat{a}\hat{b}) &= \hat{T}(\widehat{ab}) \\ &= \widehat{T(ab)} \\ &= \widehat{T(a)T(b)} \\ &= \widehat{T(a)}\widehat{T(b)} \\ &= \hat{T}(\hat{a})\hat{T}(\hat{b}). \end{aligned}$$

Similarly, if $\varphi \in \Phi$, $g \in G$, and $F \in \mathcal{F}$, we have:

$$\begin{aligned} \hat{F}(\hat{\varphi}\hat{g}) &= \hat{F}(\widehat{\varphi g}) \\ &= \widehat{F(\varphi g)} \\ &= \widehat{F(\varphi)T(g)} \\ &= \widehat{F(\varphi)}\widehat{T(g)} \\ &= \hat{F}(\hat{\varphi})\hat{T}(\hat{g}). \end{aligned}$$

So, $\hat{F}$ is $\hat{T}-$equivariant.

Now, let $\varphi_1, \varphi_2 \in \Phi$. As $i_1 : \Phi \to \hat{\Phi}$ and $i_2 : \Psi \to \hat{\Psi}$ are isometries (Theorem

3.2.1) and $F : \Phi \to \Psi$ is non-expansive,

$$\begin{aligned} D_{\hat{\Psi}}(\hat{F}(\hat{\varphi}_1), \hat{F}(\hat{\varphi}_2)) &= D_{\hat{\Psi}}(\widehat{F(\varphi_1)}, \widehat{F(\varphi_2)}) \\ &= D_{\Psi}(F(\varphi_1), F(\varphi_2)) \\ &\le D_{\Phi}(\varphi_1, \varphi_2) \\ &= D_{\hat{\Phi}}(\hat{\varphi}_1, \hat{\varphi}_2) \end{aligned}$$

whence $\hat{F}$ is non-expansive.

$\square$

Let us put

$$\mathcal{F}_1 := \{\hat{F} : \hat{\Phi} \to \hat{\Psi} \mid F \in \mathcal{F}\},$$

and define a map $f_1 : \mathcal{F} \to \mathcal{F}_1$ by setting

$$f_1(F) := \hat{F}.$$

The set $\mathcal{F}^{\mathrm{all},1}_{\hat{T}} \supseteq \mathcal{F}_1$ of all GENEOs from $(\hat{\Phi}, \hat{G})$ to $(\hat{\Psi}, \hat{H})$ with respect to the homomorphism $\hat{T} : \hat{G} \to \hat{H}$ is a metric space with the distance function D^1_{GENEO} given by

$$D^1_{\mathrm{GENEO}}(F', F'') := \sup_{\hat{\varphi} \in \hat{\Phi}} D_{\hat{\Psi}}(F'(\hat{\varphi}), F''(\hat{\varphi})), \text{ for every } F', F'' \in \mathcal{F}^{\mathrm{all},1}_{\hat{T}}.$$

Proposition 3.4.2. *The correspondence $f_1 : \mathcal{F} \to \mathcal{F}_1$ is an isometry with respect to the distances D_{GENEO} and D^1_{GENEO}.*

Proof. The map f_1 is surjective by construction. Let $F_1, F_2 \in \mathcal{F}$ be distinct GENEOs; i.e., there is a $\varphi \in \Phi$ such that $F_1(\varphi) \neq F_2(\varphi)$. As $i_2 : \Psi \to \hat{\Psi}$ is injective, $\hat{F}_1(\hat{\varphi}) := \widehat{F_1(\varphi)} \neq \widehat{F_2(\varphi)} =: \hat{F}_2(\hat{\varphi})$, whence $f_1(F_1) := \hat{F}_1 \neq \hat{F}_2 =: f_1(F_2)$, and $f_1 : \mathcal{F} \to \mathcal{F}_1$ is injective.

We now show that f_1 preserves distances. If $F_1, F_2 \in \mathcal{F}$, by applying Proposition 3.2.2 to the real-valued continuous function $f(\hat{\varphi}) := D_{\hat{\Psi}}(\hat{F}_1(\hat{\varphi}), \hat{F}_2(\hat{\varphi}))$, we get

$$\begin{aligned}
D^1_{\text{GENEO}}(\hat{F}_1, \hat{F}_2) &:= \sup_{\hat{\varphi}\in\hat{\Phi}} D_{\hat{\Psi}}(\hat{F}_1(\hat{\varphi}), \hat{F}_2(\hat{\varphi})) \\
&= \sup_{\varphi\in\Phi} D_{\hat{\Psi}}(\hat{F}_1(\hat{\varphi}), \hat{F}_2(\hat{\varphi})) \\
&= \sup_{\varphi\in\Phi} D_{\hat{\Psi}}(\widehat{F_1(\varphi)}, \widehat{F_2(\varphi)}) \\
&= \sup_{\varphi\in\Phi} D_{\Psi}(F_1(\varphi), F_2(\varphi)) \\
&=: D_{\text{GENEO}}(F_1, F_2),
\end{aligned}$$

as $i_2 : \Psi \to \hat{\Psi}$ is an isometry by Theorem 3.2.1. So, the bijection f_1 is an isometry. $\square$

From the definitions of $\hat{F}$ and $\hat{T}$, it is already clear that the following natural commutativity conditions are trivially satisfied:

Proposition 3.4.3. *For each $F \in \mathcal{F}$,*

$$i_2 \circ F = f_1(F) \circ i_1, \text{ (i.e., } \widehat{F(\varphi)} = \hat{F}(\hat{\varphi}) \text{ for every } \varphi \in \Phi)$$

and

$$k_2 \circ T = \hat{T} \circ k_1 \text{ (i.e., } \widehat{T(g)} = \hat{T}(\hat{g}) \text{ for every } g \in G).$$

3.4.2 Preservation of Pseudo-distances

In this subsection, we show that the embeddings $i : \Phi \to \hat{\Phi}$, $k : G \to \hat{G}$, and $f_1 : \mathcal{F} \to \mathcal{F}_1$ preserve the pseudo-distances d_G, $D_{\text{GENEO},H}$ and $D_{\mathcal{F},\Phi}$.

Lemma 3.4.1. *The embeddings $i : \Phi \to \hat{\Phi}$ and $k : G \to \hat{G}$ preserve the natural pseudo-distance.*

Proof. Let $\varphi_1, \varphi_2 \in \Phi$; then since $\hat{g} \in \hat{G} \iff g \in G$, Theorem 3.2.1 and

Proposition 3.3.2 imply that

$$\begin{aligned}
d_{\hat{G}}(\hat{\varphi}_1, \hat{\varphi}_2) &:= \inf_{\hat{g}\in\hat{G}} D_{\hat{\Phi}}(\hat{\varphi}_1, \hat{\varphi}_2\hat{g}) \\
&= \inf_{\hat{g}\in\hat{G}} D_{\hat{\Phi}}(\hat{\varphi}_1, \widehat{\varphi_2 g}) \\
&= \inf_{g\in G} D_{\Phi}(\varphi_1, \varphi_2 g) \\
&=: d_G(\varphi_1, \varphi_2).
\end{aligned}$$

□

Proposition 3.4.4. *The embedding $f_1 : \mathcal{F} \to \mathcal{F}_1$ preserves the pseudo-distances $D_{\mathrm{GENEO},H}$ and $D_{\mathcal{F},\Phi}$.*

Proof. Let $F_1, F_2 \in \mathcal{F}$. As $\hat{\varphi} \in \hat{\Phi} \iff \varphi \in \Phi$, from Lemma 3.4.1 and the definition of $\hat{F}$, $(F \in \mathcal{F})$, we have

$$\begin{aligned}
\hat{D}_{\mathrm{GENEO},\hat{H}}(\hat{F}_1, \hat{F}_2) &:= \sup_{\hat{\varphi}\in\hat{\Phi}} d_{\hat{H}}(\hat{F}_1(\hat{\varphi}), \hat{F}_2(\hat{\varphi})) \\
&= \sup_{\hat{\varphi}\in\hat{\Phi}} d_{\hat{H}}(\widehat{F_1(\varphi)}, \widehat{F_2(\varphi)}) \\
&= \sup_{\varphi\in\Phi} d_H(F_1(\varphi), F_2(\varphi)) \\
&=: D_{\mathrm{GENEO},H}(F_1, F_2).
\end{aligned}$$

Also, if $\varphi_1, \varphi_2 \in \Phi$, then by Theorem 3.2.1 and Proposition 3.4.2 we have

$$\begin{aligned}
\hat{D}_{\hat{\mathcal{F}},\hat{\Phi}}(\hat{\varphi}_1, \hat{\varphi}_2) &:= \sup_{\hat{F}\in\hat{\mathcal{F}}} D_{\hat{\Psi}}(\hat{F}(\hat{\varphi}_1), \hat{F}(\hat{\varphi}_2)) \\
&= \sup_{F\in\mathcal{F}} D_{\hat{\Psi}}(\widehat{F(\varphi_1)}, \widehat{F(\varphi_2)}) \\
&= \sup_{F\in\mathcal{F}} D_{\Psi}(F(\varphi_1), F(\varphi_2)) \\
&=: D_{\mathcal{F},\Phi}(\varphi_1, \varphi_2).
\end{aligned}$$

□

3.5 Compactification of the Spaces of GENEOs

We can now extend our construction from $(\hat{F},\hat{T}) : (\hat{\Phi},\hat{G}) \to (\hat{\Psi},\hat{H})$ to $(\overline{\hat{F}},\hat{T}) : (\overline{\hat{\Phi}},\hat{G}) \to (\overline{\hat{\Psi}},\hat{H})$ and $(\overline{\hat{F}},\overline{\hat{T}}) : (\overline{\hat{\Phi}},\overline{\hat{G}}) \to (\overline{\hat{\Psi}},\overline{\hat{H}})$ successively, while maintaining the assumptions of Section 3.4. First, we show that $\hat{F} : \hat{\Phi} \to \hat{\Psi}$ induces a non-expansive $\hat{T}$-equivariant map $\overline{\hat{F}} : \overline{\hat{\Phi}} \to \overline{\hat{\Psi}}$; then we will use the assumption that the family of sets $\{F(\Phi) \mid F \in \mathcal{F}\}$ covers Ψ to define a group homomorphism $\overline{\hat{T}} : \overline{\hat{G}} \to \overline{\hat{H}}$ with respect to which $\overline{\hat{F}}$ remains equivariant.

Let us define a map $\overline{\hat{F}} : \overline{\hat{\Phi}} \to \overline{\hat{\Psi}}$ as follows. Let $\overline{\varphi} \in \overline{\hat{\Phi}}$; then there is a sequence $(\hat{\varphi}_n)_{n\in\mathbb{N}}$ in $\hat{\Phi}$ such that $\hat{\varphi}_n \to \overline{\varphi}$ with respect to the uniform norm. As $\hat{F}$ is non-expansive, $(\hat{F}(\hat{\varphi}_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\hat{\Psi} \subseteq \overline{\hat{\Psi}}$; so it converges to some $\overline{\psi}$ in the complete space $\overline{\hat{\Psi}}$. Let us put

$$\overline{\hat{F}}(\overline{\varphi}) := \overline{\psi}.$$

That is,

$$\overline{\hat{F}}(\lim_{n\to\infty} \hat{\varphi}_n) := \lim_{n\to\infty} \hat{F}(\hat{\varphi}_n).$$

Note that since $\hat{F}$ is non-expansive, the map $\overline{\hat{\mathcal{F}}}$ does not depend on the sequence $(\hat{\varphi}_n)_{n\in\mathbb{N}}$ converging to $\overline{\varphi}$, and is therefore well defined. Moreover, $\overline{\hat{F}}|_{\hat{\Phi}} = \hat{F}$.

