Alma Mater Studiorum · Università di Bologna

DOTTORATO DI RICERCA IN MATEMATICA
Ciclo XXXV

Settore Concorsuale: 01/A2
Settore Scientifico Disciplinare: MAT/02

HOMOLOGICAL AND HODGE-THEORETIC ASPECTS OF MATROIDS AND POLYMATROIDS

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Esame finale anno 2023

A Giuliana, Oda, Luigi e Gino,
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Abstract

In the first part of this thesis, we study the action of the automorphism group of a matroid on the homology space of the co-independent complex. This representation turns out to be isomorphic, up to tensoring with the sign representation, with that on the homology space associated with the lattice of flats. In the case of the cographic matroid of the complete graph, this result has application in algebraic geometry: indeed De Cataldo, Heinloth and Migliorini use this outcome to study the Hitchin fibration [dCHM21]. In the second part, on the other hand, we use ideas from algebraic geometry to prove a purely combinatorial result. We construct a Leray model for a discrete polymatroid with arbitrary building set and we prove a generalized Goresky-MacPherson formula. The first row of the model is the Chow ring of the polymatroid; we prove Poincaré duality, Hard-Lefschetz theorem and Hodge-Riemann relations for the Chow ring.

Introduction

Since its introduction by Whitney [Whi35] in 1935, matroid theory has received increasing attention because of its multiple connections with algebraic geometry, algebraic topology and representation theory; but also for the countless applications in graph theory, network theory and combinatorial optimization. Particularly, in recent years long standing conjectures about log-concavity of polynomials have been brilliantly solved by studying the Chow ring of matroids [Huh12, HK12, Len13, HW17, AHK18, BES19, BHM+20b, ADH20, BEST21]. In [Rea68] Read conjectured the unimodality of the chromatic polynomial; the relative log-concavity was conjectured by Hoggar in [Hog74]. These conjectures have been extended to arbitrary matroid by Rota and Heron in [Rot71, Her72]. The first step towards proving these conjectures has been done by Huh in [Huh12]: indeed he managed to prove the log-concavity for any realizable matroid over characteristic zero. Huh was able to relate the coefficients of the characteristic polynomial of a matroid to the Milnor numbers of an arrangement of hyperplanes, linking a purely combinatorial problem to a geometric one. The geometric construction comes from a milestone article in the literature, *Wonderful models of subspace arrangements* [DCP95] by De Concini and Procesi. The authors built a smooth compactification of the complement of a subspace arrangement, in which the arrangement is replaced by a simple normal crossing divisor. Working with a realizable matroid, Huh used the Hodge-Riemann relations for the De Concini-Procesi wonderful model of a realizing hyperplane arrangement to obtain the log-concavity conjecture. The work of De Concini-Procesi was extended by Feichtner and Yuzvinsky in [FY04], by building a *Chow ring* for an arbitrary matroid who specializes to the wonderful model cohomology ring in the realizable setting. In [HK12] Huh and Katz performed a step further proving the conjecture in the case of realizable matroids over some field. Afterwards, it became more and more evident how the Hodge-Riemann relations for the Chow ring are crucial in the log-concavity test. Although the Chow ring is also defined in the non-realizable setting, it was not clear how to prove the Hodge-Riemann relations without being able to work on the underlying variety. In the great work *Hodge Theory for combinatorial geometries* [AHK18] Adiprasito, Huh and Katz proved the log-concavity conjecture of the characteristic polynomial of an arbitrary matroid, by developing a combinatorial version of Hard-Lefschetz theorems and Hodge-Riemann relations. For all these extraordinary results, revealing a new connection between algebraic geometry and combinatorics, June Huh was recently awarded the Fields Medal.

This thesis is divided into two independent parts (which correspond respectively to Chapter 2 and Chapter 3): in the first we prove a purely combinatorial result which has application to algebraic geometry ([dCHM21]); in the second, on the other hand, we use geometric intuition and ideas from algebraic geometry to obtain combinatorial results.

In the first part, in a joint work with Luca Moci [MP21], given a group G of automorphisms of a matroid M, we relate the representations of G on the homology of the independence complex of the dual matroid M^* to the representations on the homology of the lattice of flats of M, and (when M is realizable) to the top cohomology of a hyperplane arrangement. Furthermore, we analyze in detail the case of the complete graph. One motivation for this work comes from a paper by de Cataldo, Heinloth and Migliorini ([dCHM21]), that computes the supports of the perverse cohomology sheaves of the Hitchin fibration for GL_m over the locus of reduced spectral curves, studying the related Cattani-Kaplan-Schmid complex. The dual graph of such a spectral curve is the complete graph, and the action of the symmetric group on the irreducible components of the curve yields an action on its vertices, hence on the independence complex of the dual matroid of the graph. A crucial step in the analysis performed in [dCHM21] is then to determine the representations of the symmetric group on the homology of this independence complex.

In the second part of this thesis, we deal with a natural question concerning polymatroids: do the recent results about the Chow ring of matroids, mentioned above, hold also for polymatroids [BEST21, Question 1.5]? *Polymatroids* generalize arrangements of subspaces in the same way as matroids generalize hyperplane arrangements. In a joint work with Roberto Pagaria [PP21], we construct a Leray model for a discrete polymatroid with arbitrary building set and we prove a generalized Goresky-MacPherson formula. Furthermore, we prove Poincaré duality, Hard Lefschetz, and Hodge-Riemann theorems for the Chow ring of the polymatroid. Finally, we provide a relative Lefschetz decomposition with respect to the deletion of an element. Recently, in [CHL+22] Crowley, Huh, Larson, Simpson and Wang introduce the notion of Bergman fan of a polymatroid: a combinatorial model for the wonderful compactification of a subspace arrangement. They prove that the Chow ring of the Bergman fan is isomorphic to the Chow ring of the polymatroid introduced in [PP21]. Using the Bergman fan, they establish the Kähler package for the Chow ring; recovering our result and also expanding our σ-cone in which Hard-Lefschetz theorem and Hodge-Riemann relations hold (See Remark 14).

Main results

In this section we present the main results of Chapter 2 and Chapter 3 of this thesis.

Representations on the homology of matroids

In a joint work with Luca Moci, we prove that the reduced homology of the independence complex of the dual matroid M^*, up to a shift to a sign, is $\mathbb{C}[G]$-isomorphic to the reduced homology of the non-spanning complex of M, $NS(M)$, and to the reduced homology of the order complex of the lattice of flats of M, $\mathcal{L}(M)$:

Theorem A. *Let M be a matroid of rank r on n elements and let G be a group of automorphisms of M, we have the following isomorphism of $\mathbb{C}[G]$-modules for every $i \geq 0$ (and nonzero only for $i = r-2$):*

1. $\widetilde{H}_{n-3-i}(IN(M^*)) \otimes \mathrm{sgn}$
2. $\widetilde{H}_i(NS(M))$
3. $\widetilde{H}_i(\mathcal{L}(M))$

Here, the isomorphism between (1) and (2) holds more generally for any simplicial complex, being a consequence of Alexander duality. Let K be an abstract simplicial complex with vertex set V. The *Alexander dual* of K is the simplicial complex on the same vertex set defined by $K^* = \{\sigma \subseteq V \mid (V \setminus \sigma) \notin K\}$.

Theorem B. *Let K be an abstract simplicial complex and let K^* be its Alexander dual. Let G be a finite group of automorphisms of the face poset of K. Then we have the following isomorphism of $\mathbb{C}[G]$-modules:*

$$\widetilde{H}_i(K, \mathbb{C}) \simeq_G \widetilde{H}^{n-i-3}(K^*, \mathbb{C}) \otimes \mathrm{sgn}.$$

Also the isomorphism between (2) and (3) is a consequence of a more general phenomenon. Let L be a lattice and C a *cross-cut* of L, i.e., a particular subset of L satisfying three conditions (see Definition 2.6); we can associate to each cross-cut C of L an abstract simplicial complex $K(C)$ and we define the homology space of C as the homology space of $K(C)$.

Theorem C. *Let L be a lattice and G a group of automorphism of L. Let C be a G-stable cross-cut of L. Then we have the following $\mathbb{C}[G]$-module isomorphism:*

$$\widetilde{H}_i(L) \simeq_G \widetilde{H}_i(C).$$

Let M be a realizable matroid, then it is naturally associated with a hyperplane arrangement $\mathcal{A}$. The cohomology of the complement $\mathcal{M}(\mathcal{A})$ of the arrangement admits a well-known presentation in terms of M, due to Orlik and Solomon ([OS80]). We show that the top-degree part of this cohomology is isomorphic as a representation of G, up to a sign, to the reduced homology of the dual matroid $M^*_{\mathcal{A}}$ associated to $\mathcal{A}$. Using Theorem A and results of Orlik and Solomon [OS80] we get the following:

Theorem D. *Let $\mathcal{A}$ be a central essential hyperplane arrangement of dimension r and let $M_{\mathcal{A}}$ be the associated matroid with ground set of cardinality n. Then we have the following $\mathbb{C}[G]$-module isomorphism:*

$$H^r(\mathcal{M}(\mathcal{A})) \simeq_G H_{n-r-1}(IN(M^*_{\mathcal{A}})) \otimes \mathrm{sgn}.$$

Let $M(K_m)$ be the matroid associate to the complete graph K_m, which has rank $r = m-1$ and ground set of cardinality $n = \binom{m}{2}$. In fact, this is the case of interest in [dCHM21]. Notice that the lattice of flats of this matroid is isomorphic to the partition lattice Π_m. In [Sta82] Stanley describes explicitly the representations on the top homology of the partition lattice as induced representations of an m-root of the unity from a subgroup C_m generated by an m-cycle to $\mathfrak{S}_m$:

$$\widetilde{H}_{m-3}(\Pi_m) \simeq_{\mathfrak{S}_m} \mathrm{sgn} \otimes \mathrm{ind}_{C_m}^{\mathfrak{S}_m}(e^{2\pi i/m}).$$

In this case, Theorem A specializes to the following:

Theorem E. *Let $M(K_m)$ be the matroid associate to the complete graph K_m, we have the following $\mathbb{C}[\mathfrak{S}_m]$-module isomorphism*

$$\widetilde{H}_{n-3-i}(IN(M^*(K_m))) \simeq_{\mathfrak{S}_m} \mathrm{sgn} \otimes \widetilde{H}_i(\Pi_m) \simeq_{\mathfrak{S}_m} \mathrm{ind}_{C_m}^{\mathfrak{S}_m}(e^{2\pi i/m}),$$

where $n = \binom{m}{2}$ is the number of edges K_m, and C_m is the subgroup generated by an m-cycle in $\mathfrak{S}_m$.

This result is the case of interest of [dCHM21].

Hodge theory for polymatroids

Let $\mathcal{A}$ be a subspace arrangement in a $\mathbb{C}$-vector space V, we define the complement of the arrangement in the following way:

$$\mathcal{M}(\mathcal{A}) = \mathbb{C}^r \setminus \bigcup_{U \in \mathcal{A}} U.$$

The problem of computing the cohomology of the complement of a subspace arrangement was solved by Goresky and MacPherson [GM88] and by De Concini and Procesi [DCP95] with different techniques. Goresky and MacPherson used stratified Morse theory to describe the cohomology spaces of $\mathcal{M}(\mathcal{A})$ using the combinatorial data of the lattice of intersections $\mathcal{L}$:

Theorem (Goresky MacPherson '88)**.** *Let $\mathcal{A}$ be an arrangement of subspaces with complement $\mathcal{M}(\mathcal{A})$ and lattice of intersections $\mathcal{L}$, there exists an isomorphism*

$$\widetilde{H}^k(\mathcal{M}(\mathcal{A}), \mathbb{Z}) \cong \bigoplus_{W \in \mathcal{L} \setminus \hat{0}} \tilde{H}_{2\,\mathrm{cd}(W)-2-k}\Big(\Delta((\hat{0}, W)), \mathbb{Z})\Big)$$

where $\Delta((\hat{0}, W))$ is the order complex of the interval $(\hat{0}, W)$.

Instead, in [DCP95] De Concini and Procesi built a rational model for $\mathcal{M}(\mathcal{A})$ and proved that the rational cohomology algebra of $\mathcal{M}(\mathcal{A})$ is uniquely defined by the lattice of intersections. If $\mathcal{A}$ is an arrangement of subspaces with complement $\mathcal{M}(\mathcal{A})$, a *wonderful model* is a smooth projective variety Y containing $\mathcal{M}(\mathcal{A})$ as an open set, such that $Y \backslash \mathcal{M}(\mathcal{A})$ is a simple normal crossing divisor, i.e., the irreducible components are smooth and locally intersects as coordinates hyperplanes. This wonderful model $Y_{\mathcal{G}}$ is obtained from $\mathbb{P}^r$ by a sequence of blowups along some linear subspaces, the collection of those subspaces is called *building set* $\mathcal{G}$. The variety $Y_{\mathcal{G}}$ is used for studying the complement of the subspace arrangement, by considering the Leray spectral sequence for the inclusion of the complement in the wonderful model $Y_{\mathcal{G}}$. The spectral sequence collapses at the third page yielding a

Leray model $(B^{\bullet,\bullet}(\mathcal{A},\mathcal{G}),d)$ (also known as Morgan algebra [Mor78]) for the rational homotopy type. Furthermore, we have the following

$$B^{\bullet,0}(\mathcal{A},\mathcal{G}) = H^\bullet(Y_\mathcal{G}) \quad \text{and} \quad H^\bullet(B^{\bullet,\bullet}(\mathcal{A},\mathcal{G}),d) = H^\bullet(\mathcal{M}(\mathcal{A})).$$

The problem that arises spontaneously is how to combine these two results, i.e., how to explicitly find the ring structure from combinatorics. In particular how to relate the multiplication to the local homology of $\mathcal{L}$ that occurs in the Goresky-MacPherson formula. In [Yuz02] Yuzvinsky solved this problem only for a maximal building set. Yuzvinsky finds a significantly smaller subalgebra $\mathrm{CM}(\mathcal{A},\mathcal{G}_{max})$ quasi-isomorphic to $B^{\bullet,\bullet}(\mathcal{A},\mathcal{G}_{max})$ whence also a rational model of $\mathcal{M}(\mathcal{G})$. The algebra $\mathrm{CM}(\mathcal{A},\mathcal{G}_{max})$ gives a multiplicative structure on the flag complexes of $\mathcal{L}(\mathcal{A})$ that induces the ring structure on $H^\bullet(\mathcal{M}(\mathcal{A}))$.

Theorem (Yuzvinsky '02)**.** *Let $\mathcal{A}$ be a subspace arrangement with complement $\mathcal{M}(\mathcal{A})$ and lattice of intersection $\mathcal{L}=\mathcal{L}(\mathcal{A})$, there is an isomorphism*

$$\widetilde{H}^k(\mathcal{M}(\mathcal{A}),\mathbb{Q}) \cong \widetilde{H}^k(\mathrm{CM}(\mathcal{A},\mathcal{G}_{max}),\mathbb{Q}) \cong \bigoplus_{W\in\mathcal{L}\smallsetminus\hat{0}} \tilde{H}_{2\,\mathrm{cd}(W)-2-k}\Big(\Delta((\hat{0},W)),\mathbb{Q}\Big)$$

where $\Delta((\hat{0},W))$ is the order complex of the interval $(\hat{0},W)$.

In a joint work with Roberto Pagaria, we extend these results of Yuzvinsky to a non realizable setting and to an arbitrary building set. A *polymatroid* is a pair $P=(E,\mathrm{cd})$ where E is a finite ground set and $\mathrm{cd}\colon 2^E \to \mathbb{N}$ is a *increasing submodular function*. If P is realized by a subspace arrangement, then cd is the codimension of the corresponding flat. Inspired by the realizable case, we give a combinatorial definition of *building set* for polymatroids and we introduce a *Leray model* $B(P,\mathcal{G})$ for a polymatroid with building set. In the case of matroids the Leray model was recently studied by Bibby, Denham, and Feichtner [BDF20]. The last combinatorial object that we need is the $\mathcal{G}$*-nested set complex* $n(P,\mathcal{G})$. In the realizable case this complex remembers whether the intersection of the corresponding divisors in $Y_\mathcal{G}$ is non-empty.