Proposition 3.5.1. *The map $\overline{\hat{F}} : \overline{\hat{\Phi}} \to \overline{\hat{\Psi}}$ is a GENEO with respect to $\hat{T} : \hat{G} \to \hat{H}$.*

Proof. Let $\overline{\varphi}_1, \overline{\varphi}_2 \in \overline{\hat{\Phi}}$; then there are sequences $(\hat{\varphi}_{1,n})_{n\in\mathbb{N}}$ and $(\hat{\varphi}_{2,n})_{n\in\mathbb{N}}$ in $\hat{\Phi}$ such that $\hat{\varphi}_{1,n} \to \overline{\varphi}_1$ and $\hat{\varphi}_{2,n} \to \overline{\varphi}_2$. Recalling that $\hat{F}$ is non-expansive, we compute:

$$\begin{aligned}
\left\|\overline{\hat{F}}(\overline{\varphi}_1) - \overline{\hat{F}}(\overline{\varphi}_2)\right\|_\infty &= \left\|\lim_{n\to\infty} \hat{F}(\hat{\varphi}_{1,n}) - \lim_{n\to\infty} \hat{F}(\hat{\varphi}_{2,n})\right\|_\infty \\
&= \lim_{n\to\infty} \left\|\hat{F}(\hat{\varphi}_{1,n}) - \hat{F}(\hat{\varphi}_{2,n})\right\|_\infty \\
&\le \lim_{n\to\infty} \|\hat{\varphi}_{1,n} - \hat{\varphi}_{2,n}\|_\infty \\
&= \left\|\lim_{n\to\infty} \hat{\varphi}_{1,n} - \lim_{n\to\infty} \hat{\varphi}_{2,n}\right\|_\infty \\
&= \|\overline{\varphi}_1 - \overline{\varphi}_2\|_\infty;
\end{aligned}$$

so, $\overline{\hat{F}}$ is non-expansive.

Let $\overline{\varphi} \in \overline{\hat{\Phi}}$ and $\hat{g} \in \hat{G}$. Then there is a sequence $(\hat{\varphi}_n)_{n\in\mathbb{N}}$ in $\hat{\Phi}$ such that $\hat{\varphi}_n \to \overline{\varphi}$ with respect to the uniform norm; consequently, $\hat{\varphi}_n\hat{g} \to \overline{\varphi}\hat{g}$. As $\hat{F}$ is $\hat{T}-$equivariant (Proposition 3.4.1) and the action of $\hat{H}$ on $\overline{\hat{\Psi}}$ is continuous (Theorem 2.2.3), we have

$$\begin{aligned}
\overline{\hat{F}}(\overline{\varphi} \circ \hat{g}) &= \overline{\hat{F}}(\lim_{n\to\infty} \hat{\varphi}_n \circ \hat{g}) \\
&= \lim_{n\to\infty} \hat{F}(\hat{\varphi}_n \circ \hat{g}) \\
&= \lim_{n\to\infty} (\hat{F}(\hat{\varphi}_n) \circ \hat{T}(\hat{g})) \\
&= (\lim_{n\to\infty} \hat{F}(\hat{\varphi}_n)) \circ \hat{T}(\hat{g}) \\
&= \overline{\hat{F}}(\overline{\varphi}) \circ \hat{T}(\hat{g}),
\end{aligned}$$

whence $\overline{\hat{F}}$ is $\hat{T}-$equivariant and the proposition is proved. □

Let us put

$$\mathcal{F}_2 := \left\{ \overline{\hat{F}} : \overline{\hat{\Phi}} \to \overline{\hat{\Psi}} \mid F \in \mathcal{F} \right\},$$

and define a map $f_2 : \mathcal{F}_1 \to \mathcal{F}_2$ by setting

$$f_2(\hat{F}) := \overline{\hat{F}}.$$

The set $\mathcal{F}^{\mathrm{all},2}_{\hat{T}} \supseteq \mathcal{F}_2$ of all GENEOs from $(\overline{\hat{\Phi}}, \hat{G})$ to $(\overline{\hat{\Psi}}, \hat{H})$ with respect to the homomorphism $\hat{T} : \hat{G} \to \hat{H}$ is a metric space with the distance function D^2_{GENEO} given by

$$D^2_{\mathrm{GENEO}}\left(F', F''\right) := \sup_{\overline{\varphi} \in \overline{\hat{\Phi}}} D_{\overline{\hat{\Psi}}}\left(F'(\overline{\varphi}), F''(\overline{\varphi})\right), \text{ for every } F', F'' \in \mathcal{F}^{\mathrm{all},2}_{\hat{T}}.$$

Proposition 3.5.2. *The correspondence $f_2 : \mathcal{F}_1 \to \mathcal{F}_2$ is an isometry with respect to the distances D^1_{GENEO} and D^2_{GENEO}.*

Proof. The map $f_2 : \mathcal{F}_1 \to \mathcal{F}_2$ is surjective by construction. Also, if $\hat{F}_1, \hat{F}_2 \in \mathcal{F}_1$ are distinct, i.e., there is a $\hat{\varphi} \in \hat{\Phi}$ with $\hat{F}_1(\hat{\varphi}) \neq \hat{F}_2(\hat{\varphi})$, then $f_2(\hat{F}_1)(\hat{\varphi}) = \overline{\hat{F}}_1(\hat{\varphi}) \neq \overline{\hat{F}}_2(\hat{\varphi}) = f_2(\hat{F}_2)(\hat{\varphi})$ since we respectively have $\overline{\hat{F}}_1|_{\hat{\Phi}} = \hat{F}_1$ and $\overline{\hat{F}}_2|_{\hat{\Phi}} = \hat{F}_2$; whence $f_2(\hat{F}_1)(\hat{\varphi}) \neq f_2(\hat{F}_2)(\hat{\varphi})$ and f_2 is injective as well.

If $\overline{\hat{F}}_1, \overline{\hat{F}}_2 \in \mathcal{F}_2 \subseteq \mathcal{F}^{\mathrm{all},2}_{\hat{T}}$, by applying Proposition 3.2.2 to the real-valued continuous function $f(\overline{\varphi}) := D_{\overline{\hat{\Psi}}}\left(\overline{\hat{F}}_1(\overline{\varphi}), \overline{\hat{F}}_2(\overline{\varphi})\right)$, we get

$$\begin{aligned}
D^2_{\text{GENEO}}\left(\overline{\hat{F}}_1, \overline{\hat{F}}_2\right) &:= \sup_{\overline{\varphi}\in\overline{\hat{\Phi}}} D_{\overline{\hat{\Psi}}}\left(\overline{\hat{F}}_1(\overline{\varphi}), \overline{\hat{F}}_2(\overline{\varphi})\right) \\
&= \sup_{\hat{\varphi}\in\hat{\Phi}} D_{\overline{\hat{\Psi}}}\left(\overline{\hat{F}}_1(\hat{\varphi}), \overline{\hat{F}}_2(\hat{\varphi})\right) \\
&= \sup_{\hat{\varphi}\in\hat{\Phi}} D_{\hat{\Psi}}\left(\hat{F}_1(\hat{\varphi}), \hat{F}_2(\hat{\varphi})\right) \\
&=: D^1_{\text{GENEO}}\left(\hat{F}_1, \hat{F}_2\right)
\end{aligned}$$

as $D_{\hat{\Psi}}$ and $D_{\overline{\hat{\Psi}}}$ both are restrictions of the distance induced by the uniform norm on $\mathbb{R}^Y_b$ to $\hat{\Psi}$ and $\overline{\hat{\Psi}}$ respectively. So, the bijection f_2 is an isometry. $\square$

As $\overline{\hat{F}}|_{\hat{\Phi}} = \hat{F}$ for each $F \in \mathcal{F}$, by Proposition 3.4.3 we have

Proposition 3.5.3. *For each $F \in \mathcal{F}$,*

$$i_2 \circ F = (f_2 \circ f_1(F)) \circ i_1.$$

That is, for every $\varphi \in \Phi$,

$$\widehat{F(\varphi)} = \overline{\hat{F}}(\hat{\varphi}).$$

Let us now utilize the assumption that $\{F(\Phi) \mid F \in \mathcal{F}\}$ covers Ψ to define a homomorphism $\overline{\hat{T}} : \hat{G} \to \overline{\hat{H}}$. First we need

Definition 3.5.1. *We say that a space $\mathcal{F} \subseteq \mathcal{F}^{\text{all}}_T$ of GENEOs $(F,T) : (\Phi, G) \to (\Psi, H)$ is collectionwise surjective if for each $\psi \in \Psi$, there exist an $F_\psi \in \mathcal{F}$ and a $\varphi_\psi \in \Phi$ such that $F_\psi(\varphi_\psi) = \psi$; that is, $\bigcup_{F\in\mathcal{F}} F(\Phi) = \Psi$.*

The key property of collectionwise surjective spaces of GENEOs is given in

Theorem 3.5.1. *If the space $\mathcal{F} \subseteq \mathcal{F}^{\text{all}}_T$ of GENEOs $(F,T) : (\Phi, G) \to (\Psi, H)$ is collectionwise surjective, then the homomorphism T is non-expansive.*

Proof. Let $a, b \in G$; then as $\mathcal{F}$ is collectionwise surjective and each $F \in \mathcal{F}$ is a

GENEO, we have

$$\begin{aligned} D_{\text{Aut}}(T(a),T(b)) &:= \sup_{\psi\in\Psi} D_\Psi(\psi T(a),\psi T(b)) \\ &= \sup_{\psi\in\Psi} D_\Psi(F_\psi(\varphi_\psi)T(a),F_\psi(\varphi_\psi)T(b)) \\ &= \sup_{\psi\in\Psi} D_\Psi(F_\psi(\varphi_\psi a),F_\psi(\varphi_\psi b)) \\ &\le \sup_{\psi\in\Psi} D_\Phi(\varphi_\psi a,\varphi_\psi b) \\ &\le \sup_{\varphi\in\Phi} D_\Phi(\varphi a,\varphi b) \\ &= D_{\text{Aut}}(a,b). \end{aligned}$$

□

For the rest of this section, the spaces $\mathcal{F} \subseteq \mathcal{F}_T^{\text{all}}$ will be assumed to be collectionwise surjective. Clearly, $\mathcal{F}_1 := \{\hat{F} : \hat{\Phi} \to \hat{\Psi} \mid F \in \mathcal{F}\}$ is collectionwise surjective whenever $\mathcal{F}$ is so, since $\hat{F}_\psi(\hat{\varphi}_\psi) = \widehat{F_\psi(\varphi_\psi)} = \hat{\psi}$ for every $\psi \in \Psi$.

Theorem 3.5.1 implies

Corollary 3.5.1. *The homomorphism $\hat{T} : \hat{G} \to \hat{H}$ is non-expansive.*

Corollary 3.5.1 allows us to define a map $\overline{\hat{T}} : \overline{\hat{G}} \to \overline{\hat{H}}$ unambiguously: Let $\overline{g} \in \overline{\hat{G}}$ and $(\hat{g}_n)_{n\in\mathbb{N}}$ be a sequence in $\hat{G}$ that converges to $\overline{g}$ in $\overline{\hat{G}}$. As $\hat{T}$ is non-expansive and $\overline{\hat{H}}$ is a complete metric space, the sequence $(\hat{T}(\hat{g}_n))_{n\in\mathbb{N}}$ in $\hat{H}$ converges to a unique element $\overline{h} \in \overline{\hat{H}}$. We put

$$\overline{\hat{T}}(\overline{g}) := \overline{h}.$$

That is,

$$\overline{\hat{T}}\left(\lim_{n\to\infty} \hat{g}_n\right) := \lim_{n\to\infty} \hat{T}(\hat{g}_n).$$

Note that $\overline{\hat{T}}|_{\hat{G}} = \hat{T}$. So, the commutativity condition $k_2 \circ T = \hat{T} \circ k_1$ (i.e., $\widehat{T(g)} = \hat{T}(\hat{g})$ for every $g \in G$) in Proposition 3.4.3 can be rephrased as

Proposition 3.5.4. $k_2 \circ T = \overline{\hat{T}} \circ k_1$ (i.e., $\widehat{T(g)} = \overline{\hat{T}}(\hat{g})$ for every $g \in G$).

We observe that the map $\overline{\hat{T}} : \overline{\hat{G}} \to \overline{\hat{H}}$ preserves the group structure:

Theorem 3.5.2. *The function $\overline{\hat{T}} : \overline{\hat{G}} \to \overline{\hat{H}}$ is a group homomorphism.*

Proof. Let $\overline{a}, \overline{b} \in \overline{\hat{G}}$, and $(\hat{a}_n)_{n\in\mathbb{N}}, (\hat{b}_n)_{n\in\mathbb{N}}$ be sequences in $\hat{G}$ converging respectively to $\overline{a}, \overline{b}$ in $\overline{\hat{G}}$. Recalling the continuity of the composition of functions on $\mathrm{Aut}_{\hat{\Phi}}(\hat{X})$ and $\mathrm{Aut}_{\hat{\Psi}}(\hat{Y})$ (Theorem 2.2.3) and the definition of $\overline{\hat{T}}$, we compute

$$\begin{aligned}
\overline{\hat{T}}\left(\overline{a}\overline{b}\right) &= \overline{\hat{T}}\left(\lim_{n\to\infty} \hat{a}_n \lim_{n\to\infty} \hat{b}_n\right) \\
&= \overline{\hat{T}}\left(\lim_{n\to\infty} \hat{a}_n \hat{b}_n\right) \\
&= \lim_{n\to\infty} \hat{T}(\hat{a}_n \hat{b}_n) \\
&= \lim_{n\to\infty} \hat{T}(\hat{a}_n)\hat{T}(\hat{b}_n) \\
&= \lim_{n\to\infty} \hat{T}(\hat{a}_n) \lim_{n\to\infty} \hat{T}(\hat{b}_n) \\
&= \overline{\hat{T}}\left(\lim_{n\to\infty} \hat{a}_n\right) \overline{\hat{T}}\left(\lim_{n\to\infty} \hat{b}_n\right) \\
&= \overline{\hat{T}}(\overline{a})\overline{\hat{T}}(\overline{b}).
\end{aligned}$$

Therefore, $\overline{\hat{T}}$ is a group homomorphism. □

The next claim allows us to pass from $\hat{T}-$equivariance to $\overline{\hat{T}}-$equivariance.

Theorem 3.5.3. *Every GENEO $\overline{\hat{F}} \in \mathcal{F}_2 \subseteq \mathcal{F}^{\mathrm{all},2}_{\hat{T}}$ is $\overline{\hat{T}}-$equivariant as well. Hence $\left(\overline{\hat{F}}, \overline{\hat{T}}\right) : \left(\overline{\hat{\Phi}}, \overline{\hat{G}}\right) \to \left(\overline{\hat{\Psi}}, \overline{\hat{H}}\right)$ is a GENEO for each $F \in \mathcal{F}$.*

Proof. Let $\overline{\varphi} \in \overline{\hat{\Phi}}, \overline{g} \in \overline{\hat{G}}$, and $(\hat{g}_n)_{n\in\mathbb{N}}$ be a sequence in $\hat{G}$ converging to $\overline{g}$. Recalling the fact that $\overline{\hat{F}}$ is a GENEO for $\hat{T}$ (and in particular a non-expansive, and hence continuous, map) by Proposition 3.5.1, the continuity of the actions of $\mathrm{Aut}_{\overline{\hat{\Phi}}}(\hat{X})$ and $\mathrm{Aut}_{\overline{\hat{\Psi}}}(\hat{Y})$ respectively on $\overline{\hat{\Phi}}$ and $\overline{\hat{\Psi}}$ (Theorem 2.2.3), and the definition of $\overline{\hat{T}}$, we compute

$$\begin{aligned}
\overline{\hat{F}}\left(\bar{\varphi}\bar{g}\right) &= \overline{\hat{F}}\left(\overline{\varphi} \lim_{n\to\infty} \hat{g}_n\right) \\
&= \overline{\hat{F}}\left(\lim_{n\to\infty} \overline{\varphi}\hat{g}_n\right) \\
&= \lim_{n\to\infty} \overline{\hat{F}}\left(\overline{\varphi}\hat{g}_n\right) \\
&= \lim_{n\to\infty} \overline{\hat{F}}(\overline{\varphi})\hat{T}(\hat{g}_n) \\
&= \overline{\hat{F}}(\overline{\varphi}) \lim_{n\to\infty} \hat{T}(\hat{g}_n) \\
&= \overline{\hat{F}}(\overline{\varphi})\overline{\hat{T}}\left(\lim_{n\to\infty} \hat{g}_n\right) \\
&= \overline{\hat{F}}(\overline{\varphi})\overline{\hat{T}}(\overline{g}).
\end{aligned}$$

$\square$

Because of Theorems 3.5.2 and 3.5.3, we can now consider $\mathcal{F}_2$ as a set of GENEOs from $\left(\overline{\hat{\Phi}}, \overline{\hat{G}}\right)$ to $\left(\overline{\hat{\Psi}}, \overline{\hat{H}}\right)$ with respect to $\overline{\hat{T}}$, and denote this set by $\mathcal{F}_3$ to make clear that we are taking the homomorphism $\overline{\hat{T}}$ instead of $\hat{T}$.

The set $\mathcal{F}^{\mathrm{all}}_{\overline{\hat{T}}} \supseteq \mathcal{F}_3$ of all GENEOs from $(\overline{\hat{\Phi}}, \overline{\hat{G}})$ to $(\overline{\hat{\Psi}}, \overline{\hat{H}})$ with respect to $\overline{\hat{T}}$ is a metric space with the distance function D^3_{GENEO} given by

$$D^3_{\mathrm{GENEO}}(F', F'') := \sup_{\overline{\varphi} \in \overline{\hat{\Phi}}} D_{\overline{\hat{\Psi}}}(F'(\overline{\varphi}), F''(\overline{\varphi})), \;\; F', F'' \in \mathcal{F}^{\mathrm{all}}_{\overline{\hat{T}}}.$$

Moreover, since the data sets $\overline{\hat{\Phi}}$ and $\overline{\hat{\Psi}}$ are compact, the space $(\mathcal{F}^{\mathrm{all}}_{\overline{\hat{T}}}, D^3_{\mathrm{GENEO}})$ is compact as well [6, Theorem 7]. Consequently,

Proposition 3.5.5. *The closure* $\mathrm{cl}(\mathcal{F}_3)$ *of* $\mathcal{F}_3 \subseteq \mathcal{F}^{\mathrm{all}}_{\overline{\hat{T}}}$ *in the compact space* $\mathcal{F}^{\mathrm{all}}_{\overline{\hat{T}}}$ *is compact.*

As the definitions of D^2_{GENEO} and D^3_{GENEO} do not depend on the reference homomorphisms $\hat{T}$ and $\overline{\hat{T}}$ respectively, we observe that the identity from $\mathcal{F}_2$ to $\mathcal{F}_3$ is an isometry.

Propositions 3.4.2 and 3.5.2 together give

Proposition 3.5.6. *The correspondence* $f : \mathcal{F} \to \mathcal{F}_3$ *given by*

$$f := f_2 \circ f_1$$

is an isometry.

Therefore, we can rephrase Proposition 3.5.3 as

Proposition 3.5.7. *For each* $F \in \mathcal{F}$*,*

$$i_2 \circ F = f(F) \circ i_1 \text{ (i.e., } \widehat{F(\varphi)} = \overline{\hat{F}}(\hat{\varphi}) \text{ for every } \varphi \in \Phi).$$

We can now state the main result in this paper by introducing the following definition:

Definition 3.5.2. *A compact space* $\mathcal{F}^* \subseteq \mathcal{F}^{\mathrm{all}}_{T^*}$ *of GENEOs* $(F^*, T^*) : (\Phi^*, G^*) \to (\Psi^*, H^*)$ *with* $\mathrm{dom}(\Phi^*) = X^*$ *and* $\mathrm{dom}(\Psi^*) = Y^*$ *is said to be a compactification of a space* $\mathcal{F} \subseteq \mathcal{F}^{\mathrm{all}}_{T}$ *of GENEOs* $(F, T) : (\Phi, G) \to (\Psi, H)$ *with* $\mathrm{dom}(\Phi) = X$ *and* $\mathrm{dom}(\Psi) = Y$*, if the perception pairs* (Φ^*, G^*) *and* (Φ^*, G^*) *are compactifications of*

(Φ, G) *and* (Ψ, H) *respectively, and there is an isometric embedding* f *of* $\mathcal{F}$ *in* $\mathcal{F}^*$ *as a dense subspace, such that the following commutativity conditions are satisfied:* $i_2 \circ F = f(F) \circ i_1$*, for each* $F \in \mathcal{F}$*, and* $k_2 \circ T = T^* \circ k_1$.

Theorem 3.5.4. *Every collectionwise surjective space* $\mathcal{F} \subseteq \mathcal{F}_T^{\text{all}}$ *of GENEOs* $(F, T) : (\Phi, G) \to (\Psi, H)$ *with* $\text{dom}(\Phi) = X$ *and* $\text{dom}(\Psi) = Y$ *admits a compactification* $\mathcal{F}^*$*, provided the data sets* Φ *and* Ψ *are totally bounded and endow* X *and* Y *with metric structures.*

Proof. It follows from Theorem 3.3.2 and Propositions 3.5.4, 3.5.5, 3.5.6, and 3.5.7, by setting $F^* := \overline{\hat{F}}$, $T^* := \overline{\hat{T}}$, and $\mathcal{F}^* := \text{cl}(\mathcal{F}_3) \subseteq \mathcal{F}_{\overline{\hat{T}}}^{\text{all}}$. $\square$

Chapter 4

Generalized Permutants and Graph GENEOs

In this chapter, we present two models for graph GENEOs. The first model considers graphs weighted on the edges and the second one addresses the case of graphs weighted on the vertices. We introduce the notion of a generalized permutant and show how it can be used to define new GENEOs in the set theoretic setting. Finally, defining and utilizing the concept of a graph permutant, we show how the second model for graph GENEOs can be used to study the structure of simple graphs [56].

Contents

We establish a bridge between Topological Data Analysis and Geometric Deep Learning, adapting the topological theory of group equivariant non-expansive operators (GENEOs) to act on the space of all graphs weighted on vertices or edges. This is done by showing how the general concept of GENEO can be used to transform graphs and to give information about their structure. This requires the introduction of the new concepts of generalized permutant and generalized permutant measure and the mathematical proof that these concepts allow us to build GENEOs between graphs. The analysis of some simple case studies illustrates the possible use of our operators to extract structural information from graphs. This work is part of a line of research devoted to developing a compositional and geometric theory of GENEOs for Geometric Deep Learning.

Section 4.1 introduces and describes the new concepts of generalized permutant and generalized permutant measure, proving that each of them can be used to build a GENEO (Theorems 4.1.1 and 4.1.3). In Section 4.2 we introduce the concepts of vertex-weighted/edge-weighted graph GENEO and illustrate our new mathematical model with several examples. Section 4.3 is devoted to case studies showing how graph GENEOs enrich our understanding of the structure of simple graphs.

4.1 Generalized Permutants in the Set-theoretical Setting

In this section, we introduce a generalization of the concept of a permutant to the case when we may have distinct perception pairs, and show that the new concept we introduce here too can be used to populate the space of GENEOs.

Definition 4.1.1. *Let* (Φ, G)*,* $\mathrm{dom}(\Phi) = X$ *and* (Ψ, K)*,* $\mathrm{dom}(\Psi) = Y$ *be perception pairs and* $T : G \to K$ *be a group homomorphism. A finite set* $H \subseteq X^Y$ *of functions* $h : Y \to X$ *is called a generalized permutant for* T *if* $H = \emptyset$ *or* $g \circ h \circ T(g^{-1}) \in H$ *for every* $h \in H$*, and every* $g \in G$*.*

In this case, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h' = g\circ h\circ T(g^{-1})} \\ Y & \xrightarrow{T(g)} & Y \end{array}$$

We observe that the map $h \mapsto g \circ h \circ T(g^{-1})$ is a bijection from H to H, for any $g \in G$.