Theorem F. *Let P be a polymatroid. The inclusion $\mathrm{CM}(P,\mathcal{G}) \hookrightarrow B(P,\mathcal{G})$ is a quasi-isomorphism. Furthermore*

$$H^\bullet(B(P,\mathcal{G})) \cong H^\bullet(\mathrm{CM}(P,\mathcal{G})) \cong \bigoplus_{f\in L} \bigotimes_{g\in F} \tilde{H}_{2\,\mathrm{cd}(g)-2-\bullet}\Big(n((\hat{0},g),\mathcal{G})\Big).$$

In the realizable case with maximal building set, the above decomposition specializes to the Goresky-MacPherson formula.

The Leray model contains a subalgebra $\mathrm{DP}(P,\mathcal{G})$ as the first row of the spectral sequence, we call this algebra the *Chow ring* of the polymatroid. For subspace arrangements, $\mathrm{DP}(P,\mathcal{G})$ is the cohomology (indeed the Chow ring) of the wonderful model $Y_\mathcal{G}$. The combinatorial Chow ring for matroids was studied by Feichtner and Yuzvinsky [FY04] and later by Huh, Katz, and Adiprasito [Huh12, HK12, AHK18] and others. We prove that the Chow ring $\mathrm{DP}(P,\mathcal{G})$ of a polymatroid satisfies the Kähler package (see Theorems 3.31 and 3.43).

Theorem G. *The ring $\mathrm{DP}(P,\mathcal{G})$ has the Poincaré duality property. Moreover, there exists a simplicial cone $\Sigma_{P,\mathcal{G}}$ contained in $\mathrm{DP}^1(P,\mathcal{G})$ such that for each $\ell \in \Sigma_{P,\mathcal{G}}$ the Hard-Lefschetz theorem and the Hodge-Riemann relations hold.*

We proved the above theorem using methods similar to ones in [AHK18]. A second and easier proof of the Kähler package for matroid was given in [BHM+20a] using a semismall decomposition; the decomposition is the first step through the singular Hodge theory [BHM+20b]. In the realizable setting the decomposition is induced by a map between wonderful models that is semismall (for semismall maps in algebraic geometry see [dCM02, dCM09]). In the case of polymatroids the corresponding map is not semismall, hence we cannot deduce the Kahler package using this method. However we obtain a relative Lefschetz decomposition of the Chow ring.

Theorem H. *Let $\mathrm{DP}_{(a)}$ be the Chow ring for the polymatroids $P \setminus a$ where an element $a \in E$ is removed from the ground set. The Chow ring $\mathrm{DP}(P,\mathcal{G})$*

decomposes into irreducible $\mathrm{DP}_{(a)}$*-modules as*

$$\mathrm{DP}(P, \mathcal{G}) = \mathrm{DP}_{(a)} \oplus \bigoplus_{f \in S_a} \bigoplus_{k=1}^{n_f} x_f^k \, \mathrm{DP}_{(a)} \, .$$

Moreover, the irreducibles are explicitly described by:

$$x_f^k \, \mathrm{DP}_{(a)} \cong \mathrm{DP}((P \setminus a)^{f \setminus a}, (\mathcal{G} \setminus a)^{f \setminus a}) \otimes \mathrm{DP}(P_f, \mathcal{G}_f)[k].$$

The *reduced characteristic polynomial* of a polymatroid is defined by

$$\overline{\chi}_P(\lambda) = \frac{\sum_{A \subseteq E} (-1)^{|A|} \lambda^{\mathrm{cd}(E) - \mathrm{cd}(A)}}{\lambda - 1}.$$

As final step we relate the coefficients of the reduced characteristic polynomial to the Hodge-Riemann bilinear form (see Theorem 3.53). In order to do that, we restrict to the case of maximal building set and we fix an isomorphism $\deg \colon \mathrm{DP}^r(P, \mathcal{G}_{\max}) \to \mathbb{Q}$.

Theorem I. *There exist elements* $\alpha, \beta \in \mathrm{DP}^1(P, \mathcal{G}_{\max})$ *such that*

$$\overline{\chi}_P(\lambda) = \sum_{i=0}^{r} (-1)^i \deg(\alpha^i \beta^{r-i}) \lambda^i.$$

The element α belongs to the closure of the σ-cone (morally it is nef), but in general β is not in the closure of the ample cone. Hence, the coefficients of the reduced characteristic polynomial do not form a log-concave sequence (see Remark 16). Indeed every finite sequence of non-positive integers can appear as a substring of the coefficients.

Overview

In Chapter 1 we recall all the preliminaries needed for understanding the following chapters. In Section 1.1 we introduce the basic definition related to poset and its homology, also providing the example of the partition lattice Π_n. In Section 1.2 we give all the fundamental definitions of matroid theory, focusing on various way to define a matroid and on the lattice of flats associated to each matroid which will result crucial for this thesis. In Section 1.3 we introduce the notion of polymatroids with its properties, analyzing some differences with matroids. We also focus on realizable polymatroids, i.e., polymatroids induced by an arrangement of subspaces. We describe the problem of computing the cohomology of the complement of an arrangement, showing the two different approaches of Goresky-MacPherson and De Concini-Procesi.

In Chapter 2 we describe the representations of a group of automorphisms G of a matroid M on the homology of the independence complex of the dual matroid M^*. These representations are related to the homology of the lattice of flats of M. In Section 2.1 we study the representations of a group G on the homology spaces of any abstract simplicial complex Δ and its Alexander dual Δ^*, showing that the two $\mathbb{C}[G]$-module are isomorphic up to tensoring with the sign representation (see Theorem 2.2). In Section 2.2 we develop an equivariant version of Folkman's machinery of *cross-cuts* [Fol66]: see in particular our Theorem 2.11. In Section 2.3 we specialize the previous results to the case of matroids, obtaining the fundamental result of this chapter, see Theorem 2.17. In Section 2.4, we show that the top-degree part of the cohomology of the complement $\mathcal{M}(\mathcal{A})$ of a hyperplane arrangement is isomorphic as a representation of G, up to a sign, to the reduced homology of the dual matroid $M^*_{\mathcal{A}}$ associated to $\mathcal{A}$ (see Theorem 2.19). In Section 2.5 we focus on the case when M is realized by a *coned graph* or by a *complete bipartite graph*. Then our Theorem 2.17, combined with results by Kook and Lee ([Koo07, KL18]), yields isomorphisms with the $\mathbb{C}[G]$-modules of *edge-rooted* and *B-edged rooted forests* (See Equations 2.11, 2.12). Finally, in Section 2.6 we specialize our results to the case in which M is the matroid of the complete graph K_m (See Theorem 2.23).

In Chapter 3 we construct a Leray model for a discrete polymatroid and we prove a generalized Goresky-MacPherson formula. We prove the Kähler package for the Chow ring of the polymatroid. In Section 3.1 we use Gröbner basis theory in order to give two explicit bases of the Leray model (See Definition 3.4 and Theorem 3.7). In Section 3.2, by using algebraic discrete Morse theory, we compute the cohomology of the Leray model generalizing the Goresky-MacPherson formula (See Theorem 3.24). In Section 3.3, we use an inductive procedure to prove the Kähler package (See Theorem 3.31 and Theorem 3.43). The main difference with the previous methods is that we do not have partial building sets as in [BDF20] nor order filters as in [AHK18]. Our induction is based on the cardinality of the building set, and the inductive step involved completely different polymatroids. Section 3.4 is devoted to the proof of the relative Lefschetz decomposition using some lemmas from the previous sections (See Theorem 3.46). The reduced characteristic polynomial is studied in Section 3.5. We prove the claimed equality by showing that both polynomials satisfy the same recursion (See Theorem 3.53). In this proof we used the properties of the Möbius function for posets. Finally, Section 3.6 contains an explicit and exhaustive example that illustrates our definitions and properties.

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Chapter 1

Preliminaries

In this Chapter we recall all the fundamental notions needed for the understanding of this thesis.

1.1 Poset and its homology

Definition 1.1. A partially ordered set is a set X taken together with a partial order on it. Formally, a partially ordered set is defined as an ordered pair $P = (X, \leqslant)$ where X is called the ground set of P and $\leqslant$ the partial order of P.

Definition 1.2. Given two posets $(S, \leqslant_S)$ and $(T, \leqslant_T)$, an *order isomorphism* from $(S, \leqslant_S)$ to $(T, \leqslant_T)$ is a bijective function f from S to T with the property that, for every $x, y \in S$:

$$x \leqslant_S y \Longleftrightarrow f(x) \leqslant_T f(y).$$

An order isomorphism from a poset to itself is called an order automorphism.

An *upper bound* of a subset X of a poset P is an element $a \in P$ such that $a \geqslant x$, $\forall x \in X$. An upper bound b of a subset X is called *least upper bound* (*join*) if for all upper bounds z of X in P, $z \geqslant b$.

The notions of *lower bound* of X and *greatest lower bound* (*meet*) of X are defined dually.

Definition 1.3. In a poset P an element p *covers* an element q when $\nexists z \in P$ such that:

$$q < z < p.$$

An *atom* in P is an element that covers a minimal element $\widehat{0}$. A *coatom* in P is an element that is covered by a maximal element $\widehat{1}$.

Definition 1.4. A *lattice* is a poset for which any two elements x and y have a least upper bound (join) $x \vee y$ and greatest lower bound (meet) $x \wedge y$. A finite lattice is *semi-modular* if whenever x and y cover $x \wedge y$ (i.e, $\nexists z$ such that $x \wedge y \ < z < x$ or $x \wedge y \ < z < y$), then $x \vee y$ covers both x and y.

Definition 1.5. A finite lattice is *geometric* if it is semimodular and every element is a join of *atoms*.

Definition 1.6. The *Möbius function* μ of a finite lattice L is a function of two lattice-variables which for all $x, y \in L$ satisfies the following properties:

$$\mu(x,y) = \begin{cases} 1, & \text{if } x = y \\ -\sum_{x \leqslant z < y} \ \mu(x,z), & \text{if } x < y \\ 0, & \text{if } x \nleqslant y \end{cases}$$

We now introduce a poset that will be useful for the work of this thesis. We use the following notation $[n] = \{1, 2, \ldots, n\}$.

Definition 1.7. Let Π_n denote the poset of all set partitions of $[n]$, ordered by refinement. Thus the elements of Π_n are sets:

$$\beta = \{B_1, \ldots, B_k\}$$

where the B_i's are pairwise-disjoint nonempty subsets of $[n]$ with union $[n]$. Moreover:

$$\{B'_1, \ldots, B'_j\} \leqslant \{B_1, \ldots, B_k\}$$

if and only if every B'_r is contained in some B_s.

$$\begin{tikzcd}[column sep=small] & 123 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ 12|3 & 13|2 & 23|1 \\ & 1|2|3 \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

Figure 1.1: The Hasse diagram of the poset Π_3

Example 1.1. Let Π_3 be the partition lattice of $\{1, 2, 3\}$, then:

$$\Pi_3 = \{1|2|3, 12|3, 13|2, 23|1, 123\}$$

The maximal chains are:

$$a_1 = 1|2|3 \leqslant 12|3 \leqslant 123 \quad a_2 = 1|2|3 \leqslant 13|2 \leqslant 123 \quad a_3 = 1|2|3 \leqslant 23|1 \leqslant 123$$

Theorem 1.8. *The poset Π_n is a geometric lattice of rank $n - 1$.*

Proof. See [Bir67, Theorem 12; page 95]. □

1.1.1 On the homology of a poset

Let P be a finite poset. A *chain* is a totally ordered subset of a poset P. The *length* of a finite chain C is $l(C) = |C| - 1$. We assume that P has a unique minimal element $\widehat{0}$, a unique maximal element $\widehat{1}$ and that every maximal chain has the same length n; we call such a poset *ranked.* Define the rank function:

$$r : \quad P \quad \longrightarrow \quad \mathbb{N}$$

by setting $r(x)$ equal to the length of any chain in the interval $\left[\widehat{0}, x\right] = \{y \mid \widehat{0} \leqslant y \leqslant x\}$.

Definition 1.9. Let P be a ranked poset of rank n and let $S \subseteq [n-1] = \{1, 2, \ldots, n-1\}$. We define the *rank-selected subposet* P_S of P by:

$$P_S = \{x \in P \mid r(x) \in S\} \cup \{\widehat{0}, \widehat{1}\}$$

We refer to [Hat02] for all the terminology of simplicial complexes and homology.

Definition 1.10. Let Q be any poset with $\widehat{0}$ and $\widehat{1}$, then define the *order complex* $\Delta(Q)$ to be the abstract simplicial complex whose vertices are the elements of $\overline{Q} = Q \setminus \{\widehat{0}, \widehat{1}\}$ and whose faces (or simplices) are the chains

$$x_0 < x_1 < \cdots < x_k \qquad \text{in } Q \setminus \{\widehat{0}, \widehat{1}\}$$

Definition 1.11. The reduced (co)homology of a poset P, $\widetilde{H}_i(P, R)$, with coefficients in a ring R is defined as the reduced simplicial (co)homology of its order complex $\widetilde{H}_i(\Delta(P), R)$. For $x < y$ in P, we write $\widetilde{H}_i(x, y)$ for the homology of the open interval (x, y).

We review explicitly these concepts for poset in terms of chains, we use the terminology of [Wac07]. Let P be a poset with $\widehat{0}$ and $\widehat{1}$ and let j be an integer, we define the chain space

$$C_j(P, R) := R\text{-module freely generated by } j\text{-chains of } \overline{P},$$

where R is a ring. The boundary map $\partial_j : C_j(P, R) \to C_{j-1}(P, R)$ is defined by

$$\partial_j(x_1 < \cdots < x_{j+1}) = \sum_{i=1}^{j+1} (-1)^i (x_1 < \cdots \hat{x}_i < \cdots < x_{j+1}),$$

where $\hat{\cdot}$ denotes deletion. Clearly we have that $\partial_{j-1} \circ \partial_j = 0$, which makes $(C_j(P, R), \partial_j)$ an algebraic complex. Therefore we define the cycle space $Z_j(P; R) := \ker \partial_j$ and the boundary space $B_j(P, R) := \operatorname{Im} \partial_{j+1}$. The homology space of the poset P in degree j is defined by

$$\widetilde{H}_j(P, R) := {Z_j(P; R)}/{B_j(P, R)}.$$

The coboundary map $\delta_j : C_j(P, R) \leftarrow C_{j+1}(P, R)$ is defined by

$$\delta_j(x_1 < \cdots < x_j) =$$

$$\sum_{i=1}^{j+1} (-1)^i \sum_{x \in (x_{i-1}, x_i)} (x_1 < \cdots < x_{i-1} < x < x_i < \cdots < x_j),$$

for all chains $x_1 < \cdots < x_j$. We define the cocycle space to be $Z^j(P,R) := \ker \delta_j$ and the coboundary space to be $B^j(P,R) := \mathrm{Im}(\delta_{j-1})$. The cohomology space of the poset P in degree j is defined to be

$$\widetilde{H}^j(P,R) := {Z^j(P;R)}\big/{B^j(P,R)}.$$

When R is a field, $\widetilde{H}_j(P,R)$ and $\widetilde{H}^j(P,R)$ are isomorphic vector spaces; this follows from the Universal Coefficient Theorem (see [Hat02]). The j-th (reduced) Betti number of P is given by $\dim \widetilde{H}_j(P,\mathbb{C})$, which is the same as the rank of $\widetilde{H}^j(P,\mathbb{Z})$.

Throughout this section we work with complex coefficients, then we denote by $\widetilde{H}_i(Q)$ the reduced simplicial homology group $\widetilde{H}_i(Q,\mathbb{C})$. Recall that for any simplicial complex Δ, $\widetilde{H}_{-1}(\Delta,\mathbb{C}) = 0$ unless $\Delta = \emptyset$, while by definition $\widetilde{H}_{-1}(\emptyset,\mathbb{C}) \simeq \mathbb{C}$ and $\widetilde{H}_i(\emptyset,\mathbb{C}) = 0$ for $i \geqslant 0$.

Now suppose G is a subgroup of order automorphism of P (see Definition 1.2). For any $S \subseteq [n-1]$ G permutes the maximal chain of P_S. Let C_S be the free module over $\mathbb{C}$ on the set of maximal chains of P_S:

$$C_S = \langle a_1, \ldots, a_r \rangle \qquad a_i \text{ maximal chains of } P_S$$

Let α_S^P denote the permutation representation of G on C_S:

$$\begin{array}{rccc} \alpha_S^P: & G & \longrightarrow & \mathrm{GL}(C_S) \\ & g & \longmapsto & \left(\alpha_g : C_S \to C_S\right) \end{array} \tag{1.1}$$

where $\alpha_g(a_i) = g.a_i$.

Example 1.2. Let $\mathcal{P}_4 = (\{1,2,3,4\},\subseteq)$ be the Boolean lattice with four elements and let $S = \{1\} \subseteq [3]$:

$$P_S = \mathcal{P}_{4_S} = \{\emptyset, \{1,2,3,4\}, \{1\}, \{2\}, \{3\}, \{4\}\}.$$

The maximal chains of P_S are:

$$a_1 = \emptyset \subseteq \{1\} \subseteq \{1,2,3,4\} \qquad a_2 = \emptyset \subseteq \{2\} \subseteq \{1,2,3,4\}$$

$$a_3 = \emptyset \subseteq \{3\} \subseteq \{1,2,3,4\} \qquad a_4 = \emptyset \subseteq \{4\} \subseteq \{1,2,3,4\}$$

Let C_S be the free-module over $\mathbb{C}$ having $\{a_1, a_2, a_3, a_4\}$ as basis:

$$C_S = \{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4; \quad \lambda_i \in \mathbb{C}\}$$

The group of order automorphisms of $\mathcal{P}_4$ is $\mathfrak{S}_4$. We choose $G = \mathfrak{S}_4$.

$$\begin{array}{rccc} \alpha_S^P : & \mathfrak{S}_4 & \longrightarrow & \mathrm{GL}(C_S) \\ & (123) & \longmapsto & \alpha_{(123)} \\ & (34) & \longmapsto & \alpha_{(34)} \end{array}$$

$$\begin{array}{rccc} \alpha_{(123)} : & C_S & \longrightarrow & C_S \\ & a_1 & \longmapsto & a_2 \\ & a_2 & \longmapsto & a_3 \\ & a_3 & \longmapsto & a_1 \\ & a_4 & \longmapsto & a_4 \end{array} \qquad \alpha_{(123)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rccc} \alpha_{(34)} : & C_S & \longrightarrow & C_S \\ & a_1 & \longmapsto & a_1 \\ & a_2 & \longmapsto & a_2 \\ & a_3 & \longmapsto & a_4 \\ & a_4 & \longmapsto & a_3 \end{array} \qquad \alpha_{(34)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $\chi_{\alpha_S^P}$ be the character of the representation α_S^P, we have the following

$$\chi_{\alpha_S^P}((123)) = 1 \qquad \text{Number of maximal chains fixed by } (123)$$

$$\chi_{\alpha_S^P}((34)) = 2 \qquad \text{Number of maximal chains fixed by } (34)$$

As we have seen in the previous example, $\chi_{\alpha_S^P}((g))$ is the number of maximal chains of P_S fixed by g. In particular, $\chi_{\alpha_S^P}((\mathrm{Id}))$ is just the number of maximal chains of P_S.

The group G acts on each reduced homology group $\widetilde{H}_i(P_S)$ with $-1 \leqslant i \leqslant |S| - 1$. Let $\gamma_{S,i}$ denote this representation of G:

$$\gamma_{S,i} : \; G \longrightarrow \mathrm{GL}(\widetilde{H}_i(P_S))$$

Now define a virtual representation $\beta_S = \beta_S^P$ of G by:

$$\beta_S = \sum_{i=-1}^{|S|-1} (-1)^{|S|-1-i}\ \gamma_{S,i} \tag{1.2}$$

In particular, when $S = \emptyset$ then β_S is the trivial representation, i.e., $\beta_S(g) = \gamma_{S,-1}(g) = \mathrm{Id}$ for all $g \in G$.

Example 1.3. Let $\mathcal{P}_3 = (2^{\{1,2,3\}}, \subseteq)$ be the Boolean lattice with three elements and let $S = \{1, 2\}$. In this case we have that $P_S = \mathcal{P}_3$. In order to calculate the homology of $\mathcal{P}_3$ we have to consider $\overline{\mathcal{P}_3}$ and his order complex:

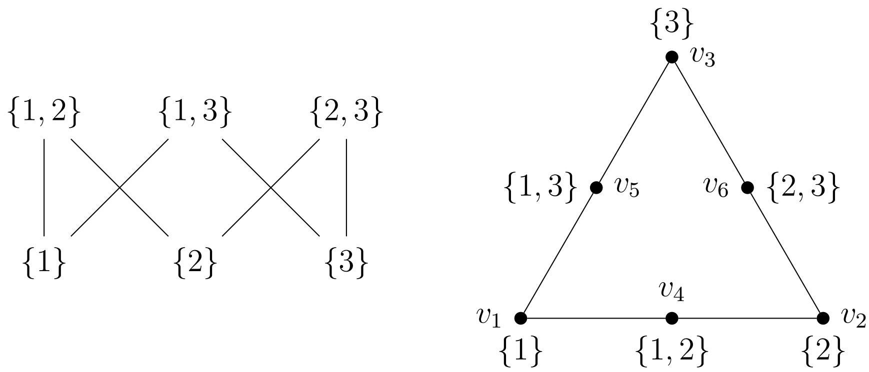


Figure 1.2: The poset $\overline{\mathcal{P}_3}$ on the left and its order complex $\Delta(\overline{\mathcal{P}_3})$ on the right

Observing the order complex of $\mathcal{P}_3$ we immediately notice that:

$$\widetilde{H}_i(\mathcal{P}_3) = 0 \quad \text{for } i = -1, 0 \qquad \text{and} \qquad \widetilde{H}_1(\mathcal{P}_3) \simeq \mathbb{C}.$$

We explicitly calculate $\widetilde{H}_1(\mathcal{P}_3)$ to see the action of the group on it.

The vertex set of $\Delta(\mathcal{P}_3)$ is $\{v_1, \ldots, v_6\}$. The 1-chains of $\overline{\mathcal{P}_3}$ are:

$$\overline{a_1} = [v_1, v_4] \qquad \overline{a_2} = [v_1, v_5] \qquad \overline{a_3} = [v_2, v_4]$$

$$\overline{a_4} = [v_2, v_6] \qquad \overline{a_5} = [v_3, v_5] \qquad \overline{a_6} = [v_3, v_6]$$

Hence:

$$C_0 = \Big\{ \sum_{i=1}^{6} \lambda_i v_i,\ \lambda_i \in \mathbb{C} \Big\} \qquad C_1 = \Big\{ \sum_{i=1}^{6} \lambda_i \overline{a_i},\ \lambda_i \in \mathbb{C} \Big\} \qquad C_2 = 0$$

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

$$\begin{array}{rcl} \partial_1 : \; C_1 & \longrightarrow & C_0 \\ \overline{a_1} & \longmapsto & v_4 - v_1 \\ \overline{a_2} & \longmapsto & v_5 - v_1 \\ \overline{a_3} & \longmapsto & v_4 - v_2 \\ \overline{a_4} & \longmapsto & v_6 - v_2 \\ \overline{a_5} & \longmapsto & v_5 - v_3 \\ \overline{a_6} & \longmapsto & v_6 - v_3 \end{array} \qquad \partial_1 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\widetilde{H}_1(\mathcal{P}_3) = \ker(\partial_1)/\mathrm{Im}(\partial_2) = \ker(\partial_1) \qquad \ker(\partial_1) = \begin{cases} x_1 = -t \\ x_2 = t \\ x_3 = t \\ x_4 = -t \\ x_5 = -t \\ x_6 = t \end{cases}$$

$$\widetilde{H}_1(\mathcal{P}_3) = \mathrm{Span}\{\underbrace{-[v_1, v_4] + [v_1, v_5] + [v_2, v_4] - [v_2, v_6] - [v_3, v_5] + [v_3, v_6]}_{l}\}$$

Let G be a subgroup of order automorphism of $\mathcal{P}_3$, so G is a subgroup of $\mathfrak{S}_3$. Choose $G = \mathfrak{S}_3$ and let's calculate $\gamma_{[2],1}$:

$$\begin{array}{rcl} \gamma_{[2],1} : \; \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(\widetilde{H}_1(\mathcal{P}_3)) \\ \mathrm{Id} & \longmapsto & \gamma_{\mathrm{Id}} \\ (12) & \longmapsto & \gamma_{(12)} \\ (123) & \longmapsto & \gamma_{(123)} \end{array}$$

$$\begin{array}{rcl} \gamma_{\mathrm{Id}} : \; \widetilde{H}_1(\mathcal{P}_3) & \longrightarrow & \widetilde{H}_1(\mathcal{P}_3) \\ l & \longmapsto & l \end{array} \qquad \begin{array}{rcl} \gamma_{(12)} : \; \widetilde{H}_1(\mathcal{P}_3) & \longrightarrow & \widetilde{H}_1(\mathcal{P}_3) \\ l & \longmapsto & -l \end{array}$$

$$\begin{array}{rcl} \gamma_{(123)} : \; \widetilde{H}_1(\mathcal{P}_3) & \longrightarrow & \widetilde{H}_1(\mathcal{P}_3) \\ l & \longmapsto & l \end{array}$$

$\gamma_{[2],1}$ is isomorphic to the sign representation:

$$\begin{array}{rcl} \rho : \; \mathfrak{S}_3 & \longrightarrow & \mathbb{C}^* \\ g & \longmapsto & \mathrm{sgn}(g) \end{array}$$

And in this case we have $\beta_{[2]} = \gamma_{[2],1}$.

To be able to enunciate the next theorem we need two results due to Baclawsky and Björner ([BB79]); we begin by setting some notation.

Definition 1.12. Given a poset P and a order automorphism f, we write P^f for the *fixed point set*:

$$P^f = \{x \in P \mid x = f(x)\}.$$

Definition 1.13. Let P be a poset and let $\epsilon_i(P)$ be the number of i-chains of $\overline{P} = P \setminus \{\widehat{0}, \widehat{1}\}$. The Euler-characteristic $\mathcal{E}(P)$ is defined by:

$$\mathcal{E}(P) = \sum_{i=0}^{+\infty} (-1)^i \ \epsilon_i(P).$$

In particular $\mathcal{E}(\emptyset) = 0$.

The well known Euler-Poincaré formula states that

$$\mathcal{E}(P) = \sum_{n=0}^{+\infty} (-1)^n \ \dim_{\mathbb{C}} H_n(P, \mathbb{C}).$$

We can also introduce the definition of reduced Euler characteristic:

$$\widetilde{\mathcal{E}}(P) = \sum_{n=-1}^{+\infty} (-1)^n \ \dim_{\mathbb{C}} \widetilde{H}_n(P, \mathbb{C}).$$

It is easy to see that:

$$\mathcal{E}(P) = \widetilde{\mathcal{E}}(P) + 1.$$

It is a theorem of P. Hall [Rot64a, Prop 6, page 346] that:

$$\mathcal{E}(P) = \mu(P) + 1, \quad \text{i.e.} \quad \widetilde{\mathcal{E}}(P) = \mu(P) \tag{1.3}$$

with $\mu(P) = \mu\big(\widehat{0}, \widehat{1}\big)$ the Möbius function of P.

Definition 1.14 (Lefschetz Number)**.** Let P be a finite poset. For an order automorphism f of P let:

$$f_n: \ H_n(P, \mathbb{C}) \longrightarrow H_n(P, \mathbb{C}) \qquad \widetilde{f}_n: \ \widetilde{H}_n(P, \mathbb{C}) \longrightarrow \widetilde{H}_n(P, \mathbb{C})$$

be the linear maps induced on homology and reduced homology respectively. The *Lefschetz number* of f is

$$\Lambda(f) = \sum_{n=0}^{+\infty} (-1)^n \operatorname{Tr}(f_n)$$

and the *reduced Lefschetz number* of f is

$$\widetilde{\Lambda}(f) = \sum_{n=-1}^{+\infty} (-1)^n \operatorname{Tr}(\widetilde{f}_n).$$

Theorem 1.15 (Hopf-Lefschetz fixed point theorem)**.** *Let P be a finite poset and let f be an order automorphism of P. Then*

$$\Lambda(f) = \mathcal{E}(P^f) \qquad \widetilde{\Lambda}(f) = \widetilde{\mathcal{E}}(P^f).$$

In particular, if $\Lambda(f) \neq 0$, then $P^f \neq \emptyset$.

Proof. See [BB79, Theorem 1.1; page 265]. The result is stated for ordinary simplicial homology, but the proof works just as well for reduced simplicial homology. □

Recall the definition of α_S and β_T respectively from Equation (1.1) and (1.2). Now we can state the following theorem of Stanley ([Sta82]):

Theorem 1.16. *The representation α_S and the virtual representation β_S are related by the formulas:*

$$\alpha_S = \sum_{T \subseteq S} \beta_T \tag{1.4}$$

$$\beta_S = \sum_{T \subseteq S} (-1)^{|S \smallsetminus T|} \alpha_T. \tag{1.5}$$

Proof. Let P be a finite poset and let G be a group of order automorphism of P. Let $\widetilde{\Lambda}_S(g)$ be the Lefschetz number of the map $g \in G$ working in P_S subposet of P:

$$\widetilde{\Lambda}_S(g) = \sum_{n=-1}^{+\infty} (-1)^n \operatorname{Tr}(\widetilde{g}_n) \qquad \widetilde{g}_n : \widetilde{H}_n(P_S, \mathbb{C}) \longrightarrow \widetilde{H}_n(P_S, \mathbb{C}).$$

Recall that:

$$\beta_S = \sum_{n=-1}^{|S|-1} (-1)^{|S|-1-n} \; \gamma_{S,n} \qquad \begin{array}{rccc} \gamma_{S,n}: & G & \longrightarrow & \mathrm{GL}(\widetilde{H}_n(P_S)) \\ & g & \longmapsto & \widetilde{g}_n. \end{array}$$

The character of the virtual representation β_S is

$$\begin{aligned} \chi_{\beta_S}(g) &= \sum_{n=-1}^{|S|-1} (-1)^{|S|-1-n} \chi_{\gamma_{S,n}}(g) = \sum_{n=-1}^{|S|-1} (-1)^{|S|-1-n} \mathrm{Tr}(\widetilde{g}_n) \\ &= (-1)^{1-|S|} \sum_{n=-1}^{|S|-1} (-1)^n \mathrm{Tr}(\widetilde{g}_n). \end{aligned}$$

Since $\widetilde{H}_n(P_S, \mathbb{C}) = 0$ for all $n > |S| - 1$, we have that

$$\widetilde{\Lambda}_S(g) = \sum_{n=-1}^{|S|-1} \; (-1)^n \; \mathrm{Tr}(\widetilde{g}_n).$$

But as far as we see before we get

$$\chi_{\beta_S}(g) = (-1)^{1-|S|} \; \widetilde{\Lambda}_S(g) \qquad \widetilde{\Lambda}_S(g) = (-1)^{|S|-1} \; \chi_{\beta_S}(g). \tag{1.6}$$

Let $\widetilde{\mathcal{E}}(P_S^g)$ be the reduced Euler-characteristic of the subposet P_S^g of P_S. By applying Theorem 1.15 to the poset P_S we have that:

$$\widetilde{\Lambda}_S(g) = \widetilde{\mathcal{E}}(P_S^g). \tag{1.7}$$

By definition of the Euler characteristic, recalling that $\chi_{\alpha_S}(g)$ is the number of maximal chains of P_S fixed by g, we claim that

$$\widetilde{\mathcal{E}}(P_S^g) = \sum_{T \subseteq S} \; (-1)^{|T|-1} \; \chi_{\alpha_T}(g).$$

Hence for (1.7):

$$(-1)^{|S|-1} \; \chi_{\beta_S}(g) = \sum_{T \subseteq S} \; (-1)^{|T|-1} \; \chi_{\alpha_T}(g)$$

$$\chi_{\beta_S}(g) = \sum_{T \subseteq S} \; (-1)^{|T|-|S|} \; \chi_{\alpha_T}(g) = \sum_{T \subseteq S} \; (-1)^{|S \setminus T|} \; \chi_{\alpha_T}(g)$$

for all $g \in G$, so Equation (1.5) follows. For obtaining Equation (1.4), it suffices to apply the Inclusion-Exclusion principle. $\square$

Example 1.4. Let $\mathcal{P}_3 = (2^{\{1,2,3\}}, \subseteq)$ be the Boolean lattice with three elements. We want to calculate explicitly β_S with the new characterization provided by Theorem 1.16 and compare the result with that of the previous example. Since $\text{length}(P) = 3$, we take $S = \{1, 2\} \subseteq [2]$.