Definition 4.1.1 extends Definition 2.3.1 in two different directions. First of all, it does not require that the origin perception pair (Φ, G) and the target perception pair (Ψ, K) coincide. Secondly, it does not require that the elements of the set H are bijections. In Section 4.3.2 we will see how the concept of generalized permutant can be applied.

Example 4.1.1. *Let X, Y be two nonempty finite sets, with $Y \subseteq X$. Let G be the group of all permutations of X that preserve Y, and K be the group of all permutations of Y. Set $\Phi = \mathbb{R}^X$ and $\Psi = \mathbb{R}^Y$. Assume that $T : G \to K$ takes each permutation of X to its restriction to Y. Define H as the set of all functions $h : Y \to X$ such that the cardinality of* Im *h is smaller than a fixed integer m. Then H is a generalized permutant for T.*

In the following two subsections, we will express two other ways to look at generalized permutants, beyond their definition. To this end, we will assume that two perception pairs (Φ, G), (Ψ, K) and a group homomorphism $T : G \to K$ are given, with $\mathrm{dom}(\Phi) = X$ and $\mathrm{dom}(\Psi) = Y$.

4.1.1 Generalized Permutants as unions of equivalence classes

In view of Definition 4.1.1, we can define an equivalence relation $\sim$ on X^Y:

Definition 4.1.2. *Let $h, h' \in X^Y$. We say that h is equivalent to h', and write $h \sim h'$, if there is a $g \in G$ such that $h' = g \circ h \circ T(g^{-1})$.*

It is easy to see that $\sim$ is indeed an equivalence relation on X^Y.

Proposition 4.1.1. *A subset H of X^Y is a generalized permutant for T if and only if H is a (possibly empty) union of equivalence classes for $\sim$.*

Proof. Assume that H is a generalized permutant for T. If $h \in H$ and $h \sim h' \in X^Y$, then the definition of the relation $\sim$ and the definition of generalized permutant imply that $h' \in H$ as well, and therefore H is a union of equivalence classes for $\sim$. Conversely, if H is a union of equivalence classes for the relation $\sim$, $h \in H$ and $g \in G$, then $g \circ h \circ T(g^{-1}) \in H$, since $g \circ h \circ T(g^{-1}) \sim h$. As a consequence, H is a generalized permutant for T. $\square$

4.1.2 Generalized Permutants as unions of orbits

The map $\alpha : G \times X^Y \to X^Y$ taking (g, f) to $g \circ f \circ T(g^{-1})$ is a left group action, since $\alpha(\mathrm{id}_X, f) = \mathrm{id}_X \circ f \circ T(\mathrm{id}_X^{-1}) = f$ and $\alpha(g_2, \alpha(g_1, f)) = \alpha(g_2, g_1 \circ f \circ T(g_1^{-1})) =$

$g_2 \circ (g_1 \circ f \circ T(g_1^{-1})) \circ T(g_2^{-1}) = (g_2 \circ g_1) \circ f \circ T((g_2 \circ g_1)^{-1}) = \alpha(g_2 \circ g_1, f)$. For every $f \in X^Y$, the set $O(f) := \{\alpha(g, f) : g \in G\}$ is called the *orbit* of f. By observing that $O(f)$ is the equivalence class of f in X^Y for $\sim$, from Proposition 4.1.1 the following result immediately follows.

Proposition 4.1.2. *A subset H of X^Y is a generalized permutant for T if and only if H is a (possibly empty) union of orbits for the group action α.*

The main use of the concept of generalized permutant is expressed by the following theorem, extending Proposition 2.3.1.

Theorem 4.1.1. *Let (Φ, G), $\mathrm{dom}(\Phi) = X$ and (Ψ, K), $\mathrm{dom}(\Psi) = Y$ be perception pairs, $T : G \to K$ a group homomorphism, and H be a generalized permutant for T. Then the restriction to Φ of the operator $F : \mathbb{R}^X \to \mathbb{R}^Y$ defined by*

$$F(\varphi) := \frac{1}{|H|} \sum_{h \in H} \varphi \circ h$$

is a GENEO from (Φ, G) to (Ψ, K) with respect to T provided $F(\Phi) \subseteq \Psi$.

Proof. Let $\varphi \in \Phi$ and $g \in G$. Then by the definition of a generalized permutant and the change of variable $h' = g \circ h \circ T(g^{-1})$, we have

$$\begin{aligned} F(\varphi \circ g) &:= \frac{1}{|H|} \sum_{h \in H} (\varphi \circ g) \circ h \\ &= \frac{1}{|H|} \sum_{h \in H} \varphi \circ g \circ h \circ T(g^{-1}) \circ T(g) \\ &= \frac{1}{|H|} \sum_{h' \in H} \varphi \circ h' \circ T(g) \\ &= F(\varphi) \circ T(g) \end{aligned}$$

whence F is equivariant.

If $\varphi_1, \varphi_2 \in \Phi$, then

$$\begin{aligned}
\|F(\varphi_1) - F(\varphi_2)\|_\infty &= \left\| \frac{1}{|H|} \sum_{h \in H} \varphi_1 \circ h - \frac{1}{|H|} \sum_{h \in H} \varphi_2 \circ h \right\|_\infty \\
&= \frac{1}{|H|} \left\| \sum_{h \in H} (\varphi_1 \circ h - \varphi_2 \circ h) \right\|_\infty \\
&\leq \frac{1}{|H|} \sum_{h \in H} \|\varphi_1 \circ h - \varphi_2 \circ h\|_\infty \\
&\leq \frac{1}{|H|} \sum_{h \in H} \|\varphi_1 - \varphi_2\|_\infty \\
&= \frac{1}{|H|} |H| \, \|\varphi_1 - \varphi_2\|_\infty \\
&= \|\varphi_1 - \varphi_2\|_\infty
\end{aligned}$$

whence F is non-expansive, and hence a GENEO. □

4.1.3 Generalized permutant measures

As shown in [41], the concept of a permutant can be extended to the one of a *permutant measure*, provided that the set X under consideration is finite. This is done by using the following definition, referring to a subgroup G of the group $\mathrm{Aut}(X)$ of all permutations of the set X, and to the perception pair $(\mathbb{R}^X, G)$.

Definition 4.1.3. *[41] A finite signed measure μ on* $\mathrm{Aut}(X)$ *is called a* permutant measure *with respect to G if each subset H of* $\mathrm{Aut}(X)$ *is measurable and μ is invariant under the conjugation action of G (i.e., $\mu(H) = \mu(gHg^{-1})$ for every $g \in G$).*

With a slight abuse of notation, we will denote by $\mu(h)$ the signed measure of the singleton $\{h\}$ for each $h \in \mathrm{Aut}(X)$. The next example shows how we can apply Definition 4.1.3.

Example 4.1.2. *Let us consider the set X of the vertices of a cube in $\mathbb{R}^3$, and the group G of the orientation-preserving isometries of $\mathbb{R}^3$ that take X to X. Set $T = \mathrm{id}_G$. Let π_1, π_2, π_3 be the three planes that contain the center of mass of X and are parallel to a face of the cube. Let $h_i : X \to X$ be the orthogonal symmetry with respect to π_i, for $i \in \{1, 2, 3\}$. We have that the set $\{h_1, h_2, h_3\}$ is an orbit under the action expressed by the map α defined in Section 4.1.2. We can now define a* *permutant measure μ on* $\mathrm{Aut}(X)$ *by setting* $\mu(h_1) = \mu(h_2) = \mu(h_3) = c$*, where* c *is a positive real number, and* $\mu(h) = 0$ *for any* $h \in \mathrm{Aut}(X)$ *with* $h \notin \{h_1, h_2, h_3\}$*. We also observe that while the cardinality of* G *is* 24*, the cardinality of the support* $\mathrm{supp}(\mu) := \{h \in \mathrm{Aut}(X) : \mu(h) \neq 0\}$ *of the signed measure* μ *is* 3*.*

The concept of permutant measure is important because it makes available an important representation result for linear GENEOs. But first we need

Definition 4.1.4. *[40] Let* X *be a finite set and* G *be a subgroup of* $\mathrm{Aut}(X)$*. We say that* G *acts on* X *transitively if for every* $x_1, x_2 \in X$ *there is a* $g \in \mathrm{Aut}(X)$ *such that* $g(x_1) = x_2$*.*

Theorem 4.1.2. *[41] Assume that* $G \subseteq \mathrm{Aut}(X)$ *acts transitively on the finite set* X *and* F *is a map from* $\mathbb{R}^X$ *to* $\mathbb{R}^X$*. The map* F *is a linear group equivariant non-expansive operator from* $(\mathbb{R}^X, G)$ *to* $(\mathbb{R}^X, G)$ *with respect to the homomorphism* $\mathrm{id}_G : G \to G$ *if and only if a permutant measure* μ *exists such that* $F(\varphi) = \sum_{h \in \mathrm{Aut}(X)} \varphi \circ h^{-1}\ \mu(h)$ *for every* $\varphi \in \mathbb{R}^X$*, and* $\sum_{h \in \mathrm{Aut}(X)} |\mu(h)| \leq 1$*.*

We now state a definition that extends the concept of permutant measure.

Definition 4.1.5. *Let* X *and* Y *be finite nonempty sets. Let us choose a subgroup* G *of* $\mathrm{Aut}(X)$*, a subgroup* K *of* $\mathrm{Aut}(Y)$*, and a homomorphism* $T : G \to K$*. A finite signed measure* μ *on* X^Y *is called a* generalized permutant measure *with respect to* T *if each subset* H *of* X^Y *is measurable and* $\mu\left(g \circ H \circ T(g^{-1})\right) = \mu\left(H\right)$ *for every* $g \in G$*.*

Definition 4.1.5 extends Definition 4.1.3 in two different directions. First of all, it does not require that the origin perception pair $(\mathbb{R}^X, G)$ and the target perception pair $(\mathbb{R}^Y, K)$ coincide. Secondly, the measure μ is not defined on $\mathrm{Aut}(X)$ but on the set X^Y.

Example 4.1.3. *Let* X *and* Y *be nonempty finite sets, with* $Y \subseteq X$*. Let* G *be the group of all permutations of* X *that preserve* Y*, and* K *be the group of all permutations of* Y*. Set* $\Phi = \mathbb{R}^X$ *and* $\Psi = \mathbb{R}^Y$*. Assume that* $T : G \to K$ *takes each permutation of* X *to its restriction to* Y*. For any positive integer* m*, define* H_m *as the set of all functions* $h : Y \to X$ *such that the cardinality of* $\mathrm{Im}\ h$ *is equal to* m*. For each* $h \in H_m$*, let us set* $\mu(h) := \frac{1}{m|H_m|}$*. Then* μ *is a generalized permutant measure with respect to* T*.*

We can prove the following result, showing that every generalized permutant measure allows us to build a GENEO between perception pairs.

Theorem 4.1.3. *Let X and Y be finite nonempty sets. Let us choose a subgroup G of $\mathrm{Aut}(X)$, a subgroup K of $\mathrm{Aut}(Y)$, and a homomorphism $T : G \to K$. If μ is a generalized permutant measure with respect to T, then the map $F_\mu : \mathbb{R}^X \to \mathbb{R}^Y$ defined by setting $F_\mu(\varphi) := \sum_{f\in X^Y} \varphi \circ f \ \mu(f)$ is a linear GEO from (Φ, G) to (Ψ, K) with respect to T. If $\sum_{f\in X^Y} |\mu(f)| \le 1$, then F is a GENEO.*

Proof. It is immediate to check that F_μ is linear. Moreover, by applying the change of variable $\hat{f} = g \circ f \circ T(g^{-1})$ and the equality $\mu\left(g \circ f \circ T(g^{-1})\right) = \mu(f)$, for every $\varphi \in \mathbb{R}^X$ and every $g \in G$ we get

$$
\begin{aligned}
F_\mu(\varphi \circ g) &= \sum_{f\in X^Y} \varphi \circ g \circ f \ \mu(f) \\
&= \sum_{f\in X^Y} \varphi \circ g \circ f \circ T(g^{-1}) \circ T(g) \ \mu(g \circ f \circ T(g^{-1})) \\
&= \sum_{\hat{f}\in X^Y} \varphi \circ \hat{f} \circ T(g) \ \mu(\hat{f}) \\
&= F_\mu(\varphi) \circ T(g)
\end{aligned}
$$

since the map $f \mapsto g \circ f \circ T\left(g^{-1}\right)$ is a bijection from X^Y to X^Y. This proves that F_μ is equivariant.