In this case we have that $P_S = \mathcal{P}_3$. We want to calculate:

$$\beta_{[2]} = \sum_{T \subseteq [2]} (-1)^{|[2] \smallsetminus T|} \ \alpha_T$$

i) For the first element of the sum let us consider $T = S = [2]$. The maximal chains of P_S are:

$$a_1 = \emptyset \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \qquad a_2 = \emptyset \subseteq \{1\} \subseteq \{1,3\} \subseteq \{1,2,3\}$$

$$a_3 = \emptyset \subseteq \{2\} \subseteq \{1,2\} \subseteq \{1,2,3\} \qquad a_4 = \emptyset \subseteq \{2\} \subseteq \{2,3\} \subseteq \{1,2,3\}$$

$$a_5 = \emptyset \subseteq \{3\} \subseteq \{1,3\} \subseteq \{1,2,3\} \qquad a_6 = \emptyset \subseteq \{3\} \subseteq \{2,3\} \subseteq \{1,2,3\}$$

Let C_S be the free-module over $\mathbb{C}$ with $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ as basis:

$$C_S = \{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 + \lambda_5 a_5 + \lambda_6 a_6; \quad \lambda_i \in \mathbb{C}\}$$

Choose $G = \mathfrak{S}_3$.

$$\begin{array}{rccc} \alpha_S : & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(C_S) \\ & \mathrm{Id} & \longmapsto & I_6 \\ & (12) & \longmapsto & \alpha^S_{(12)} \\ & (123) & \longmapsto & \alpha^S_{(123)} \end{array}$$

$$\begin{array}{rccc} \alpha^S_{(12)} : & C_S & \longrightarrow & C_S \\ & a_1 & \longmapsto & a_3 \\ & a_2 & \longmapsto & a_4 \\ & a_3 & \longmapsto & a_1 \\ & a_4 & \longmapsto & a_2 \\ & a_5 & \longmapsto & a_6 \\ & a_6 & \longmapsto & a_5 \end{array} \qquad \alpha^S_{(12)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{rccc} \alpha^S_{(123)}: & C_S & \longrightarrow & C_S \\ & a_1 & \longmapsto & a_4 \\ & a_2 & \longmapsto & a_3 \\ & a_3 & \longmapsto & a_6 \\ & a_4 & \longmapsto & a_5 \\ & a_5 & \longmapsto & a_1 \\ & a_6 & \longmapsto & a_2 \end{array} \qquad \alpha^S_{(123)} = \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}$$

It turns out that α_S is the regular representation of $\mathfrak{S}_3$.

ii) We now consider $T = T_1 = \emptyset \subseteq S$. The only maximal chain of P_{T_1} is

$$b_1 = \emptyset \subseteq \{1, 2, 3\}.$$

Let C_{T_1} be the free-module over $\mathbb{C}$ with $\{b_1\}$ as basis:

$$C_{T_1} = \{\lambda_1 b_1;\quad \lambda_1 \in \mathbb{C}\} \qquad \begin{array}{rccc} \alpha_{T_1}: & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(C_{T_1}). \\ & g & \longmapsto & \mathrm{Id} \end{array}$$

iii) Let $T = T_2 = \{1\} \subseteq S$. The maximal chains of P_{T_2} are

$$c_1 = \emptyset \subseteq \{1\} \subseteq \{1, 2, 3\} \quad c_2 = \emptyset \subseteq \{2\} \subseteq \{1, 2, 3\} \quad c_3 = \emptyset \subseteq \{3\} \subseteq \{1, 2, 3\}$$

Let C_{T_2} be the free-module over $\mathbb{C}$ with $\{c_1, c_2, c_3\}$ as basis:

$$C_{T_2} = \{\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3;\quad \lambda_i \in \mathbb{C}\}$$

$$\begin{array}{rccc} \alpha_{T_2}: & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(C_{T_2}) \\ & \mathrm{Id} & \longmapsto & I_3 \\ & (12) & \longmapsto & \alpha^{T_2}_{(12)} \\ & (123) & \longmapsto & \alpha^{T_2}_{(123)} \end{array}$$

$$\begin{array}{rccc} \alpha^{T_2}_{(12)}: & C_{T_2} & \longrightarrow & C_{T_2} \\ & c_1 & \longmapsto & c_2 \\ & c_2 & \longmapsto & c_1 \\ & c_3 & \longmapsto & c_3 \end{array} \qquad \alpha^{T_2}_{(12)} = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$$

$$\begin{array}{rccc} \alpha^{T_2}_{(123)}: & C_{T_2} & \longrightarrow & C_{T_2} \\ & c_1 & \longmapsto & c_2 \\ & c_2 & \longmapsto & c_3 \\ & c_3 & \longmapsto & c_1 \end{array} \qquad \alpha^{T_2}_{(123)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The representation α_{T_2} is isomorphic to the direct sum of the trivial and the standard representations of $\mathfrak{S}_3$, i.e.,

$$\chi_{\alpha_{T_2}} = \chi_{\ydiagram{2,1}} + \chi_{\ydiagram{3}}.$$

iv) Let $T = T_3 = \{2\} \subseteq S$. The maximal chains of P_{T_3} are:

$$d_1 = \emptyset \subseteq \{1,2\} \subseteq \{1,2,3\} \qquad d_2 = \emptyset \subseteq \{1,3\} \subseteq \{1,2,3\}$$

$$d_3 = \emptyset \subseteq \{2,3\} \subseteq \{1,2,3\}$$

Let C_{T_3} be the free-module over $\mathbb{C}$ with $\{d_1, d_2, d_3\}$ as basis:

$$C_{T_3} = \{\lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3; \quad \lambda_i \in \mathbb{C}\}$$

$$\begin{array}{rccc} \alpha_{T_3}: & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(C_{T_3}) \\ & \mathrm{Id} & \longmapsto & I_3 \\ & (12) & \longmapsto & \alpha^{T_3}_{(12)} \\ & (123) & \longmapsto & \alpha^{T_3}_{(123)} \end{array}$$

$$\begin{array}{rccc} \alpha^{T_3}_{(12)}: & C_{T_3} & \longrightarrow & C_{T_3} \\ & d_1 & \longmapsto & d_1 \\ & d_2 & \longmapsto & d_3 \\ & d_3 & \longmapsto & d_2 \end{array} \qquad \alpha^{T_3}_{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{rccc} \alpha^{T_3}_{(123)}: & C_{T_3} & \longrightarrow & C_{T_3} \\ & d_1 & \longmapsto & d_3 \\ & d_2 & \longmapsto & d_1 \\ & d_3 & \longmapsto & d_2 \end{array} \qquad \alpha^{T_3}_{(123)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

It turns out that the representation α_{T_3} is isomorphic to the representation α_{T_2}, hence:

$$\chi_{\alpha_{T_3}} = \chi_{\ydiagram{2,1}} + \chi_{\ydiagram{3}}.$$

Now we compute the character of the representation $\beta_{[2]}$:

$$\beta_{[2]} = \alpha_{T_1} - \alpha_{T_2} - \alpha_{T_3} + \alpha_S$$

$$\chi_{\beta_{[2]}}(\mathrm{Id}) = \chi_{\alpha_{T_1}}(\mathrm{Id}) - \chi_{\alpha_{T_2}}(\mathrm{Id}) - \chi_{\alpha_{T_3}}(\mathrm{Id}) + \chi_{\alpha_S}(\mathrm{Id}) = 1 - 3 - 3 + 6 = 1$$

$$\chi_{\beta_{[2]}}((12)) = 1 - 1 - 1 + 0 = -1$$

$$\chi_{\beta_{[2]}}((123)) = 1 - 0 - 0 + 0 = 1$$

$\beta_{[2]}$ is isomorphic to the sign representation:

$$\begin{array}{rccc} \rho: & \mathfrak{S}_3 & \longrightarrow & \mathbb{C}^* \\ & g & \longmapsto & \mathrm{sgn}(g) \end{array}$$

The result is consistent with the previous example. In this case, $\beta_{[2]}$ coincides with the calculation of $\gamma_{[2],1}$.

We try to generalize this result for some types of poset:

Definition 1.17. A finite ranked poset P with $\widehat{0}$ and $\widehat{1}$ is said to be *Cohen–Macaulay* (over $\mathbb{C}$) if for every interval $I = [x, y] = \{z \ : \ x \leqslant z \leqslant y\}$ of P we have:

$$\widetilde{H}_i(I) = 0 \qquad \text{whenever } i \neq \dim\,\Delta(I).$$

Theorem 1.18. *If P is a Cohen–Macaulay poset of rank n and if $S \subseteq [n-1]$, then P_S is also Cohen–Macaulay.*

Proof. See [Bac80, Theorem 6.4; page 247]. □

Let P be a Cohen–Macaulay poset with $\widehat{0}$ and $\widehat{1}$, it follows from (1.2), that

$$\beta_S = \gamma_{S,s-1} \qquad s = |S|.$$

In other words:

Theorem 1.19. *If P is a Cohen–Macaulay poset then β_S is isomorphic to the representation $\gamma_{S,s-1}$ of G on the top reduced homology group $\widetilde{H}_{s-1}(P_S)$.*

Now we report a result due to Folkman which applies to any geometric lattice:

Theorem 1.20. *Let L be a geometric lattice of rank r and let μ denote the Möbius function of L. Then:*

$$\dim\left(\widetilde{H}_i(L)\right) = \begin{cases} |\mu(\widehat{0},\widehat{1})|, & \text{if } i = r-2 \\ 0, & \text{if } i \neq r-2 \end{cases}$$

Proof. See [Fol66, Theorem 4.1; page 634]. □

1.1.2 On the homology of the partition lattice

Let Π_n denote the poset of all partitions of $[n]$, ordered by refinement introduced in Definition 1.7.

Proposition 1.21. *Let μ be the Möbius function of the lattice of partitions Π_n, then:*

$$\mu(\widehat{0},\widehat{1}) = (-1)^{n-1}\,(n-1)!.$$

Proof. See [Rot64a, Proposition 3; page 359]. □

From Theorem 1.20 it follows that every geometric lattice L is a Cohen-Macaulay poset and then the only non-zero reduced homology group of L is the $(r-2)$-th. Hence, Π_n is a Cohen-Macaulay poset.

The symmetric group $\mathfrak{S}_n$ acts as an order automorphism group on Π_n by permuting the letters of the partitions. For example, the transposition (12) acting on the partition $13|2|4$ yields the partition $23|1|4$. Let's see an example of how (12) acts on a chain:

$$1|2|3|4 < 23|1|4 < 234|1 < 1234 \quad\longmapsto\quad 1|2|3|4 < 13|2|4 < 134|2 < 1234$$

Our aim is to study the representation γ_{n-3} of $\mathfrak{S}_n$ on the top homology group $\widetilde{H}_{n-3}(\Pi_n)$ of Π_n .

If we take $S = [n-2]$, we obtain from Theorem 1.19:

$$(\Pi_n)_S = \{x \in \Pi_n \mid x = \widehat{0} \;\vee\; x = \widehat{1} \;\vee\; r(x) \in S\} = \Pi_n$$

$$\beta_{[n-2]} = \sum_{i=-1}^{n-3} \ (-1)^{n-3-i} \ \gamma_{S,i} = \gamma_{S,n-3} = \gamma_{n-3}.$$

So in the case of the partition lattice the representation we are looking for is $\beta_{[n-2]} = \gamma_{n-3}$:

$$\beta_{[n-2]} = \gamma_{n-3} : \ \ \mathfrak{S}_n \ \ \longrightarrow \ \ \mathrm{GL}(\widetilde{H}_{n-3}(\Pi_n)).$$

By Theorem 1.20 and Proposition 1.21 we have that:

$$\dim(\widetilde{H}_{n-3}(\Pi_n)) = |\mu(\widehat{0}, \widehat{1})| = |(-1)^{n-1} \ (n-1)!| = (n-1)!.$$

Thus, $\beta_{[n-2]}$ is a representation of $\mathfrak{S}_n$ of dimension $(n-1)!$. Before describing $\beta_{[n-2]}$ more explicitly, let's make an example:

Example 1.5. Consider the partition lattice Π_4 of $[4]$ with every maximal chain of length 3 and with $S = [2]$, we always consider S maximal since we want that:

$$P_S = P = \Pi_4.$$

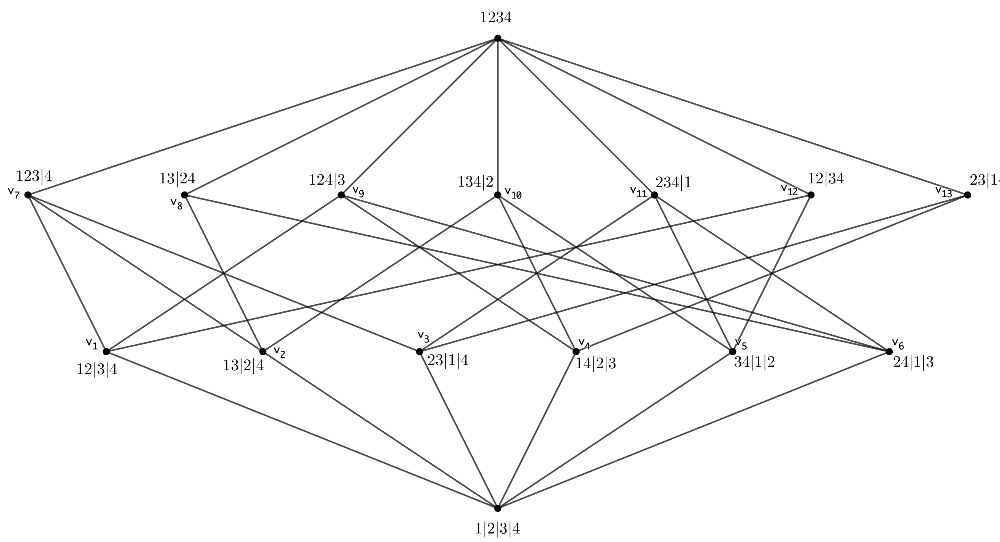


Figure 1.3: The partition lattice Π_4

$$\mu(\Pi_4) = \mu(\widehat{0}, \widehat{1}) = -3! = -6 \qquad \widetilde{H}_1(\Pi_4) \simeq \mathbb{C}^6$$

$$C_1 \simeq \mathbb{C}^{18} \qquad C_0 \simeq \mathbb{C}^{13}$$

$$\mathcal{B}_{C_1} = \Big\{ \overset{a_1}{[v_1, v_7]}, \overset{a_2}{[v_1, v_9]}, \overset{a_3}{[v_1, v_{12}]}, \overset{a_4}{[v_2, v_7]}, \overset{a_5}{[v_2, v_8]}, \overset{a_6}{[v_2, v_{10}]}, \overset{a_7}{[v_3, v_7]}, \overset{a_8}{[v_3, v_{11}]}, \overset{a_9}{[v_3, v_{13}]},$$

$$\overset{a_{10}}{[v_4, v_9]}, \overset{a_{11}}{[v_4, v_{10}]}, \overset{a_{12}}{[v_4, v_{13}]}, \overset{a_{13}}{[v_5, v_{10}]}, \overset{a_{14}}{[v_5, v_{11}]}, \overset{a_{15}}{[v_5, v_{12}]}, \overset{a_{16}}{[v_6, v_8]}, \overset{a_{17}}{[v_6, v_9]}, \overset{a_{18}}{[v_6, v_{11}]} \Big\}$$

$$\mathcal{B}_{C_0} = \Big\{ v_1, v_2, \ldots, v_{13} \Big\}$$

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \qquad \widetilde{H}_1(\Pi_4) = \ker(\partial_1)/\mathrm{Im}(\partial_2) = \ker(\partial_1)$$

$$\partial_1 = \begin{pmatrix}
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\widetilde{H}_1(\Pi_4) = \mathrm{Span}\{ -a_4 + a_5 + a_7 \overset{b_1}{-} a_8 - a_{16} + a_{18}; -a_4 + a_5 + a_7 - a_9 \overset{b_2}{-} a_{10} + a_{12} - a_{16} + a_{17};$$

$$-a_4 + a_6 + a_7 \overset{b_3}{-} a_9 - a_{11} + a_{12}; a_2 - a_3 - a_{10} \overset{b_4}{+} a_{11} - a_{13} + a_{15};$$

$$-a_8 + a_9 + a_{11} \overset{b_5}{-} a_{12} - a_{13} + a_{14}; a_1 - a_2 - a_7 \overset{b_6}{+} a_9 + a_{10} - a_{12} \} =$$

$$= \{ b_1, b_2, b_3, b_4, b_5, b_6 \}.$$

We indicate with $\overline{C_S}$ the free-module over $\mathbb{C}$ on the set of maximal chains of $P_S = \Pi_4$ without the minimal and maximal element. Thus,

$$\overline{C_S} = \overline{C_{[2]}} = C_1$$

Before we can calculate γ_1 we need to see how $\alpha = \alpha_{[2]}$ works on C_1:

$$\begin{array}{rccc} \alpha : & \mathfrak{S}_4 & \longrightarrow & \mathrm{GL}(C_1) \\ & (12) & \longmapsto & \alpha_{(12)} \\ & (123) & \longmapsto & \alpha_{(123)} \\ & (1234) & \longmapsto & \alpha_{(1234)} \\ & (12)(34) & \longmapsto & \alpha_{(12)(34)} \end{array}$$

$\alpha_{(12)}(a_1) = a_1 \quad \alpha_{(12)}(a_2) = a_2 \quad \alpha_{(12)}(a_3) = a_3 \quad \alpha_{(12)}(a_4) = a_7 \quad \alpha_{(12)}(a_5) = a_9$

$\alpha_{(12)}(a_6) = a_8 \quad \alpha_{(12)}(a_7) = a_4 \quad \alpha_{(12)}(a_8) = a_6 \quad \alpha_{(12)}(a_9) = a_5 \quad \alpha_{(12)}(a_{10}) = a_{17}$

$\alpha_{(12)}(a_{11}) = a_{18} \quad \alpha_{(12)}(a_{12}) = a_{16} \quad \alpha_{(12)}(a_{13}) = a_{14} \quad \alpha_{(12)}(a_{14}) = a_{13}$

$\alpha_{(12)}(a_{15}) = a_{15} \quad \alpha_{(12)}(a_{16}) = a_{12} \quad \alpha_{(12)}(a_{17}) = a_{10} \quad \alpha_{(12)}(a_{18}) = a_{11}$

$\alpha_{(123)}(a_1) = a_7 \quad \alpha_{(123)}(a_2) = a_8 \quad \alpha_{(123)}(a_3) = a_9 \quad \alpha_{(123)}(a_4) = a_1 \quad \alpha_{(123)}(a_5) = a_3$

$\alpha_{(123)}(a_6) = a_2 \quad \alpha_{(123)}(a_7) = a_4 \quad \alpha_{(123)}(a_8) = a_6 \quad \alpha_{(123)}(a_9) = a_5 \quad \alpha_{(123)}(a_{10}) = a_{18}$

$\alpha_{(123)}(a_{11}) = a_{17} \quad \alpha_{(123)}(a_{12}) = a_{16} \quad \alpha_{(123)}(a_{13}) = a_{10} \quad \alpha_{(123)}(a_{14}) = a_{11}$

$\alpha_{(123)}(a_{15}) = a_{12} \quad \alpha_{(123)}(a_{16}) = a_{15} \quad \alpha_{(123)}(a_{17}) = a_{14} \quad \alpha_{(123)}(a_{18}) = a_{13}$

$\alpha_{(1234)}(a_1) = a_8 \quad \alpha_{(1234)}(a_2) = a_7 \quad \alpha_{(1234)}(a_3) = a_9 \quad \alpha_{(1234)}(a_4) = a_{18} \quad \alpha_{(1234)}(a_5) = a_{16}$

$\alpha_{(1234)}(a_6) = a_{17} \quad \alpha_{(1234)}(a_7) = a_{14} \quad \alpha_{(1234)}(a_8) = a_{13} \quad \alpha_{(1234)}(a_9) = a_{15} \quad \alpha_{(1234)}(a_{10}) = a_1$

$\alpha_{(1234)}(a_{11}) = a_2 \quad \alpha_{(1234)}(a_{12}) = a_3 \quad \alpha_{(1234)}(a_{13}) = a_{10} \quad \alpha_{(1234)}(a_{14}) = a_{11}$

$\alpha_{(1234)}(a_{15}) = a_{12} \quad \alpha_{(1234)}(a_{16}) = a_5 \quad \alpha_{(1234)}(a_{17}) = a_4 \quad \alpha_{(1234)}(a_{18}) = a_6$

$$\alpha_{(12)(34)}(a_1)=a_8 \quad \alpha_{(12)(34)}(a_2)=a_7 \quad \alpha_{(12)(34)}(a_3)=a_9 \quad \alpha_{(12)(34)}(a_4)=a_{18}$$

$$\alpha_{(12)(34)}(a_5)=a_{16} \quad \alpha_{(12)(34)}(a_6)=a_{17} \quad \alpha_{(12)(34)}(a_7)=a_{14} \quad \alpha_{(12)(34)}(a_8)=a_{13}$$

$$\alpha_{(12)(34)}(a_9)=a_{15} \quad \alpha_{(12)(34)}(a_{10})=a_1 \quad \alpha_{(12)(34)}(a_{11})=a_2 \quad \alpha_{(12)(34)}(a_{12})=a_3$$

$$\alpha_{(12)(34)}(a_{13})=a_{10} \quad \alpha_{(12)(34)}(a_{14})=a_{11} \quad \alpha_{(12)(34)}(a_{15})=a_{12}$$

$$\alpha_{(12)(34)}(a_{16})=a_5 \quad \alpha_{(12)(34)}(a_{17})=a_4 \quad \alpha_{(12)(34)}(a_{18})=a_6$$

$$\begin{array}{rccc} \gamma_1 : & \mathfrak{S}_4 & \longrightarrow & \mathrm{GL}(\widetilde{H}_1(\Pi_4)) \\ & \mathrm{Id} & \longmapsto & I_6 \\ & (12) & \longmapsto & \gamma^1_{(12)} \\ & (123) & \longmapsto & \gamma^1_{(123)} \\ & (1234) & \longmapsto & \gamma^1_{(1234)} \\ & (12)(34) & \longmapsto & \gamma^1_{(12)(34)} \end{array}$$

$$\begin{array}{rccc} \gamma^1_{(12)} : & \widetilde{H}_1(\Pi_4) & \longrightarrow & \widetilde{H}_1(\Pi_4) \\ & b_1 & \longmapsto & \text{-}b_3 \\ & b_2 & \longmapsto & \text{-}b_2 \\ & b_3 & \longmapsto & \text{-}b_1 \\ & b_4 & \longmapsto & b_1\text{-}b_2+b_4\text{-}b_5 \\ & b_5 & \longmapsto & b_1\text{-}b_3\text{-}b_5 \\ & b_6 & \longmapsto & b_2+b_6 \end{array} \qquad \gamma^1_{(12)} = \begin{pmatrix} 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rccc} \gamma^1_{(123)} : & \widetilde{H}_1(\Pi_4) & \longrightarrow & \widetilde{H}_1(\Pi_4) \\ & b_1 & \longmapsto & \text{-}b_3\text{-}b_4\text{-}b_6 \\ & b_2 & \longmapsto & \text{-}b_1\text{-}b_4+b_5\text{-}b_6 \\ & b_3 & \longmapsto & \text{-}b_2\text{-}b_6 \\ & b_4 & \longmapsto & \text{-}b_1+b_2 \\ & b_5 & \longmapsto & b_2\text{-}b_3 \\ & b_6 & \longmapsto & b_1 \end{array} \qquad \gamma^1_{(123)} = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{rccc} \gamma^1_{(1234)}: & \widetilde{H}_1(\Pi_4) & \longrightarrow & \widetilde{H}_1(\Pi_4) \\ & b_1 & \longmapsto & \text{-}b_1 + b_3 + b_5 \\ & b_2 & \longmapsto & \text{-}b_1\text{-}b_4 + b_5\text{-}b_6 \\ & b_3 & \longmapsto & \text{-}b_1 + b_2\text{-}b_4 + b_5 \\ & b_4 & \longmapsto & \text{-}b_6 \\ & b_5 & \longmapsto & b_4 \\ & b_6 & \longmapsto & b_4\text{-}b_5 + b_6 \end{array} \qquad \gamma^1_{(1234)} = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rccc} \gamma^1_{(12)(34)}: & \widetilde{H}_1(\Pi_4) & \longrightarrow & \widetilde{H}_1(\Pi_4) \\ & b_1 & \longmapsto & \text{-}b_2 + b_3 \\ & b_2 & \longmapsto & \text{-}b_2 \\ & b_3 & \longmapsto & b_1\text{-}b_2 \\ & b_4 & \longmapsto & b_4\text{-}b_5 + b_6 \\ & b_5 & \longmapsto & \text{-}b_5 \\ & b_6 & \longmapsto & \text{-}b_6 \end{array} \qquad \gamma^1_{(12)(34)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\begin{array}{rccc} \chi_{\gamma_1}: & \mathfrak{S}_4 & \longrightarrow & \mathbb{C} \\ & \mathrm{Id} & \longmapsto & 6 \\ & (12) & \longmapsto & 0 \\ & (123) & \longmapsto & 0 \\ & (1234) & \longmapsto & 0 \\ & (12)(34) & \longmapsto & -2 \end{array}$$

To do less calculations you could directly find β_S using Theorem 1.16. We note that the representation γ_1 we found is isomorphic to the induced representation $\mathrm{ind}_{C_4}^{\mathfrak{S}_4}(i)$.

We want to show that the result we obtained is not a case but extends to all the partitions lattices $\Pi_n,\ n \in \mathbb{N}$.

We need the following results of P. Hall and P. Hanlon:

Proposition 1.22. *Let L be a finite lattice with atoms $\{a_1, \ldots, a_n\}$ and coatoms $\{b_1, \ldots, b_m\}$.*

a) If $\widehat{0}$ is not the meet of coatoms, i.e.

$$b_1 \wedge b_2 \wedge \cdots \wedge b_m \neq \widehat{0}$$

then:

$$\mu(\widehat{0}, \widehat{1}) = 0$$

b) If $\widehat{1}$ is not the join of atoms, i.e.

$$a_1 \vee a_2 \vee \cdots \vee a_m \neq \widehat{1}$$

then:

$$\mu(\widehat{0}, \widehat{1}) = 0$$

Proof. See [Rot64a, Corollary (Ph. Hall); page 349]. □

Lemma 1.23. *Let $\pi \in \mathfrak{S}_n$, and let Π_n^π denote the sublattice of Π_n fixed pointwise by π. Let μ_π denote the Möbius function of Π_n^π. Then:*

$$\mu_\pi(\widehat{0}, \widehat{1}) = \begin{cases} (-1)^{d-1}\mu(n/d)(d-1)!(n/d)^{d-1}, & \textit{if } \pi \textit{ is a product of } d \textit{ cycles of length } n/d \\ 0, & \textit{otherwise} \end{cases}$$

Here $\mu(n/d)$ denotes the usual number-theoretic Möbius function.

Proof. See [Han81, Theorem 4; page 338].

Hanlon actually computes $\mu_\pi(x_\pi, \widehat{1})$, where x_π is the meet of all coatoms of Π_n^π. It follows from [Han81] Lemma 2 that:

$$x_\pi = \widehat{0} \qquad \Longleftrightarrow \qquad \text{all cycles of } \pi \text{ have the same length}$$

Combining the previous results with Proposition 1.22 we obtain the proof of the lemma. □

Theorem 1.24. *Let μ denote the usual number-theoretic Möbius function. Then:*

$$\mu(n) = \sum_{\substack{1 \leq h \leq n \\ (h,n)=1}} e^{\frac{2\pi i h}{n}}$$

i.e. the sum of the primitive n-th roots of unity.

Proof. See [HW79], Theorem 271 and Equation (16.6.4), page 239. □

In order to prove Lemma 1.26 we need the following result on the character of an induced representation:

Theorem 1.25. *Let χ be the character of the representation ρ of G induced by the representation θ of H whose character is χ_θ. Let x be an element of G and C_j its conjugacy class in G with h_j elements, and let $g = g_j h_j$ where g is the order of G. Let h be the order of H. Then:*

$$\chi(x) = \frac{g_j}{h} \sum_{z \in C_j \cap H} \chi_\theta(z).$$

Proof. If G is a finite group, for every $a \in G$ the elements in the conjugacy class of a are in $1-1$ correspondence with the cosets of the centralizer:

$$C_G(a) = \{g \in G \mid ga = ag\}.$$

This can be seen by observing that any two elements b, c belonging to the same coset of $C_G(a)$, i.e. there exists an element z in $C_G(a)$ such that $b = zc$, give rise to the same element while conjugating a:

$$b^{-1}ab = c^{-1}z^{-1}azc = c^{-1}ac.$$

Thus, the number of elements in the conjugacy class of a is the index $[G : C_G(a)]$. The cardinality of $|C_G(a)|$ and its cosets is $g/h_j = g_j$. We have seen that two elements that belong to the same coset of $C_G(a)$ give rise to the same element while conjugating a.

We define:

$$\chi_1(w) = \begin{cases} \chi_\theta(w) & w \in H \\ 0 & w \notin H. \end{cases}$$

From [Ser77, Theorem 12; page 30] we know that:

$$\chi(x) = \frac{1}{h} \sum_{\substack{y \in G \\ y^{-1}xy \in H}} \chi_\theta(y^{-1}xy) = \frac{1}{h} \sum_{y \in G} \chi_1(y^{-1}xy). \tag{1.8}$$

As y ranges over G, $y^{-1}xy$ ranges over C_j and give the same $z \in C_j$ exactly g_j times. From Equation (1.8) we obtain

$$\chi(x) = \frac{1}{h}\, g_j \sum_{z\in C_j} \chi_1(z) = \frac{1}{h}\, g_j \sum_{z\in C_j\cap H} \chi_\theta(z).$$

□

Lemma 1.26. *Let C_n be a cyclic subgroup of $\mathfrak{S}_n$ of order n generated by an n-cycle σ. Let $\zeta = e^{2\pi i/n}$ and let ρ_n be the associated representation of C_n:*

$$\begin{array}{rccc} \rho_n: & C_n & \longrightarrow & GL(V) \simeq \mathbb{C}^*. \\ & \sigma & \longmapsto & \zeta \end{array}$$

Define the induced representation $\psi_n = ind_{C_n}^{\mathfrak{S}_n}(\rho_n)$ and let $\pi \in \mathfrak{S}_n$. We have the following:

$$\chi_{\psi_n}(\pi) = \begin{cases} \mu(n/d)(d-1)!(n/d)^{d-1}, & \textit{if } \pi \textit{ is a product of } d \textit{ cycles of length } n/d \\ 0. & \textit{otherwise} \end{cases}$$

Proof. Theorem 1.25 on the character of induced representation yields

$$\chi_{\psi_n}(\pi) = \frac{|\mathfrak{S}_n|}{|C_n||C_\pi|} \sum_{\tau\in C_\pi \bigcap C_n} \chi_{\rho_n}(\tau) = \frac{(n-1)!}{|C_\pi|} \sum_{\tau\in C_\pi \bigcap C_n} \chi_{\rho_n}(\tau) \qquad (1.9)$$

where C_π is the conjugacy class of $\mathfrak{S}_n$ containing π.

Suppose that π has d cycles, if $d \nmid n$ and π has not d cycles of length n/d then $C_\pi \bigcap C_n = \emptyset$. Hence,

$$\chi_{\psi_n}(\pi) = 0 \qquad \text{unless } d|n \text{ and } \pi \text{ has } d \text{ cycles of length } n/d.$$

Let π have d cycles of length n/d, if $\tau \in C_\pi \bigcap C_n$ then exists k with $\gcd(n,k) = d$ such that $\sigma^k = \tau$; thus we have that $\rho_n(\tau) = \chi_{\rho_n}(\tau) = \zeta^k$.