If $\sum_{f\in X^Y} |\mu(f)| \le 1$,

$$
\begin{aligned}
\|F_\mu(\varphi)\|_\infty &= \left\| \sum_{f\in X^Y} \varphi \circ f \ \mu(f) \right\|_\infty \\
&\le \sum_{f\in X^Y} \|\varphi \circ f\|_\infty |\mu(f)| \\
&\le \sum_{f\in X^Y} \|\varphi\|_\infty |\mu(f)| \\
&= \|\varphi\|_\infty \sum_{f\in X^Y} |\mu(f)| \\
&\le \|\varphi\|_\infty .
\end{aligned}
$$

This implies that the linear map F_μ is non-expansive, and concludes the proof of our theorem. □

The condition $|\mathrm{supp}(\mu)| \ll |G|$ is not rare in applications (cf., e.g., Example 4.1.2) and is the main reason to build GEOs by means of (generalized) permutant measures, instead of using the representation of GEOs as G-convolutions and integrating on possibly large groups.

In the following, we will apply the aforementioned concepts to graphs. For the sake of simplicity, we will drop the word “generalized” and use the expression “graph permutant”.

4.2 GENEOs on Graphs

The notions of perception pair, permutant, and GENEO can be easily applied in a graph-theoretic setting. In this section, we develop a model for graphs with weights assigned to their vertices (vw-graphs, for short), and another for graphs with weights assigned to their edges (ew-graphs, for short), often called "weighted graphs" in literature. Our vertex model has implications for the rapidly growing field of graph convolutional neural networks, while the edge model we propose owes its significance to the widely recognized importance of weighted graphs.

As a graph [57] we shall mean a triple $\Gamma = (V_\Gamma, E_\Gamma, \psi_\Gamma)$, where ψ_Γ assigns to each edge of E_Γ the unordered pair of its end vertices in V_Γ. Since we only consider simple graphs (i.e. with no loops and no multiple edges) we write $e = \{A, B\}$ to mean $\psi_\Gamma(e) = \{A, B\}$. Let us recall that an automorphism g of Γ is a pair $g = (g_V, g_E)$, where $g_V : V_\Gamma \to V_\Gamma$ and $g_E : E_\Gamma \to E_\Gamma$ are bijections respecting the incidence function ψ_Γ. The group $\mathrm{Aut}(\Gamma)$ of all automorphisms of Γ induces two particular subgroups, here denoted as $\mathrm{Aut}(V_\Gamma)$ and $\mathrm{Aut}(E_\Gamma)$, of the groups of permutations of V_Γ and of E_Γ. We represent permutations as cycle products.

For any $k \in \mathbb{N}$, put

$$\mathbb{N}_k := \{1 \leq i \leq k \mid i \in \mathbb{N}\}.$$

Let a graph $\Gamma = (V_\Gamma, E_\Gamma, \psi_\Gamma)$ with n vertices and m edges be given. By fixing an indexing of the vertices (resp. edges), we can identify $\mathrm{Aut}(V_\Gamma)$ (resp. $\mathrm{Aut}(E_\Gamma)$) with some subgroup of S_n (resp. S_m), for the sake of simplicity. Analogously, a real function defined on V_Γ (resp. E_Γ) will be represented as an n-tuple (resp. m-tuple) of real numbers. Anyway, we shall denote vertices (resp. edges) by consecutive capital (resp. lowercase) letters and not by numerical indexes.

In this section, we will consider a space Φ_{V_Γ} of real valued functions on V_Γ, as a subspace of $\mathbb{R}^n$ endowed with the sup-norm $\|\cdot\|_\infty$; i.e., the real valued functions $\varphi \in \Phi_{V_\Gamma}$ on the vertex set V_Γ are given by column vectors $\varphi = (\varphi^1, \varphi^2, \cdots, \varphi^n)$ of length n. Analogously, the symbol Φ_{E_Γ} will refer to a subspace of $\mathbb{R}^m$ endowed with the sup-norm; i.e., the real valued functions $\varphi \in \Phi_{E_\Gamma}$ on the edge set E_Γ are given by column vectors $\varphi = (\varphi^1, \varphi^2, \cdots, \varphi^m)$ of length m.

Let G be a subgroup of the group $\mathrm{Aut}(V_\Gamma)$ (resp. $\mathrm{Aut}(E_\Gamma)$) corresponding to the group of all graph automorphisms of Γ. By the previous convention, the elements of G can be considered to be permutations of the set $\mathbb{N}_n$ (resp. $\mathbb{N}_m$).

4.2.1 GENEOs on graphs weighted on vertices

The concepts of perception pair, GEO/GENEO, and (generalized) permutant can be applied to vw-graphs.

Definition 4.2.1. *Let Φ_{V_Γ} be a set of functions from V_Γ to $\mathbb{R}$ and G be a subgroup of* $\mathrm{Aut}(V_\Gamma)$*. If (Φ_{V_Γ}, G) is a perception pair, we will call it a* vw*-graph perception pair for $\Gamma = (V_\Gamma, E_\Gamma, \psi_\Gamma)$, and will write* $\mathrm{dom}(\Phi_{V_\Gamma}) = V_\Gamma$.

Definition 4.2.2. *Let $(\Phi_{V_{\Gamma_1}}, G_1)$ and $(\Phi_{V_{\Gamma_2}}, G_2)$ be two* vw*-graph perception pairs and $T : G_1 \to G_2$ be a group homomorphism. If $F : \Phi_{V_{\Gamma_1}} \to \Phi_{V_{\Gamma_2}}$ is a GEO (resp. GENEO) from $(\Phi_{V_{\Gamma_1}}, G_1)$ to $(\Phi_{V_{\Gamma_2}}, G_2)$ with respect to T, we will say that F is a* vw-graph GEO *(resp.* vw-graph GENEO*).*

Definition 4.2.3. *Let $(\Phi_{V_{\Gamma_1}}, G_1)$ and $(\Phi_{V_{\Gamma_2}}, G_2)$ be two* vw*-graph perception pairs and $T : G_1 \to G_2$ be a group homomorphism. We say that $H \subseteq {V_{\Gamma_1}}^{V_{\Gamma_2}}$ is a* vw*-graph permutant for T if $\alpha_g(H) \subseteq H$ for every $g \in G_1$; that is, $\alpha_g(f) = g \circ f \circ T(g^{-1}) \in H$ for every $f \in H$ and $g \in G_1$.*

Let us consider some examples of vw-graph perception pairs, vw-graph GENEOs and vw-graph permutants.

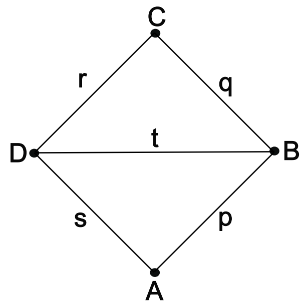

Figure 4.2.1: The graph of Example 4.2.1.

Example 4.2.1. *Consider the graph $\Gamma = (V_\Gamma, E_\Gamma, \psi_\Gamma)$ with vertex set $V_\Gamma = \{A, B, C, D\}$ and edge set $E_\Gamma = \{p = \{A, B\}, q = \{B, C\}, r = \{C, D\}, s = \{A, D\},$*

$t = \{B, D\}\}$ *(see Fig. 4.2.1). Its automorphism group* $\mathrm{Aut}(V_\Gamma)$ *is given by*

$$\mathrm{Aut}(V_\Gamma) = \{\mathrm{id}_{\mathbb{N}_4}, (A, C), (B, D), (A, C)(B, D)\}.$$

Let

$$G = \{\mathrm{id}_{\mathbb{N}_4}, \delta = (B, D)\},$$

and Φ_{V_Γ} *be the subspace of* $\mathbb{R}^4$ *given by*

$$\Phi_{V_\Gamma} := \{\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4) \in \mathbb{R}^4 \mid \varphi^1 + \varphi^3 = 0\}.$$

Clearly, $\varphi \circ \delta = (\varphi^1, \varphi^4, \varphi^3, \varphi^2) \in \Phi_{V_\Gamma}$ *for all* $\varphi \in \Phi_{V_\Gamma}$*; so,* (Φ_{V_Γ}, G) *is a* vw*-graph perception pair for* Γ*.*

The next example shows that we can have different perception pairs with the same graph and the same group.

Example 4.2.2. *Let* G *be as in Example 4.2.1 and*

$$\Phi_{V_\Gamma} = \{\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4) \in \mathbb{R}^4 \mid \sum_{i \in \mathbb{N}_4} (\varphi^i)^2 \leq 1\}.$$

Then (Φ_{V_Γ}, G) *is a* vw*-graph perception pair.*

We can now define a simple class of GENEOs.

Example 4.2.3. *Let* (Φ_{V_Γ}, G) *be as in Example 4.2.1 and a map* F *be defined by*

$$F(\varphi) = (\varphi^1/d_1, \varphi^2/d_2, \varphi^3/d_3, \varphi^4/d_4),\ \varphi \in \Phi_{V_\Gamma},\ \text{and } d_1, d_2, d_3, d_4 \in [1, \infty).$$

If, for all $\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4) \in \Phi_{V_\Gamma}$ *and* $g \in G$*, we have* $F(\varphi \circ g) = F(\varphi) \circ g$*, then*

$$\left(\frac{\varphi^1}{d_1}, \frac{\varphi^4}{d_2}, \frac{\varphi^3}{d_3}, \frac{\varphi^2}{d_4}\right) = \left(\frac{\varphi^1}{d_1}, \frac{\varphi^4}{d_4}, \frac{\varphi^3}{d_3}, \frac{\varphi^2}{d_2}\right)$$

whence $d_2 = d_4$*; and the requirement that* $F(\varphi) \in \Phi_{V_\Gamma}$ *entails* $d_1 = d_3$*.*

Moreover,

$$\|F(\varphi_1) - F(\varphi_2)\|_\infty \leq \frac{1}{\min\{d_1, d_2\}} \|\varphi_1 - \varphi_2\|_\infty \leq \|\varphi_1 - \varphi_2\|_\infty$$

for all $\varphi_1 = (\varphi_1^1, \varphi_1^2, \varphi_1^3, \varphi_1^4)$*,* $\varphi_2 = (\varphi_2^1, \varphi_2^2, \varphi_2^3, \varphi_2^4) \in \Phi_{V_\Gamma}$*, whence* F *is non-expansive.*

Therefore, the map F defined above is a vw-*graph GENEO if and only if $d_1 = d_3$ and $d_2 = d_4$.*

We now prepare for the first instances of graph permutants in Examples 4.2.6 and 4.2.7.