ζ is a primitive n-th root of unity. A power $w = \zeta^k$ of ζ is a primitive a-th root of unity for

$$a = \frac{n}{\gcd(n,k)}.$$

Hence $\chi_{\rho_n}(\tau)$ is a primitive n/d-th root of unity, so $\chi_{\rho_n}(\tau)$ runs through all primitive n/d-th root of unity. From Theorem 1.24 we obtain that

$$\sum_{\tau \in C_\pi \bigcap C_n} \chi_{\rho_n}(\tau) = \mu(n/d).$$

We can compute the size of the conjugacy class C_π using [Sag01, Proposition 1.1.1] and we obtain

$$|C_\pi| = \frac{n!}{(n/d)^d\ d!} = \frac{(n-1)!}{(n/d)^{d-1}\ (d-1)!}.$$

Substituting what obtained in Equation (1.9) we have the following

$$\chi_{\psi_n}(\pi) = \frac{(n-1)!\ (n/d)^{d-1}\ (d-1)!}{(n-1)!}\ \mu(n/d) = (n/d)^{d-1}\ (d-1)!\ \mu(n/d).$$

□

Theorem 1.27. *Let $G = \mathfrak{S}_n$ acts on $P = \Pi_n$ in the canonical way. Using the notations described in the previous Lemma we claim that*

$$\beta_{[n-2]} = \psi_n \otimes \operatorname{sgn}.$$

Proof. From Equation (1.6) and Equation (1.7) we get:

$$\chi_{\beta_{[n-2]}}(\pi) = (-1)^{n-1}\ \widetilde{\Lambda}_{[n-2]}(\pi);$$

$$\widetilde{\Lambda}_{[n-2]}(\pi) = \widetilde{\mathcal{E}}(P^\pi_{[n-2]}) = \widetilde{\mathcal{E}}(P^\pi) = \widetilde{\mathcal{E}}(\Pi^\pi_n).$$

From Equation (1.3) we have also:

$$\widetilde{\mathcal{E}}(\Pi^\pi_n) = \mu_\pi(\Pi^\pi_n) = \mu_\pi(\widehat{0}, \widehat{1}).$$

By combining these last two results with Lemma 1.23 we get that:

$$\chi_{\beta_{[n-2]}}(\pi) = (-1)^{n-1}\ \widetilde{\Lambda}_{[n-2]}(\pi) = (-1)^{n-1}\ \mu_\pi(\widehat{0}, \widehat{1})$$

$$\chi_{\beta_{[n-2]}}(\pi) = \begin{cases} (-1)^{n+d}\mu(n/d)(d-1)!(n/d)^{d-1}, & \text{if } \pi \text{ is a product of } d \text{ cycles of length } n/d \\ 0, & \text{otherwise} \end{cases}$$

Recall π is a d-cycle, using Lemma 1.26 we have

$$\chi_{\beta_{[n-2]}}(\pi) = (-1)^{n+d}\ \chi_{\psi_n}(\pi) = (-1)^{n-d}\ \chi_{\psi_n}(\pi).$$

□

Remark 1. Note that if $n \not\equiv 2 (\mathrm{mod}\ 4)$, then $(-1)^{n-d} = 1$ for all $d|n$. Thus in this case:

$$\gamma_{n-3} = \beta_{[n-2]} = \psi_n.$$

1.2 Matroids

In this Section we recall the fundamental definitions of matroid theory following [Oxl06].

1.2.1 Basic definitions

Definition 1.28. A *matroid* M is an ordered pair (E, I) consisting of a finite set E and a collection I of subsets of E satisfying the three following conditions:

(I1) $\emptyset \in I$

(I2) *If* $A \in I$ *and* $A' \subseteq A$, *then* $A' \in I$

(I3) *If* A *and* B *are in* I *and* $|A| < |B|$, *then there is an element* $e \in B \setminus A$ *such that* $A \cup \{e\} \in I$.

The first two properties define an abstract simplicial complex.

The members of I are the *independent sets* of M, and E is the *ground set* of M. We shall often write $\mathrm{IN}(M)$ for I and $E(M)$ for E. A subset of E that is not in I is called *dependent*.

Proposition 1.29. *Let* E *be the set of column labels of an* $m \times n$ *matrix* A *over a field* $\mathbb{K}$, *and let* I *be the set of subsets* X *of* E *for which the set of columns labeled by* X *is linearly independent in the* $\mathbb{K}$-*vector space* $V(m, \mathbb{K})$ *of dimension* m. *Then* $M = (E, I)$ *is a matroid.*

The matroid obtained as above from the matrix A will be denoted by $M[A]$. This matroid is called the $\mathbb{K}$-*vector matroid* of A. We say that a matroid M is *representable on a field* $\mathbb{K}$ if there exists a matrix A over $\mathbb{K}$ such that $M = M[A]$. The list of column vectors in the matrix A, in the vector space $V(m, \mathbb{K})$, is called a $\mathbb{K}$-*realization* of M.

Definition 1.30. A *hyperplane arrangement* $\mathcal{A}$ is a finite family of linear hyperplanes $\mathcal{A} = \{V_1, \dots, V_n\}$ in a $\mathbb{K}$-vector space V. If V_i is a hyperplane, write n_i for some (nonzero) normal vector to V_i. Let A be the matrix with the n_i as column vectors, we define the matroid associate to $\mathcal{A}$ as $M_{\mathcal{A}} = M[A]$.

Definition 1.31. A minimal dependent set in an arbitrary matroid M will be called a *circuit* of M and we shall denote the set of circuits of M by $\mathcal{C}$ or $\mathcal{C}(M)$. A circuit of M having n elements will also be called an n-*circuit*. A 1-circuit of M is called a *loop*, equivalently an element is a loop if it belongs to no basis. An element that belongs to no circuit is called a *coloop* or *isthmus*; equivalently an element is a coloop if it belongs to every basis. If a two-element set $\{a, b\}$ is a circuit of M, then a and b are called *parallel* in M.

Definition 1.32. A matroid is called *simple* if it has no circuits consisting of one or two elements. Analogously, a matroid is simple if it does not contain loops nor parallel elements.

Example 1.6. Let A be the matrix:

$$\begin{array}{ccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{array}$$

over the field $\mathbb{R}$. Then $E = \{1, 2, 3, 4, 5\}$ and

$$I = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}.$$

Thus the set of dependent sets of this matroid is

$$\{\{3\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\} \cup \{X \subseteq E \ : \ |X| \geqslant 3\}.$$

The set of circuits, i.e., dependent sets all of whose proper subsets are independent is

$$\{\{3\}, \{1, 4\}, \{1, 2, 5\}, \{2, 4, 5\}\}.$$

Evidently, as in the last example, once I has been specified, $\mathcal{C}(M)$ can be determined. Similarly, I can be determined from $\mathcal{C}(M)$: the members of I are those subsets of E that contain no member of $\mathcal{C}(M)$. Thus a matroid is uniquely determined by its set $\mathcal{C}$ of circuits.

We can associate a matroid to a graph:

Proposition 1.33. *Let E be the set of edges of a graph Γ and $\mathcal{C}$ be the set of edge sets of simple cycles of Γ. Then $\mathcal{C}$ is the set of circuits of a matroid on E.*

Proof. See [Oxl06, Proposition 1.1.7; page 11]. □

Definition 1.34. The matroid derived above from the graph Γ is called the *cycle matroid* of Γ. It is denoted by $M(\Gamma)$. Clearly a set X of edges is independent in $M(\Gamma)$ if and only if X does not contain the edge set of a cycle or, equivalently, $\Gamma[X]$, the subgraph induced by X, is a forest.

Definition 1.35. Two matroids M_1 and M_2 are *isomorphic*, written $M_1 \cong M_2$, if there is a bijection:

$$\psi : \quad E(M_1) \quad \longrightarrow \quad E(M_2)$$

such that, for all $X \subseteq E(M_1)$, $\psi(X)$ is independent in M_2 if and only if X is independent in M_1.

Example 1.7. Let Γ be the graph shown in Figure 1.4 and let $M = M(\Gamma)$. Then:

$$E(M) = \{e_1, e_2, e_3, e_4, e_5\} \quad \mathcal{C}(M) = \{\{e_3\}, \{e_1, e_4\}, \{e_1, e_2, e_5\}, \{e_2, e_4, e_5\}\}.$$

Comparing M with the matroid $M[A]$ in the previous example, we see that, under the bijection:

$$\begin{array}{rccc} \psi : & \{1, 2, 3, 4, 5\} & \longrightarrow & \{e_1, e_2, e_3, e_4, e_5\} \\ & i & \longmapsto & e_i \end{array}$$

a set X is a circuit in $M[A]$ if and only if $\psi(X)$ is a circuit in M. Equivalently, a set Y is independent in $M[A]$ if and only if $\psi(Y)$ is independent in M. Thus the matroids M and $M[A]$ are isomorphic.

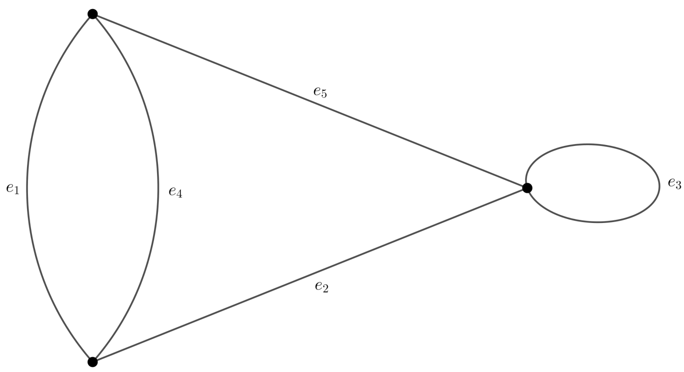


Figure 1.4

A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*. So for instance the matroid $M[A]$ is graphic.

Definition 1.36. If Γ is a graph, we can form a directed graph $D(\Gamma)$ by arbitrarily assigning a direction to each edge. Let $A_{D(\Gamma)}$ denote the *incidence matrix* of $D(\Gamma)$, that is, $A_{D(\Gamma)}$ is the matrix $[a_{ij}]$ whose rows and columns are indexed by the vertices and edges, respectively, of $D(\Gamma)$, where:

$$a_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ is the tail of non-loop edge } j \\ -1, & \text{if vertex } i \text{ is the head of non-loop edge } j \\ 0, & \text{otherwise} \end{cases}$$

Proposition 1.37. *If Γ is a graph, then $M(\Gamma) \cong M[A_{D(\Gamma)}]$ over any field $\mathbb{K}$ for any $D(\Gamma)$ formed by Γ.*

Proof. See [Oxl06, Proposition 5.1.2; page 138]. □

1.2.2 Basis, Rank and Closure Operator

Definition 1.38. A maximal independent set in a matroid M is called *basis* of M.

Lemma 1.39. *If B_1 and B_2 are bases of a matroid M, then $|B_1| = |B_2|$.*

Proof. See [Oxl06, Lemma 1.2.1; page 16]. □

If M is a matroid and $\mathcal{B}$ is its collection of bases, then, by (I1):

(B1) *$\mathcal{B}$ is non-empty.*

Lemma 1.40. *$\mathcal{B}$ satisfies the following condition:*

(B2) *If B_1 and B_2 are members of $\mathcal{B}$ and $x \in B_1 \smallsetminus B_2$, then there is an element y of $B_2 \smallsetminus B_1$ such that:*

$$(B_1 \smallsetminus \{x\}) \cup \{y\} \ \in \mathcal{B}$$

Proof. See [Oxl06, Lemma 1.2.2; page 17]. □

Theorem 1.41. *Let E be a set and $\mathcal{B}$ be a collection of subsets of E satisfying* (B1) *and* (B2)*. Let I be the collection of subsets of E that are contained in some member of $\mathcal{B}$. Then (E, I) is a matroid having $\mathcal{B}$ as its collection of bases.*

Proof. See [Oxl06, Theorem 1.2.3; page 17]. □

Corollary 1.42. *Let $\mathcal{B}$ a set of subsets of a set E. Then $\mathcal{B}$ is the collection of bases of a matroid on E if and only if it satisfies* (B1)-(B2)*.*

Definition 1.43. Let M be the matroid (E, I) and suppose that $X \subseteq E$. Define:

$$I|X = \{A \subseteq X \ : \ A \in I\}.$$

Then it is easy to see that the pair $(X, I|X)$ is a matroid. We call this matroid the *restriction of* M to X. It is denoted by $M|X$. We define the *rank* $\operatorname{rk}(X)$ of X to be the size of a basis B of $M|X$.

In other words, the rank of $X \subseteq E$ is the maximal cardinality of an element of I contained in X. It is clear that rk has the following properties:

(R1) *If $X \subseteq E$, then* $\operatorname{rk}(X) \leqslant |X|$.

(R2) *If* $X \subseteq Y \subseteq E$*, then* $\operatorname{rk}(X) \leqslant \operatorname{rk}(Y)$.

Lemma 1.44. *The rank function* rk *of a matroid* M *on a set* E *satisfies the following condition (submodularity):*

(R3) *If* X *and* Y *are subsets of* E*, then:*

$$\operatorname{rk}(X \cup Y) + \operatorname{rk}(X \cap Y) \leqslant \operatorname{rk}(X) + \operatorname{rk}(Y).$$

Proof. See [Oxl06, Lemma 1.3.1; page 23]. □

Theorem 1.45. *Let* E *be a set and* rk *be a function that maps* 2^E *into the set of non-negative integers and satisfies* (R1)-(R3)*. Let* I *be the collection of subsets* X *of* E *for which* $\operatorname{rk}(X) = |X|$*. Then* (E, I) *is a matroid having rank function* rk*.*

Proof. See [Oxl06, Theorem 1.3.2; page 23]. □

Corollary 1.46. *Let* E *be a set. A function*

$$\operatorname{rk} : \quad 2^E \quad \longrightarrow \quad \mathbb{Z}^+$$

is the rank function of a matroid on E *if and only if* rk *satisfies* (R1)-(R3).

Independent sets, bases and circuits are easily characterized in terms of the rank function:

Proposition 1.47. *Let* M *be a matroid with rank function* rk *and suppose that* $X \subseteq E(M)$*. Then:*

i) X *is independent if and only if* $|X| = \operatorname{rk}(X)$

ii) X *is a basis if and only if* $|X| = \operatorname{rk}(X) = \operatorname{rk}(M)$

iii) X *is a circuit if and only if* X *is non-empty and, for all* $x \in X$,

$$\operatorname{rk}(X \smallsetminus \{x\}) = |X| - 1 = \operatorname{rk}(X)$$

Definition 1.48. Let M be an arbitrary matroid having ground set E and rank function rk. Let cl be the function from 2^E into 2^E defined for all $X \subseteq E$, by

$$\mathrm{cl}(X) = \big\{x \in E \ : \ \mathrm{rk}(X \cup \{x\}) = \mathrm{rk}(X)\big\}.$$

This function is called the *closure operator* of M.

Definition 1.49. If M is a matroid and $X \subseteq E(M)$, we call $\mathrm{cl}(X)$ the *closure* of X in M. If $X = \mathrm{cl}(X)$, then X is called a *flat* of M. A *hyperplane* of M is a flat of rank $(\mathrm{rk}(M) - 1)$.

1.2.3 Duality

In this subsection we define the dual of a matroid.

Theorem 1.50. *Let M be a matroid and define*

$$\mathcal{B}^* = \{E(M) \smallsetminus B \ : \ B \in \mathcal{B}(M)\}.$$

Then $\mathcal{B}^*$ is the set of bases of a matroid on $E(M)$. This matroid is called dual matroid and is denoted by M^*.

Proof. See [Oxl06, Theorem 2.1.1; page 68]. $\square$

The bases of M^* are called *cobases* of M. A similar convention applies to other distinguished subsets of $E(M^*)$. Hence, for example, the circuits, hyperplanes, independent set of M^* are called *cocircuits, cohyperplanes, coindipendent sets* of M.

Remark 2. If Γ is a planar graph, and Γ^* is its dual, then:

$$M(\Gamma^*) = M^*(\Gamma).$$

Note that Γ^* depends on the planar embedding of Γ, but $M(\Gamma^*)$ not. If Γ is not planar, then the dual graph is not defined, but we still have a dual matroid $M^*(\Gamma)$. This class of matroids is called *cographic matroid.*

In general, we attach an asterisk to a symbol to denote association with the dual. Thus, for example, rk^* will denote the rank function of M^* while $\mathcal{C}^*$ denotes its set of circuits. Evidently:

$$\mathrm{rk}(M) + \mathrm{rk}^*(M) = |E(M)| \tag{1.10}$$

The next result generalizes Equation (1.10) to give a formula for rk^*, the *corank function* of M.

Lemma 1.51. *Let $M = (E, I)$ be a matroid and $M^* = (E, I^*)$ its dual. Let A be a subset of the ground set E, then:*

$$\mathrm{rk}^*(A) = \mathrm{rk}(A^c) + |A| - \mathrm{rk}(E)$$

Proof. See [Oxl06, Proposition 2.1.9; page 72]. □

1.2.4 Lattice of flats of matroids

Let M be a matroid, we denote by $\mathcal{L}(M)$ the poset of flats of M ordered by inclusion $(\mathcal{L}(M), \subseteq)$.

Lemma 1.52. *The poset $(\mathcal{L}(M), \subseteq)$ is a geometric lattice and, for all flats X and Y of M, we have*

$$X \wedge Y = X \cap Y \quad \textit{and} \quad X \vee Y = \mathrm{cl}(X \cup Y).$$

Proof. See [Oxl06, Lemma 1.7.3 and Theorem 1.7.5; page 54/55]. □

Example 1.8. Let K_n be the complete graph on n vertices. A particularly important example of geometric lattice is the lattice of flats of the matroid $M(K_n)$. Let V be the vertex set of K_n. If F is a flat of $M(K_n)$, we denote by π_F the partition of V in which i and j are in the same partition if and only if the edge ij is in F. Conversely, if $\beta \in \Pi_n$ we denote by F_β the flat of $M(K_n)$ in which the edge ij is in F_β if and only if i and j are in the same partition of β. This determines a map from the set $\mathcal{L}(M(K_n))$ of flats of $M(K_n)$ and the partition lattice Π_n of the n-set V:

$$\begin{array}{rccc} \phi: & \mathcal{L}(M(K_n)) & \longrightarrow & \Pi_n \\ & F & \longmapsto & \pi_F \end{array}$$

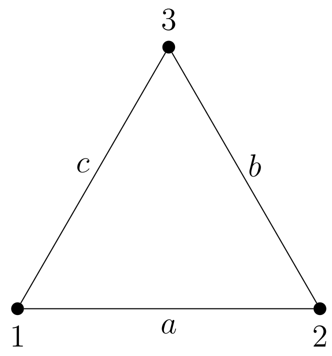


Figure 1.5

Moreover, ϕ is easily shown to be an order isomorphism. For $F_1, F_2 \in \mathcal{L}(M(K_n))$ we have

$$F_1 \subseteq F_2 \quad \Longleftrightarrow \quad \pi_{F_1} \leqslant \pi_{F_2}$$

where $\leqslant$ indicates the order relationship introduced in Definition 1.7.

Let $M(K_3)$ be the matroid associated to the graph K_3 (See Figure 1.5) with:

$$E = \{a, b, c\} \quad I = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

$$\mathcal{L}(M(K_3)) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}.$$

In this case the order isomorphism between $\mathcal{L}(M(K_3))$ and Π_3 is the following

$$\begin{array}{rccc} \phi: & \mathcal{L}(M(K_3)) & \longrightarrow & \Pi_3 \\ & \emptyset & \longmapsto & 1|2|3 \\ & \{a\} & \longmapsto & 12|3 \\ & \{b\} & \longmapsto & 23|1 \\ & \{c\} & \longmapsto & 13|1 \\ & \{a, b, c\} & \longmapsto & 123. \end{array}$$

1.2.5 Characteristic polynomial

The characteristic polynomial of a matroid is one of its most fundamental invariants, it is the matroid analog of the chromatic polynomial of a graph.

Definition 1.53. Let $M = (E, I)$ be a matroid with rank function rk. We define its *characteristic polynomial* to be

$$\chi_M(\lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{\operatorname{rk}(M) - \operatorname{rk}(A)}.$$

It follows immediately that this polynomial is an invariant since, given two matroids with the same rank function (hence isomorphic), they have the same characteristic polynomial.

Definition 1.54. We denote the absolute value of the coefficients of $\lambda^{\operatorname{rk}(M)-k}$ in $\chi_M(\lambda)$ with ω_k and we call it *k-th Whitney number of the first kind.*

It is easy to see that any matroid with at least one loop has characteristic polynomial $\chi_M(\lambda) \equiv 0$. Furthermore, one can see that adding parallel elements to M does not change $\chi_M(\lambda)$. These observations lead us to believe that all we need to study is simple matroids. Now we provide another powerful presentation of this polynomial for simple matroids:

Theorem 1.55. *Let $M = (E, I)$ be a simple matroid and let $\mathcal{L}(M)$ be its lattice of flats, then we have the following*

$$\chi_M(\lambda) = \sum_{F \in \mathcal{L}(M)} \mu(\emptyset, F) \lambda^{\operatorname{rk}(M) - \operatorname{rk}(F)};$$

where μ is Möbius function of $\mathcal{L}(M)$ and rk *is the rank function of M. Equivalently, we have*

$$\omega_k = (-1)^k \sum_{\substack{F \in \mathcal{L}(M) \\ \operatorname{rk}(F) = k}} \mu(\emptyset, F).$$

Proof. See [Zas87, Proposition 7.1.4 and 7.2.1]. □

From the definition of Möbius function (See Definition 1.6), we know that $\mu(\emptyset, E) = -\sum_{F \neq E} \mu(\emptyset, F)$; hence it follows immediately that $\chi_M(1) = 0$.

Definition 1.56. We define the *reduced characteristic polynomial* of M as

$$\bar{\chi}_M(\lambda) = \frac{\chi_M(\lambda)}{\lambda - 1}.$$

The characteristic polynomial of a matroid contains important information, let's see some examples. If $M(G)$ is a matroid of a connected graph G, then $\lambda\chi_{M(G)}(\lambda)$ is *the chromatic polynomial* of G, it counts the proper colorings of vertices of G with λ given colors, where no two neighboring vertices have the same color.

Another important application is when the matroid is realized by an arrangement of hyperplanes. Let $\mathcal{A}$ be an arrangement of hyperplanes in a $\mathbb{K}$-vector space and let $M_{\mathcal{A}}$ be the associate matroid (see Definition 1.30); we define the complement of the arrangement:

$$\mathcal{M}(\mathcal{A}) = \mathbb{K}^r \smallsetminus \bigcup_{U\in\mathcal{A}} U.$$

Depending on the underlying field, $\chi_{M_{\mathcal{A}}}(\lambda)$ stores different information about $\mathcal{M}(\mathcal{A})$. If $\mathbb{K} = \mathbb{R}$ is the field of real numbers, we have that $\mathcal{M}(\mathcal{A})$ consists exactly of $|\chi_{M_{\mathcal{A}}}(-1)|$ regions. If we work with complex numbers we have that

$$\omega_k = \beta_k(\mathcal{M}(\mathcal{A})),$$

where $\beta_k(\mathcal{M}(\mathcal{A}))$ are the k-th Betti numbers of the complement $\mathcal{M}(\mathcal{A})$ in the cohomology ring $H^*(\mathcal{M}(\mathcal{A}))$. Lastly when $\mathbb{K} = \mathbb{K}_q$ is a finite field, the complement $\mathcal{M}(\mathcal{A})$ has exactly $\chi_{M_{\mathcal{A}}}(q)$ points.

In [Rot71, Her72] Rota and Heron conjecture that the sequence of coefficients of the characteristic polynomial is unimodal and log-concave. This conjecture is now a theorem by Adiprasito, Huh and Katz:

Theorem 1.57. *Let M be a matroid, the sequence of its Whitney numbers of the first kind ω_k form a sequence that is log-concave, i.e.,*

$$\omega_{k-1}\omega_{k+1} \leq \omega_k^2;$$

for all $1 \leq k \leq \operatorname{rk}(M)$. In particular, the sequence is unimodal.

Proof. See [AHK18, Theorem 9.9]. □

1.3 Polymatroids

For general references about polymatroids we suggest [Wel76].

1.3.1 Basic definitions

Definition 1.58. A *polymatroid* P is an ordered pair (E, cd) consisting of a finite set E and a *codimension* function $\mathrm{cd}\colon 2^E \to \mathbb{N}$ satisfying:

(C1) $\mathrm{cd}(\emptyset) = 0$,

(C2) if $A \subseteq B$, then $\mathrm{cd}(A) \leq \mathrm{cd}(B)$, and

(C3) if $A, B \subseteq E$, then $\mathrm{cd}(A \cup B) + \mathrm{cd}(A \cap B) \leq \mathrm{cd}(A) + \mathrm{cd}(B)$.

A polymatroid is a matroid if the codimension of singletons are either zero or one.

The closure of a subset $A \subseteq E$ is the subset

$$\mathrm{cl}(A) = \{a \in E \mid \mathrm{cd}(A \cup \{a\}) = \mathrm{cd}(A)\}.$$

A *flat* is a closed set and the collection of flats with the inclusion forms a poset $\mathcal{L}(P)$, that we call the *poset of flats*. We will use the notation $\max(X)$ for X a subset of a poset as the set of maximal elements of X. Edmonds showed that the set of flats of a polymatroid is closed under intersection ([Edm70], Theorem 25). Since the set of flats of a polymatroid is finite and has a maximal element, namely E, this implies that the set of flats ordered under inclusion forms a lattice. The meet in this lattice is intersection and the join is given by $X \vee Y = \mathrm{cl}(X \cup Y)$.

Definition 1.59. Let $P = (E, \mathrm{cd})$ be a polymatroid, we define the *independence polytope* of P to be

$$P_{ind}(\mathrm{cd}) = \{x \in \mathbb{R}^E : x_i \geq 0 \text{ for all } i \in E, \sum_{i \in A} x_i \leq \mathrm{cd}(A) \text{ for all } A \subseteq E\}$$

and its *base polytope* to be

$$P_{base}(\mathrm{cd}) = \{x \in \mathbb{R}^E : \sum_{i\in A} x_i \leq \mathrm{cd}(A) \text{ for all } A \subseteq E, \sum_{i \in E} x_i = \mathrm{cd}(E)\}.$$

A vector in $P_{base}(\mathrm{cd})$ is called a *base vector*. If cd is the rank function of a matroid M then this defines the matroid independence polytope $P_{ind}(M)$ and the matroid base polytope $P_{base}(M)$. When Edmonds in [Edm70] introduced polymatroids, he defined the polymatroid itself to be the polytope $P_{ind}(\mathrm{cd})$.

Definition 1.60. Two polymatroids $P_1 = (E_1, \mathrm{cd}_{P_1})$ and $P_2 = (E_2, \mathrm{cd}_{P_2})$ are *isomorphic*, written $P_1 \cong P_2$, if there is a bijection $\psi : E_1 \longrightarrow E_2$ such that $\mathrm{cd}_{P_2} \circ \psi = \mathrm{cd}_{P_1}$.

1.3.2 Subspace arrangements

Definition 1.61. A *subspace arrangement* $\mathcal{A}$ is a finite family of linear subspaces $\mathcal{A} = \{V_1, \ldots, V_n\}$ in a $\mathbb{K}$-vector space V. Such $\mathcal{A}$ gives rise to a polymatroid $P_\mathcal{A} = (\{1, \ldots, n\}, \mathrm{cd}_\mathcal{A})$ by defining for each $A \subseteq [n]$:

$$\mathrm{cd}_\mathcal{A}(A) = \mathrm{cd}\left(\bigcap_{i\in A} V_i\right),$$

where cd is the codimension function in the $\mathbb{K}$-vector space V. A polymatroid is said to be *realizable* (or *representable*) over a field $\mathbb{K}$ if it is isomorphic to a polymatroid $(\{1, \ldots, n\}, \mathrm{cd}_\mathcal{A}))$ for some subspace arrangement $\mathcal{A}$ in a $\mathbb{K}$-vector space V.

Example 1.9. Let $E = \{a, b, c\}$ and $\mathrm{cd}\colon 2^E \to \mathbb{N}$ the function defined by

$$\mathrm{cd}(a) = \mathrm{cd}(b) = 1, \qquad \mathrm{cd}(ab) = \mathrm{cd}(c) = 2,$$
$$\mathrm{cd}(ac) = \mathrm{cd}(bc) = \mathrm{cd}(abc) = 3.$$

This function defines a polymatroid P with poset of flats $\mathcal{L}(P)$ shown in Figure 1.6. This polymatroid is realizable: a realization is a collection in $\mathbb{R}^3$ of two subspace of dimension 2 and a line in general position (See Figure 1.6).

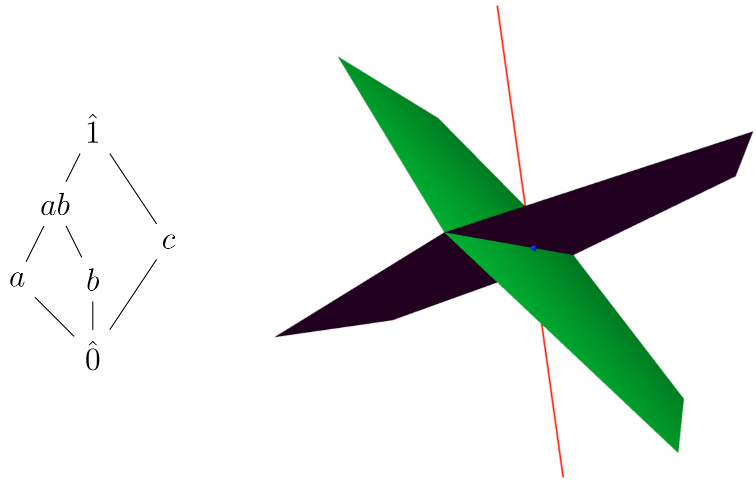


Figure 1.6: The poset $\mathcal{L}(P)$ and a realization of P in $\mathbb{R}^3$

Definition 1.62. Let $\mathcal{A}$ be a subspace arrangement in a complex vector space V of dimension r, we define the *complement* of the arrangement:

$$\mathcal{M}(\mathcal{A}) = \mathbb{C}^r \setminus \bigcup_{U \in \mathcal{A}} U.$$

The combinatorial data associated with an arrangement is recorded in a partially ordered set, the *intersection lattice* $\mathcal{L}(\mathcal{A})$. It is the set of intersection of subspaces in $\mathcal{A}$ ordered by reversed inclusion. We can consider the *order complex* $\Delta(\overline{\mathcal{L}})$ of the proper part $\overline{\mathcal{L}} := \mathcal{L} \setminus \{\hat{0}, \hat{1}\}$, i.e., the abstract simplicial complex formed by the linearly ordered subsets in $\overline{\mathcal{L}}$. The topology of $\Delta(\overline{\mathcal{L}})$ plays a crucial role for describing the topology of arrangement complements. For instance, in [GM88] Goresky and MacPherson have used the stratified Morse theory to describe the additive cohomology with integer coefficients of $\mathcal{M}(\mathcal{A})$ in terms of the intersection lattice $\mathcal{L}$:

Theorem 1.63 (Goresky MacPherson '88)**.** *Let $\mathcal{A}$ be a subspace arrangement with complement $\mathcal{M}(\mathcal{A})$ and lattice of intersection $\mathcal{L}$, there is an additive isomorphism*

$$\widetilde{H}^k(\mathcal{M}(\mathcal{A}), \mathbb{Z}) \cong \bigoplus_{W \in \mathcal{L} \setminus \hat{0}} \tilde{H}_{2\,\mathrm{cd}(W)-2-k}\Big(\Delta((\hat{0}, W)), \mathbb{Z})\Big)$$

where $\Delta((\hat{0}, W))$ is the order complex of the interval $(\hat{0}, W)$. We used the convention that $\tilde{H}_{-1}((\emptyset, \mathbb{Z})) = \mathbb{Z}$.

A completely different approach was used by De Concini and Procesi in [DCP95]. They constructed a rational model for $\mathcal{M}(\mathcal{A})$ using only the intersection lattice $\mathcal{L}(\mathcal{A})$ and proved that the rational cohomology algebra and rational homotopy type of $\mathcal{M}(\mathcal{A})$ are defined by this lattice.

Definition 1.64. Let $\mathcal{A}$ be a subspace arrangement with complement $\mathcal{M}(\mathcal{A})$, a *wonderful model* is a smooth projective variety Y containing $\mathcal{M}(\mathcal{A})$ as open subset such that $Y \backslash \mathcal{M}(\mathcal{A})$ is a simple normal crossing divisor, i.e., the irreducible components are smooth and intersect locally as coordinate hyperplanes.

Definition 1.65. Let $\mathcal{L}$ be the intersection lattice of an arrangement of subspaces in a vector space V and let $\mathrm{cd} : \mathcal{L} \to \mathbb{N}$ be the corresponding codimension function. A subset $\mathcal{G}$ in $\mathcal{L} \setminus \{\hat{0}\}$ is a *geometric building set* if for all $x \in \mathcal{L}$

$$[\hat{0}, x] = \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y]$$

and

$$\mathrm{cd}(x) = \sum_{y \in \max(\mathcal{G}_{\leq x})} \mathrm{cd}(y).$$

We define $F(P, \mathcal{G}, x) = \max(\mathcal{G}_{\leq x})$ the set of $\mathcal{G}$*-factors* of x.