Example 4.2.4. *Let $\Gamma = (V_\Gamma, E_\Gamma, \psi_\Gamma)$ be the cycle graph C_4 with $V_\Gamma = \{A, B, C, D\}$. Its automorphism group is given by*

$$\mathrm{Aut}(V_\Gamma) = \{\mathrm{id}_{\mathbb{N}_4}, \alpha = (A, B, C, D), \alpha^2, \alpha^3, (A, C), (B, D), (A, B)(C, D), (A, D)(B, C)\}$$

and

$$G = \{\mathrm{id}_{\mathbb{N}_4}, \alpha, \alpha^2, \alpha^3\}$$

is a subgroup of $\mathrm{Aut}(V_\Gamma)$*.*

If Φ_{V_Γ} is the same as in Example 4.2.1, then (Φ_{V_Γ}, G) is not a vw-*graph perception pair. However, if we define*

$$\Phi_{V_\Gamma} = \{\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4) \in \mathbb{R}^4 \mid \varphi^1 + \varphi^3 = 0 = \varphi^2 + \varphi^4\}$$

then $(\Phi_{V_\Gamma}, \mathrm{Aut}(V_\Gamma))$, and therefore (Φ_{V_Γ}, G), are vw-*graph perception pairs.*

Example 4.2.5. *Let G be as in Example 4.2.4 and*

$$\Phi_{V_\Gamma} = \{\varphi \in \mathbb{R}^4 \mid \|\varphi\|_\infty \leq 1\}.$$

Then (Φ_{V_Γ}, G) is a vw-*graph perception pair.*

Example 4.2.6. *Let G be as in Example 4.2.4 and*

$$H = \{h_1 = (A, B)(C, D), h_2 = (A, D)(B, C)\} \subseteq \mathrm{Aut}(V_\Gamma).$$

Then H is a vw-*graph permutant for $T = \mathrm{id}_G$.*

Example 4.2.7. *Let Γ be as in Example 4.2.4 and*

$$G = \{\mathrm{id}_{\mathbb{N}_4}, \alpha^2, (A, B)(C, D), (A, D)(B, C)\}$$

be the Klein 4-group contained in $\mathrm{Aut}(V_\Gamma)$*. If*

$$H = \{(A, C), (B, D)\} \subseteq \mathrm{Aut}(V_\Gamma)$$

then H is a vw-*graph permutant for $T = \mathrm{id}_G$.*

As usual, in the following we will denote by K_n the complete graph on n vertices.

Proposition 4.2.1. *Let $\Gamma := K_n$ and $H \subseteq G = \mathrm{Aut}(V_\Gamma) \cong S_n$ be the set of all transpositions of V_Γ. Then H is a* vw-*graph permutant for $T = \mathrm{id}_G$.*

Proof. Let $f \in H$ and $g \in G$; we show that $g \circ f \circ g^{-1} \in H$. Let us put $f := (A, B)$ for some $A, B \in V_\Gamma$, $C := g(A)$ and $D := g(B)$. Then

$$g \circ f \circ g^{-1}(C) = g \circ f(A) = g(B) = D$$

$$g \circ f \circ g^{-1}(D) = g \circ f(B) = g(A) = C.$$

While if $L \in V_\Gamma$ is different from both C and D, then as g is bijective, $g^{-1}(L) \neq g^{-1}(C) = A$ and $g^{-1}(L) \neq g^{-1}(D) = B$. We thus have

$$g \circ f \circ g^{-1}(L) = g \circ g^{-1}(L) = L$$

whence $g \circ f \circ g^{-1} = (C, D) \in H$, as required. □

As stated in Theorem 4.1.1, the concept of a vw-graph permutant can be used to define vw-graph GENEOs.

Example 4.2.8. *Let (Φ_{V_Γ}, G) be the same as in Example 4.2.5 and H be the same as in Example 4.2.6. Set $F(\varphi) = \frac{1}{2}(\varphi \circ h_1 + \varphi \circ h_2)$. Then $F(\Phi_{V_\Gamma}) \subseteq \Phi_{V_\Gamma}$; therefore by Theorem 4.1.1, F is a* vw-*graph GENEO.*

4.2.2 GENEOs on graphs weighted on edges

The concepts of perception pair, GEO/GENEO, and (generalized) permutant can be applied to ew-graphs as well.

Definition 4.2.4. *Let Φ_{E_Γ} be a set of functions from E_Γ to $\mathbb{R}$ and G be a subgroup of* $\mathrm{Aut}(E_\Gamma)$*. If (Φ_{E_Γ}, G) is a perception pair, we will call it an* ew-*graph perception pair for $\Gamma = (V_\Gamma, E_\Gamma, \psi_\Gamma)$, and will write* $\mathrm{dom}(\Phi_{E_\Gamma}) = E_\Gamma$.

Definition 4.2.5. *Let $(\Phi_{E_{\Gamma_1}}, G_1)$ and $(\Phi_{E_{\Gamma_2}}, G_2)$ be two* ew-*graph perception pairs and $T : G_1 \to G_2$ be a group homomorphism. If $F : \Phi_{E_{\Gamma_1}} \to \Phi_{E_{\Gamma_2}}$ is a GEO (resp. GENEO) from $(\Phi_{E_{\Gamma_1}}, G_1)$ to $(\Phi_{E_{\Gamma_2}}, G_2)$ with respect to T, we will say that F is an* ew-graph GEO *(resp.* ew-graph GENEO*).*

Definition 4.2.6. *Let $(\Phi_{E_{\Gamma_1}}, G_1)$ and $(\Phi_{E_{\Gamma_2}}, G_2)$ be two* ew-*graph perception pairs and $T : G_1 \to G_2$ be a group homomorphism. We say that $H \subseteq {E_{\Gamma_1}}^{E_{\Gamma_2}}$ is an* ew-*graph permutant for T if $\alpha_g(H) \subseteq H$ for every $g \in G_1$; that is, $\alpha_g(f) = g \circ f \circ T(g^{-1}) \in H$ for every $f \in H$ and $g \in G_1$.*

The group $\mathrm{Aut}(\Gamma)$ of all graph automorphisms of a graph Γ induces a particular subgroup $\mathrm{Aut}(E_\Gamma)$ of the group S_m of all permutations of E_Γ. The elements of $\mathrm{Aut}(E_\Gamma)$ can be considered to be those permutations of E_Γ that directly correspond to the permutations of V_Γ defining all graph automorphisms of Γ.

If $\Gamma = K_n$, the group $\mathrm{Aut}(\Gamma)$ is isomorphic to S_n, and we have

$$S_n \cong \mathrm{Aut}(V_\Gamma) \cong \mathrm{Aut}(E_\Gamma) \subseteq S_m.$$

Therefore, we will consider $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(E_\Gamma)$ to be the same in this case.

Let us consider some examples of perception pairs and GENEOs in the context of ew-graphs.

Example 4.2.9. *Let $\Gamma = K_4 = (V_\Gamma, E_\Gamma, \psi_\Gamma)$ with*

$$V_{K_4} = \{A, B, C, D\}$$

$$E_{K_4} = \{p = \{A, B\}, q = \{B, C\}, r = \{A, C\}, s = \{A, D\}, t = \{B, D\}, u = \{C, D\}\}$$

(see Fig. 4.2.2), and consider the group $G = \{\mathrm{id}_{E_\Gamma},\ \delta = (r\ s)(q\ t)\} \subseteq \mathrm{Aut}(E_\Gamma)$ together with the space $\Phi_{E_\Gamma} = \{\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4, \varphi^5, \varphi^6) \mid \varphi^1 + \varphi^6 = 0\} \subseteq \mathbb{R}^m$ of the functions with opposite values on the two edges fixed by the elements of G. Clearly, $\varphi \circ \delta \in \Phi_{E_\Gamma}$, and (Φ_{E_Γ}, G) is an ew-*graph perception pair.*

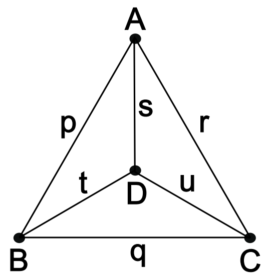

Figure 4.2.2: The complete graph K_4.

Example 4.2.10. *Let (Φ_{E_Γ}, G) be as in Example 4.2.9 and consider the map F defined by*

$$F(\varphi) := (\varphi^1/d_1, \varphi^2/d_2, \varphi^3/d_3, \varphi^4/d_4, \varphi^5/d_5, \varphi^6/d_6),$$
$$\varphi \in \Phi_{E_\Gamma},\ \textit{and}\, d_i \geq 1,\ \forall i \in \mathbb{N}_6.$$

In order that $F(\varphi) \in \Phi_{E_\Gamma}$ we should have $d_1 = d_6$, and the requirement that F be equivariant with respect to G entails $d_3 = d_4$ and $d_2 = d_5$.

Moreover, a simple computation shows that

$$\begin{aligned}\|F(\varphi_1) - F(\varphi_2)\|_\infty &\leq \frac{1}{\min\{d_1, d_2, d_3\}} \|\varphi_1 - \varphi_2\|_\infty \\ &\leq \|\varphi_1 - \varphi_2\|_\infty\end{aligned}$$

for all $\varphi_1 = (\varphi_1^i \ / \ i \in \mathbb{N}_6), \varphi_2 = (\varphi_2^i \ / \ i \in \mathbb{N}_6) \in \Phi_{E_\Gamma}$, whence F is non-expansive.

Therefore, the map F defined above is an ew*-graph GENEO if and only if $d_1 = d_6, d_2 = d_5$, and $d_3 = d_4$.*

The proof of the following proposition follows the one of Prop. 4.2.1.

Proposition 4.2.2. *Let $\Gamma := K_n$ and $H \subseteq G = \mathrm{Aut}(E_\Gamma) \cong S_n$ be the set of all edge permutations corresponding to transpositions of V_Γ. Then H is an* ew*-graph permutant for $T = \mathrm{id}_G$.*

4.3 Case Studies

We illustrate the model of Section 4.2.2 and show how graph GENEOs allow us to extract useful information from graphs. This can be done by “smart forgetting” of differences: by some sort of average, but keeping the same dimension of the space of functions (as in Sect. 4.3.1) or by dimension reduction (as in Sect. 4.3.2).

4.3.1 Subgraphs of K_4

The choice of a permutant determines how different functions are mapped to the same “signature” by the corresponding GENEO. In this subsection, we consider functions on the edge set of a complete graph K_n, taking values that are either 0 or 1; this means that each such a function identifies a subgraph of K_n. A GENEO will, in general, produce functions that can have any real value, so not representing subgraphs anymore. With the aim of getting equal results for “similar” subgraphs, we have chosen as a permutant the set of edge permutations produced by swapping two vertices in any possible way.

Let Γ be the complete graph K_4 (Fig. 4.2.2) with $\Phi_{E_\Gamma} := \mathbb{R}^6$. We have

$$S_4 \cong \mathrm{Aut}(K_4) \cong \mathrm{Aut}(E_{K_4}) \subseteq S_6.$$

The subset

$$H := \{(q,r)(s,t), (p,q)(s,u), (p,t)(r,u), (p,r)(t,u), (p,s)(q,u), (q,t)(r,s)\}$$

of $G = \mathrm{Aut}(E_{K_4})$ consisting of permutations of E_{K_4} induced by all transpositions of V_{K_4} is an ew-graph permutant for $T = \mathrm{id}_G$ by Prop. 4.2.2. Therefore, the operator $F\colon \mathbb{R}^6 \to \mathbb{R}^6$ defined by

$$F(\varphi) := \frac{1}{6}\sum_{h\in H} \varphi \circ h$$

is an ew-graph GENEO.

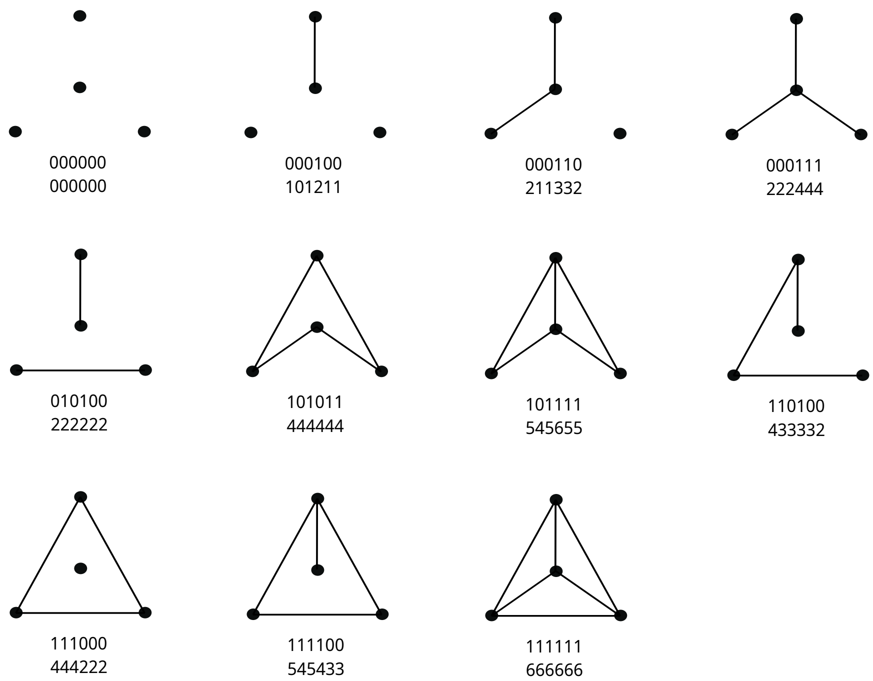

Figure 4.3.1: The subgraphs of K_4 up to isomorphisms, the 6-tuples representing each of them (above) and their F_4-codes (multiplied by 6, below).

Subgraphs of K_4 can be represented by elements of

$$\Phi_4 := \{\varphi = (\varphi^1, \cdots, \varphi^6) \in \Phi_{E_{K_4}} \mid \varphi^r \in \{0,1\}, r \in \mathbb{N}_6\}$$

and the restriction F_4 of F to $\Phi_4 \subseteq \Phi_{E_{K_4}}$ can be used to draw meaningful comparisons between them (see Fig. 4.3.1).

Definition 4.3.1. *We say that the image $F_4(\varphi)$ is an F_4-code for the subgraph $\varphi \in \Phi_4$ of K_4.*

Definition 4.3.2. *We say that an F_4-code c_1 is F_4-equivalent to an F_4-code c_2, and write $c_1 \sim_4 c_2$, if c_2 is the result of a permutation of c_1.*

Clearly, $\sim_4$ is an equivalence relation.

Definition 4.3.3. *We say that $\varphi' := (\varphi^6, \cdots, \varphi^1)$ is the reversal of $\varphi := (\varphi^1, \cdots, \varphi^6)$ $\in \Phi_{E_{K_4}}$.*

Definition 4.3.4. *Let $\varphi_1, \varphi_2 \in \Phi_{E_{K_4}}$. We say that φ_1 and φ_2 are complementary if $\varphi_1 + \varphi_2 = (1, \cdots, 1)$*

We wrote a simple program to compute all F_4-codes and found that

1. Naturally enough, isomorphic subgraphs have F_4-equivalent codes. Therefore, in some cases, it suffices to consider only the 11 non-isomorphic subgraphs of K_4.

2. Complementary subgraphs have complementary codes.

3. There is only one case, up to graph isomorphisms, in which non-isomorphic subgraphs of K_4 have F_4-equivalent codes: $\varphi_1 := (1,1,1,0,0,0)$ and $\varphi_2 := (0,0,0,1,1,1)$ with $F_4(\varphi_1) := (4,4,4,2,2,2)/6$ and $F_4(\varphi_2) := (2,2,2,4,4,4)/6$. In this case, the graphs are complementary as well, which explains why we have equivalent codes despite the graphs being non-isomorphic. Moreover, φ_1 and φ_2 are reversals of each other, and so are the corresponding codes.

4. If $\varphi_1 \in \Phi_4$ is a reversal of $\varphi_2 \in \Phi_4$, then $F_4(\varphi_1)$ is a reversal of $F_4(\varphi_2)$.

We wrote a program to compute F_5-codes for the 34 non-isomorphic subgraphs of K_5 as well and found that they were never F_5-equivalent. A similar statement holds for the complete graph K_3.

4.3.2 Graph GENEOs for C_6 and C_3

A more drastic way of quotienting differences away is dimension reduction of the space of functions. In this subsection, we use the generalized notion of permutant (Sect. 4.1), by mapping the edges of a small, auxiliary graph to the edges of the graph of interest. Note that we have great freedom, in that we are not bound to stick to graph homomorphisms.

Let $X := (V_X, E_X, \psi_X)$ be the cycle graph C_6 (see Fig. 4.3.2) with

$$V_X := \{A, B, C, D, E, F\},$$

$$E_X := \{a = \{A, B\}, b = \{B, C\}, c = \{C, D\}, d = \{D, E\}, e = \{E, F\}, f = \{A, F\}\}$$

and $Y := (V_Y, E_Y, \psi_Y)$ be the cycle graph C_3 with

$$V_Y := \{G, H, I\},$$

$$E_Y := \{g = \{G, H\}, h = \{H, I\}, i = \{G, I\}\}.$$

Their automorphisms groups respectively are the dihedral groups

$$D_6 := \{\alpha, \beta \mid \alpha^6 = \beta^2 = (\beta\alpha)^2 = 1\},$$

$$D_3 := \{\gamma, \delta \mid \gamma^3 = \delta^2 = (\delta\gamma)^2 = 1\},$$

where $\alpha := (a, b, c, d, e, f), \beta := (a, f)(b, e)(c, d), \gamma := (g, h, i)$, and $\delta := (g, i)$.

Let us put $\Phi_{E_X} := \mathbb{R}^6$, $\Phi_{E_Y} := \mathbb{R}^3$; also put $G := \mathrm{Aut}(E_X) = D_6$ and $K := \mathrm{Aut}(E_Y) = D_3$ and consider the group homomorphism $T : G \to K$ given by $T(\alpha) := \gamma$ and $T(\beta) := \delta$.

There are 216 functions $p : E_Y \to E_X$ and the equivalence class of each is an ew-graph permutant H_p (Sect. 4.1.1). For the sake of conciseness, we will denote the function $p := \{(g \mapsto e_1), (h \mapsto e_2), (i \mapsto e_3)\}$ simply by $p := \mathrm{e_1e_2e_3}$. For example, $p := \{(g \mapsto c), (h \mapsto a), (i \mapsto f)\}$ will be written as $p := \mathrm{caf}$.

The ew-graph permutants $H_p, p \in E_X^{E_Y}$ are of four possible sizes:

1. There is only 1 ew-graph permutant with 2 elements. It corresponds to the function aec.

2. There is only 1 ew-graph permutant with 4 elements. It is induced by bfd.

3. There are 5 ew-graph permutants with 6 elements each that correspond to the functions aaa, abc, ace, add, and afb.

4. There are 15 ew-graph permutants with 12 elements each that correspond to the functions aab, aac, aad, aae, aaf, abd, acb, acd, adb, adc, baa, bad, bca, bce, and bdb.

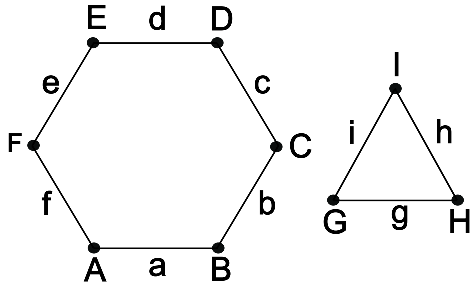

Figure 4.3.2: The cycle graphs C_6 and C_3.

Considering only the weights in $\{0, 1\}$, we wrote programs for computing the ew-graph GENEOs corresponding to the functions aec and bfd. Similar computations can be made for the rest of the functions listed above. This detailed analysis on particular functions raised a number of questions and conjectures that we plan to study in the near future.

Conclusions

We have shown that when the spaces of measurements are totally bounded and rich enough to ensure that any two points can be distinguished by our measurements, we can always embed our perception pairs and the space of GENEOs into compact perception pairs and compact spaces of GENEOs, provided that the set of our operators is collectionwise surjective. This result makes available a sound basis for further research concerning spaces of GENEOs, and paves the way for possible applications of the theory.

Of course, the computation costs might be higher while working with compactifications, but that should not be considered to be a drawback. In fact, in practical applications, one does not necessarily need to work with compactifications in an explicitly concrete manner. The mere recognition that certain spaces of GENEOs can be nicely embedded in compact ones is all that one needs most of the time.

Our research has raised several questions as well. For example, it is not clear whether the assumption of collectionwise surjectivity could be removed or made milder. Furthermore, we could wonder if our approach could be extended to the case when X and Y are endowed with a pseudo-metric instead of a metric structure, thereby extending the range of applicability of our constructions. We are planning to follow these lines of research in the near future.

GENEOs represent the possibility to formalize observer dependence in Machine Learning processes, one step towards Explainable AI [36, 37, 38]. On the other hand, they are precisely defined mathematical tools; as such, they were first conceived in a topological setting for TDA. In the same environment, permutants turn out to be effective gears for the production of GENEOs. The ever-growing use of graphs in data representation and Geometrical Deep Learning [9, 58, 59, 60, 61] motivated the present extension to the graph-theoretical domain, mirroring an analogous generalization already carried out in Topological Persistence [62].

The extension to graphs was performed here on two different lines: the first sees data as functions defined on the vertices of a graph; this is perhaps the most common use of graphs in Machine Learning. The second development line deals with functions defined on the graph edges; this is best suited for processing filtered graphs and for comparing different graphs with the same vertex set. In both models, we applied a graph-theoretical analogue of a generalized definition of permutant.

The examples studied here are just meant to show some realizations of the abstract concepts we introduced: our goal was mainly to establish a solid mathematical background. Extensions to digraphs, and possibly to hypergraphs, will follow. We hope that experimenters, interested in concrete applied problems, will make good use of the flexibility and modularity of this theory, for translating their viewpoints and biases into the language of GENEOs and permutants in the graph-theoretical setting as well.

The C++ programs used here are available at the repository:

https://gitlab.com/patrizio.frosini/graph-geneos.

Bibliography

[1] Fabio Anselmi, Lorenzo Rosasco, and Tomaso Poggio. On invariance and selectivity in representation learning. *Information and Inference: A Journal of the IMA*, 5(2):134–158, 2016.

[2] Taco Cohen and Max Welling. Group equivariant convolutional networks. In *International Conference on Machine Learning*, pages 2990–2999, 2016.

[3] Daniel E Worrall, Stephan J Garbin, Daniyar Turmukhambetov, and Gabriel J Brostow. Harmonic networks: Deep translation and rotation equivariance. In *Proc. IEEE Conf. on Computer Vision and Pattern Recognition (CVPR)*, volume 2, pages 7168–7177, 2017.

[4] Yoshua Bengio, Aaron Courville, and Pascal Vincent. Representation learning: A review and new perspectives. *IEEE Trans. Pattern Anal. Mach. Intell.*, 35(8):1798–1828, August 2013.

[5] David W. Romero, Erik J. Bekkers, Jakub M. Tomczak, and Mark Hoogendoorn. Attentive group equivariant convolutional networks. In *Proceedings of the 37th International Conference on Machine Learning (ICML'20), JMLR.org*, pages 8188–8199, 2020.

[6] Mattia G. Bergomi, Patrizio Frosini, Daniela Giorgi, and Nicola Quercioli. Towards a topological–geometrical theory of group equivariant non-expansive operators for data analysis and machine learning. *Nature Machine Intelligence*, 1:423–433, sep 2019.

[7] P. Frosini. Position paper: Towards an observer-oriented theory of shape comparison. In A. Ferreira, A. Giachetti, and D. Giorgi, editors, *Proceedings of the Eurographics 2016 Workshop on 3D Object Retrieval*, 3DOR '16, pages 5–8. Eurographics Association, 2016.

[8] Giovanni Bocchi, Patrizio Frosini, Alessandra Micheletti, Alessandro Pedretti, Carmen Gratteri, Filippo Lunghini, Andrea Rosario Beccari, and Carmine Talarico. GENEOnet: A new machine learning paradigm based on group equivariant non-expansive operators. an application to protein pocket detection. 2022. https://arxiv.org/abs/2202.00451.

[9] Michael M. Bronstein, Joan Bruna, Yann LeCun, Arthur Szlam, and Pierre Vandergheynst. Geometric deep learning: Going beyond Euclidean data. *IEEE Signal Processing Magazine*, 34(4):18–42, 2017.

[10] Michael Bronstein. Geometric foundations of deep learning, Jun 2022.

[11] Nicola Quercioli. *On the topological theory of Group Equivariant Non-Expansive Operators*. PhD thesis, Alma Mater Studiorum - Università di Bologna, Italy, 2021. http://amsdottorato.unibo.it/9770/1/Tesi-Dottorato-Quercioli.pdf.

[12] Hensel F., Moor M., and Rieck B. A survey of topological machine learning methods. *Front. Artif. Intell. 4:681108*, 2021.

[13] C. Hofer, R. Kwitt, M. Niethammer, and M. Dixit. Connectivity optimized representation learning via persistent homology. In *Proceedings of the 36th International Conference on Machine Learning, eds K. Chaudhuri and R. Salakhutdinov (PMLR)*, page 2751–2760, 2019.

[14] M. Moor, M. Horn, B. Rieck, and K. Borgwardt. Topological autoencoders. In *Proceedings of the 37th International Conference on Machine Learning, eds H. Daumé III and A. Singh (PMLR)*, pages 7045–7054, 2020.

[15] C. Chen, X. Ni, Q. Bai, and Y. Wang. A topological regularizer for classifiers via persistent homology. In *Proceedings of Machine Learning Research, eds K. Chaudhuri and M. Sugiyama (PMLR)*, pages 2573–2582, 2019.

[16] C. Hofer, F. Graf, M. Niethammer, and R. Kwitt. Topologically densified distributions. In *Proceedings of the 37th International Conference on Machine Learning, eds H. Daumé III and A. Singh (PMLR)*, pages 4304–4313, 2020.

[17] Q. Zhao, Z. Ye, C. Chen, and Y. Wang. Persistence enhanced graph neural network. In *Proceedings of the Twenty Third International Conference on Artificial Intelligence and Statistics, eds S. Chiappa and R. Calandra (PMLR)*, pages 2896–2906, 2020.

[18] C. Hofer, F. Graf, B. Rieck, M. Niethammer, and R. Kwitt. Graph filtration learning. In *Proceedings of the 37th International Conference on Machine Learning, eds H. Daumé III and A. Singh (PMLR)*, pages 4314–4323, 2020.

[19] V. Khrulkov and I. Oseledets. Geometry score: a method for comparing generative adversarial networks. In *Proceedings of the 35th International Conference on Machine Learning, eds J. Dy and A. Krause (Stockholm: PMLR)*, pages 2621–2629, 2018.

[20] B. Rieck, M. Togninalli, C. Bock, M. Moor, M. Horn, T. Gumbsch, and et al. Neural persistence: a complexity measure for deep neural networks using algebraic topology. In *International Conference on Learning Representations*, 2019.

[21] K. N. Ramamurthy, K. Varshney, and K. Mody. Topological data analysis of decision boundaries with application to model selection. In *Proceedings of the 36th International Conference on Machine Learning, eds K. Chaudhuri and R. Salakhutdinov (PMLR)*, pages 5351–5360, 2019.

[22] R. B. Gabrielsson and G. Carlsson. Exposition and interpretation of the topology of neural networks. In *18th IEEE International Conference On Machine Learning And Applications (ICMLA)*, pages 1069–1076, 2019.

[23] Fabio Anselmi, Georgios Evangelopoulos, Lorenzo Rosasco, and Tomaso Poggio. Symmetry-adapted representation learning. *Pattern Recognition*, 86:201 – 208, 2019.

[24] Stéphane Mallat. Group invariant scattering. *Communications on Pure and Applied Mathematics*, 65(10):1331–1398, 2012.

[25] Stéphane Mallat. Understanding deep convolutional networks. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 374(2065):20150203, 2016.

[26] Daniel E Worrall, Stephan J Garbin, Daniyar Turmukhambetov, and Gabriel J Brostow. Harmonic networks: Deep translation and rotation equivariance. In *Proc. IEEE Conf. on Computer Vision and Pattern Recognition (CVPR)*, volume 2, 2017.

[27] Chiyuan Zhang, Stephen Voinea, Georgios Evangelopoulos, Lorenzo Rosasco, and Tomaso Poggio. Discriminative template learning in group-convolutional

networks for invariant speech representations. In *INTERSPEECH-2015*, Dresden, Germany, 09/2015 2015. International Speech Communication Association (ISCA), International Speech Communication Association (ISCA).

[28] Herbert Edelsbrunner, David Letscher, and Afra Zomorodian. Topological persistence and simplification. *Discrete & Computational Geometry*, 28(4):511–533, 2002.

[29] David Cohen-Steiner, Herbert Edelsbrunner, and John L Harer. Stability of persistence diagrams. *Discrete & Computational Geometry*, 37(1):103–120, 2007.

[30] Gunnar Carlsson. Topology and data. *Bull. Amer. Math. Soc.*, 46(2):255–308, 2009.

[31] Herbert Edelsbrunner and John L. Harer. *Computational Topology. An introduction.* American Mathematical Society, Providence, RI, USA, 2010.

[32] Herbert Edelsbrunner and Dmitriy Morozov. Persistent homology: theory and practice. In *European Congress of Mathematics*, pages 31–50. Eur. Math. Soc., Zürich, 2013.

[33] Patrizio Frosini and Grzegorz Jabłoński. Combining persistent homology and invariance groups for shape comparison. *Discrete & Computational Geometry*, 55(2):373–409, 2016.

[34] Patrizio Frosini and Nicola Quercioli. Some remarks on the algebraic properties of group invariant operators in persistent homology. In Andreas Holzinger, Peter Kieseberg, A Min Tjoa, and Edgar Weippl, editors, *Machine Learning and Knowledge Extraction*, pages 14–24, Cham, 2017. Springer International Publishing.

[35] Francesco Conti, Patrizio Frosini, and Nicola Quercioli. On the construction of group equivariant non-expansive operators via permutants and symmetric functions. *Frontiers in Artificial Intelligence*, 5, 2022.

[36] Cynthia Rudin. Stop explaining black box machine learning models for high stakes decisions and use interpretable models instead. *Nature Machine Intelligence*, 1(5):206–215, May 2019.

[37] Ribana Roscher, Bastian Bohn, Marco F. Duarte, and Jochen Garcke. Explainable machine learning for scientific insights and discoveries. *IEEE Access*, 8:42200–42216, 2020.

[38] Luca Longo, Randy Goebel, Freddy Lecue, Peter Kieseberg, and Andreas Holzinger. Explainable artificial intelligence: Concepts, applications, research challenges and visions. In *International Cross-Domain Conference for Machine Learning and Knowledge Extraction*, pages 1–16. Springer, 2020.

[39] Francesco Camporesi, Patrizio Frosini, and Nicola Quercioli. On a new method to build group equivariant operators by means of permutants. In Andreas Holzinger, Peter Kieseberg, A Min Tjoa, and Edgar Weippl, editors, *Machine Learning and Knowledge Extraction*, pages 265–272, Cham, 2018. Springer International Publishing.

[40] Risi Kondor and Shubhendu Trivedi. On the generalization of equivariance and convolution in neural networks to the action of compact groups. In Jennifer Dy and Andreas Krause, editors, *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 2747–2755. PMLR, 10–15 Jul 2018.

[41] Giovanni Bocchi, Stefano Botteghi, Martina Brasini, Patrizio Frosini, and Nicola Quercioli. On the finite representation of group equivariant operators via permutant measures. 2020. https://arxiv.org/abs/2008.06340.

[42] Chad Giusti, Eva Pastalkova, Carina Curto, and Vladimir Itskov. Clique topology reveals intrinsic geometric structure in neural correlations. *Proceedings of the National Academy of Sciences*, 112(44):13455–13460, 2015.

[43] Louis-David Lord, Paul Expert, Henrique M Fernandes, Giovanni Petri, Tim J Van Hartevelt, Francesco Vaccarino, Gustavo Deco, Federico Turkheimer, and Morten L Kringelbach. Insights into brain architectures from the homological scaffolds of functional connectivity networks. *Frontiers in Systems Neuroscience*, 10:85, 2016.

[44] Bastian Rieck, Ulderico Fugacci, Jonas Lukasczyk, and Heike Leitte. Clique community persistence: A topological visual analysis approach for complex networks. *IEEE Transactions on Visualization and Computer Graphics*, 24(1):822–831, 2017.

[45] Alexander Port, Iulia Gheorghita, Daniel Guth, John M Clark, Crystal Liang, Shival Dasu, and Matilde Marcolli. Persistent topology of syntax. *Mathematics in Computer Science*, 12(1):33–50, 2018.

[46] Xiang Liu, Huitao Feng, Jie Wu, and Kelin Xia. Hom-complex-based machine learning (HCML) for the prediction of protein–protein binding affinity changes upon mutation. *Journal of Chemical Information and Modeling*, 62(17):3961–3969, 2022.

[47] Pietro Donatini and Patrizio Frosini. Natural pseudo-distances between closed curves. *Forum Math.*, 21(6):981–999, 2009.

[48] Pietro Donatini and Patrizio Frosini. Natural pseudodistances between closed surfaces. *J. Eur. Math. Soc. (JEMS)*, 9(2):331–353, 2007.

[49] Pietro Donatini and Patrizio Frosini. Natural pseudodistances between closed manifolds. *Forum Math.*, 16(5):695–715, 2004.

[50] Pasquale Cascarano, Patrizio Frosini, Nicola Quercioli, and Amir Saki. On the geometric and riemannian structure of the spaces of group equivariant non-expansive operators. 2021. https://arxiv.org/abs/2103.02543.

[51] J.R. Munkres. *Topology*. Featured Titles for Topology. Prentice Hall, Incorporated, 2000.

[52] Steven A. Gaal. *Point set topology*. Pure and Applied Mathematics, Vol. XVI. Academic Press, New York-London, 1964.

[53] R. Li, S. Zhong, and C. Swartz. An improvement of the Arzelà-Ascoli theorem. *Topology and its Applications*, 159:2058–2061, 2012.

[54] R. Engelking. *General Topology*, volume 6 of *Sigma Series in Pure Mathematics*. Herldermann Verlag, Berlin, 1989.

[55] Faraz Ahmad. Compactification of perception pairs and spaces of group equivariant non-expansive operators. 2022. https://arxiv.org/abs/2210.04043.

[56] Faraz Ahmad, Massimo Ferri, and Patrizio Frosini. Generalized permutants and graph GENEOs. 2022. https://arxiv.org/abs/2206.14798v1.

[57] John Adrian Bondy and Uppaluri Siva Ramachandra Murty. *Graph theory with applications*, volume 290. Macmillan London, 1976.

[58] Weihua Hu, Matthias Fey, Marinka Zitnik, Yuxiao Dong, Hongyu Ren, Bowen Liu, Michele Catasta, and Jure Leskovec. Open graph benchmark: Datasets for machine learning on graphs. *Advances in Neural Information Processing Systems*, 33:22118–22133, 2020.

[59] Pierre Latouche and Fabrice Rossi. Graphs in machine learning: an introduction. In *European Symposium on Artificial Neural Networks, Computational Intelligence and Machine Learning (ESANN), Proceedings of the 23-th European Symposium on Artificial Neural Networks, Computational Intelligence and Machine Learning (ESANN 2015)*, pages 207–218, 2015.

[60] Ilya Makarov, Dmitrii Kiselev, Nikita Nikitinsky, and Lovro Subelj. Survey on graph embeddings and their applications to machine learning problems on graphs. *PeerJ Computer Science*, 7:e357, 2021.

[61] Xiao-Shuang Li, Xiang Liu, Le Lu, Xian-Sheng Hua, Ying Chi, and Kelin Xia. Multiphysical graph neural network (MP-GNN) for covid-19 drug design. *Briefings in Bioinformatics*, 2022.

[62] Mattia G Bergomi, Massimo Ferri, Pietro Vertechi, and Lorenzo Zuffi. Beyond topological persistence: Starting from networks. *Mathematics*, 9(23):3079, 2021.