The above definition of geometric building set is motivated by the following construction of De Concini and Procesi:

Definition 1.66. Let $\mathcal{A}$ be a complex subspace arrangement, $\mathcal{G}$ a geometric building set in $\mathcal{L}(\mathcal{A})$, and $G_1, \ldots, G_t$ a non-increasing linear order on $\mathcal{G}$. The *De Concini-Procesi wonderful model* for $\mathcal{A}$, $Y_{\mathcal{G}}$, is the result of successively blowing up subspaces $G_1, \ldots, G_t$, respectively their proper transforms.

Theorem 1.67 (De Concini Procesi '95)**.** *The variety $Y_{\mathcal{G}}$ is a wonderful model in the sense of Definition 1.64.*

Definition 1.68 ($\mathcal{G}$-nested set complex)**.** A subset S of $\mathcal{G}$ is called $\mathcal{G}$-nested if, for any set of incomparable elements $x_1, \dots, x_t$ in S of cardinality at least two, the join $x_1 \vee \cdots \vee x_t$ is not contained in $\mathcal{G}$. The $\mathcal{G}$-nested sets form an abstract simplicial complex, called the $\mathcal{G}$-nested set complex.

The geometric building set $\mathcal{G}$ is a good choice of some elements to blow up, in order to obtain a wonderful model $Y_{\mathcal{G}}$ with some exceptional divisors $D_g \subset Y_{\mathcal{G}}$, $g \in \mathcal{G}$ indexed by $\mathcal{G}$. A subset S of $\mathcal{G}$ is $\mathcal{G}$-nested if and only if the corresponding divisors $\{D_W\}_{W \in S}$ have nonempty intersection.

The variety $Y_{\mathcal{G}}$ is used for studying the complement of the subspace arrangement, by considering the Leray spectral sequence for the inclusion of the complement in the wonderful model $Y_{\mathcal{G}}$.

The Leray model $(B^{\bullet,\bullet}(\mathcal{A}, \mathcal{G}), d)$ is the second page of the Leray spectral sequence for the natural inclusion

$$V \setminus \cup_{A \in \mathcal{A}} A \cong Y_{\mathcal{G}} \setminus \cup_{g \in \mathcal{G}} D_g \hookrightarrow Y_{\mathcal{G}}.$$

This spectral sequence collapses at the third page (See for instance [Bib16, Lemma 3.2]), hence it becomes a differential bigraded algebra also known as the Morgan algebra (See [DCP95]). Furthermore,

$$B^{\bullet,0}(\mathcal{A}, \mathcal{G}) = H^{\bullet}(Y_{\mathcal{G}}) \quad \text{and} \quad H^{\bullet}(B^{\bullet,\bullet}(\mathcal{A}, \mathcal{G}), d) = H^{\bullet}(\mathcal{M}(\mathcal{A})).$$

Explicitly, $B^{\bullet,\bullet}(\mathcal{A}, \mathcal{G})$ is a $\mathbb{Q}$-differential bigraded algebra generated by e_W, x_W for $W \in \mathcal{G}$ with bidegree $(0, 1)$ and $(2, 0)$ respectively and relations:

$$e_T x_S \Big(\sum_{Z \leq W} x_Z \Big)^b = 0$$

for $S, T \subset \mathcal{G}, W \in \mathcal{G}$ and $b = \operatorname{cd}(W) - \operatorname{cd}(\bigvee (T \cup S)_{<W})$, with differential defined by $d(e_W) = x_W$. In Chapter 3 (See Definition 3.4) we extend this definition for any arbitrary polymatroid (not necessarily realizable).

De Concini and Procesi constructed this rational model for $\mathcal{M}(\mathcal{A})$ using only the lattice $\mathcal{L}$. The natural problem that was left open is how to recover the ring structure explicitly from the combinatorics. In particular how to relate the multiplication to the local homology of $\mathcal{L}$ that occurs in the Goresky-MacPherson formula (see Theorem 1.63).

In [Yuz02] Yuzvinsky solve this problem only for a maximal building set. In this paper Yuzvinsky finds a significantly smaller subalgebra $\mathrm{CM}(\mathcal{A}, \mathcal{G}_{max})$ quasi-isomorphic to $B^{\bullet,\bullet}(\mathcal{A}, \mathcal{G}_{max})$ whence also a rational model of $\mathcal{M}(\mathcal{G})$. The algebra $\mathrm{CM}(\mathcal{A}, \mathcal{G}_{max})$ provides a multiplicative structure on the flag complexes of $\mathcal{L}(\mathcal{A})$ that induces the ring structure on $H^\bullet(\mathcal{M}(\mathcal{A}))$.

Theorem 1.69 (Yuzvinsky '02)**.** *Let $\mathcal{A}$ be a subspace arrangement with complement $\mathcal{M}(\mathcal{A})$ and lattice of intersection $\mathcal{L} = \mathcal{L}(\mathcal{A})$, there is an isomorphism*

$$\widetilde{H}^k(\mathcal{M}(\mathcal{A}), \mathbb{Q}) \cong \widetilde{H}^k(\mathrm{CM}(\mathcal{A}, \mathcal{G}_{max}), \mathbb{Q}) \cong \bigoplus_{W \in \mathcal{L} \smallsetminus \hat{0}} \tilde{H}_{2\,\mathrm{cd}(W)-2-k}\Big(\Delta((\hat{0}, W)), \mathbb{Q}\Big)$$

where $\Delta((\hat{0}, W))$ is the order complex of the interval $(\hat{0}, W)$. We used the convention that $\tilde{H}_{-1}((\emptyset, \mathbb{Z})) = \mathbb{Z}$.

In [DGM00, dLS01] Pierre Deligne, Mark Goresky, Robert MacPherson and, respectively, Mark de Longueville and Carsten A. Schultz generalize Theorem 1.69 working with integer coefficients.

In Chapter 3 we extend the results of Theorem 1.69 to the non-realizable setting and to arbitrary building sets, see Theorems 3.23 and 3.24, by using the *critical monomial algebra* $\mathrm{CM}(P, \mathcal{G})$. In the case the polymatroid P is generated from an arrangement of subspaces and $\mathcal{G}$ is the maximal building set then $\mathrm{CM}(P, \mathcal{G}_{max}) = \mathrm{CM}(\mathcal{A}, \mathcal{G}_{max})$. In relation to the work of [DGM00, dLS01] we left open the problem with integer coefficients (See Conjecture 3.25).

Chapter 2

Representations on the homology of matroids

In this Chapter, given a group G of automorphisms of a matroid M, we describe the representations of G on the homology of the independence complex of the dual matroid M^*. These representations are related to the homology of the lattice of flats of M, and (when M is realizable) to the top cohomology of a hyperplane arrangement. Finally, we analyze in detail the case of the cographic matroid of the complete graph, which has applications to algebraic geometry.

2.1 Representations and Alexander duality

We recall here some basic facts in combinatorial topology. For more details the reader can refer to [Hat02]. Let K be an abstract simplicial complex with vertex set V with $|V| = n$. For $\sigma \in K$, let

$$\overline{\sigma} = V \setminus \sigma.$$

Definition 2.1. The *Alexander dual* of K is the simplicial complex on the same vertex set defined by

$$K^* = \{\sigma \subseteq V \mid \overline{\sigma} \notin K\}.$$

It is easy to see that $K^{**} = K$.

Let G be a finite group of automorphisms of the face poset $(K, \subseteq)$. Then G is a subgroup of the symmetric group $\mathfrak{S}_n$, made out of the vertex maps $g : V \longrightarrow V$ such that whenever the vertices $j_1, \ldots, j_{i+1}$ span an i-simplex of K, the points $g(j_1), \ldots, g(j_{i+1})$ span an i-simplex of K. Therefore g induces a simplicial homeomorphism $\tilde{g}$, and $\tilde{g}$ induces a chain-isomorphism $\tilde{g}_\#$ on the group of oriented i-chains in the following way:

$$\begin{array}{rccc} \tilde{g}_{\#,i} : & C_i(K, \mathbb{C}) & \longrightarrow & C_i(K, \mathbb{C}). \\ & [j_1, \ldots, j_{i+1}] & \longmapsto & [g(j_1), \ldots, g(j_{i+1})] \end{array}$$

Moreover $\tilde{g}_\#$ induces an isomorphism on the reduced homology groups $\widetilde{H}_i(K)$ (and the reduced cohomology groups $\widetilde{H}^i(K)$):

$$\rho_{i,g} : \ \widetilde{H}_i(K, \mathbb{C}) \ \longrightarrow \ \widetilde{H}_i(K, \mathbb{C}).$$

This defines representations of G on the $\mathbb{C}$-vector spaces $\widetilde{H}_i(K, \mathbb{C})$, i.e., homomorphisms

$$\begin{array}{rccc} \rho_i : & G & \longrightarrow & \mathrm{GL}(\widetilde{H}_i(K, \mathbb{C})). \\ & g & \longmapsto & \rho_{i,g} \end{array}$$

It follows from the definition of K^* that G is also a finite group of automorphisms of the *face poset* of K^*. Therefore, following the construction above, we get representations ρ^{*^i} of G on the reduced cohomology of K^*:

$$\begin{array}{rccc} \rho^{*^i} : & G & \longrightarrow & \mathrm{GL}(\widetilde{H}^i(K^*, \mathbb{C})). \\ & g & \longmapsto & \rho_g^{*i} \end{array}$$

The following theorem is the main result of this subsection; in order to develop its proof we need few lemmas (Lemma 2.4 and Lemma 2.5).

Theorem 2.2. *Let K be an abstract simplicial complex and let K^* be its Alexander dual. Let G be a finite group of automorphisms of the face poset of K. Then:*

$$\rho_i \simeq \rho^{*^{n-i-3}} \otimes \ \mathrm{sgn}$$

where $n = |V|$ and sgn *is the sign representation (restricted from $\mathfrak{S}_n$ to its subgroup G). Or, equivalently, we have the following isomorphism of $\mathbb{C}[G]$-modules:*

$$\widetilde{H}_i(K, \mathbb{C}) \simeq_G \widetilde{H}^{n-i-3}(K^*, \mathbb{C}) \otimes \mathrm{sgn}.$$

Our proof follows from Björner and Tancer ([BT09]), but carefully records the action G. We introduce some notations: let us denote by $\{1, 2, \ldots, n\}$ the elements of V, let also

$$p(\sigma) = \prod_{j \in \sigma} (-1)^{j-1}.$$

For $j \in \sigma \in K$, we define the *sign*

$$\mathrm{sgn}(j, \sigma) = (-1)^{i-1}$$

where j is the i-th smallest element of the set σ.

For $\sigma \in K$ we write e_σ to denote the oriented simplex associated to σ considered with an increasing order of its elements:

$$\sigma = \{j_1, \ldots, j_{i+1}\} \qquad e_\sigma = [j_1, \ldots, j_{i+1}] \text{ where } j_1 < \cdots < j_{i+1}.$$

For every $g \in G$, if $e_\sigma = [j_1, \ldots, j_{i+1}]$ we denote

$$g.\sigma = \{g(j_1), \ldots, g(j_{i+1})\} \qquad g.e_\sigma = [g(j_1), \ldots, g(j_{i+1})].$$

The $g(j_1), \ldots, g(j_{i+1})$ are not necessarily in ascending order: let $\tau \in \mathfrak{S}_{i+1} \subseteq \mathfrak{S}_n$ be the permutation that rearranges the elements in ascending order, and fixes the elements that are not in $g.\sigma$, so that $\tau.(g.e_\sigma) = e_{g.\sigma}$. We also define:

$$c(g, \sigma) = \mathrm{sgn}(\tau).$$

Since τ^{-1} permutes the elements of $e_{g.\sigma}$ we obtain:

$$g.e_\sigma = \tau^{-1}.(e_{g.\sigma}) = \mathrm{sgn}(\tau^{-1}) e_{g.\sigma} = \mathrm{sgn}(\tau) e_{g.\sigma} = c(g, \sigma) e_{g.\sigma}.$$

Similarly, we define a permutation $\overline{\tau} \in \mathfrak{S}_{n-1-i} \subseteq \mathfrak{S}_n$ which rearranges the elements of $g.e_{\overline{\sigma}}$ in ascending order:

$$\overline{\tau}.(g.e_{\overline{\sigma}}) = e_{g.\overline{\sigma}} \qquad g.e_{\overline{\sigma}} = c(g, \overline{\sigma}) e_{g.\overline{\sigma}}.$$

We can now formulate an important lemma that will prove to be crucial for the proof of Lemma 2.4:

Lemma 2.3. *Let $V = \{1, \dots, n\}$ and let $\sigma \subseteq V$. Then, for every $g \in \mathfrak{S}_n$, we have the following:*

$$p(\sigma)\ \mathrm{sgn}(g)\ c(g, \overline{\sigma}) = c(g, \sigma)\ p(g.\sigma). \tag{2.1}$$

Proof. For every $g \in \mathfrak{S}_n$, we define a permutation $g' = \overline{\tau}\tau g$. First we apply the permutation g to σ and $\overline{\sigma}$. Then applying τ and $\overline{\tau}$, we rearrange in ascending order both $g.e_\sigma$ and $g.e_{\overline{\sigma}}$.

As we have defined it, g' is a permutation of $\mathfrak{S}_n$ such that:

if $i, j \in \sigma$ with $i < j$ then $g'(i) < g'(j)$ and

if $i, j \in \overline{\sigma}$ with $i < j$ then $g'(i) < g'(j)$.

In particular, we have that:

$$g'.e_\sigma = e_{g'.\sigma} \qquad \text{and} \qquad g'.e_{\overline{\sigma}} = e_{g'.\overline{\sigma}}.$$

We can express g in the following way $g = \tau^{-1} \circ \overline{\tau}^{-1} \circ g'$. It easy to see that $p(g.\sigma) = p(g'.\sigma)$. Thus, Equation (2.1) becomes:

$$p(\sigma)\ \mathrm{sgn}(\tau^{-1}\overline{\tau}^{-1}g')\ c(g, \overline{\sigma}) = c(g, \sigma)\ p(g.\sigma)$$

$$p(\sigma)\ \mathrm{sgn}(\tau^{-1})\mathrm{sgn}(\overline{\tau}^{-1})\mathrm{sgn}(g')\ \mathrm{sgn}(\overline{\tau}) = \mathrm{sgn}(\tau)\ p(g'.\sigma)$$

$$p(\sigma)\ \mathrm{sgn}(g') = p(g'.\sigma)$$

$$\prod_{i\in\sigma} (-1)^{i-1}\ \mathrm{sgn}(g') = \prod_{i\in\sigma} (-1)^{g'(i)-1}$$

$$\prod_{i\in\sigma} (-1)^{i-g'(i)}\ \mathrm{sgn}(g') = 1$$

$$\prod_{i\in\sigma} (-1)^{i-g'(i)} = \mathrm{sgn}(g'). \tag{2.2}$$

In order to prove Equation (2.2), let $i \in \sigma$ be the k-th element of e_σ and we define:

$$A_i = \{(i,j) \mid j \in \overline{\sigma},\ i < j,\ g'(i) > g'(j)\} \qquad \text{and}$$

$$B_i = \{(j,i) \mid j \in \overline{\sigma},\ j < i,\ g'(j) > g'(i)\}.$$

We have that:

$$\operatorname{sgn}(g') = (-1)^{\sum_{i\in\sigma}(|A_i|+|B_i|)}.$$

Let us assume that $i < g'(i)$. It is easy to see that $|B_i| = 0$. Furthermore:

$$|\{(i,j) \mid j \in \overline{\sigma},\ i < j\}| = (n-i) - (|\sigma| - k) = n - i - |\sigma| + k$$

and

$$\begin{aligned} |\{(i,j) \mid j \in \overline{\sigma},\ i < j,\ g'(i) < g'(j)\}| &= |\{(i,j) \mid j \in \overline{\sigma},\ g'(i) < g'(j)\}| \\ &= (n - g'(i)) - (|\sigma| - k) = n - g'(i) - |\sigma| + k. \end{aligned}$$

By subtracting term by term the two equalities above, we get:

$$|A_i| = (n - i - |\sigma| + k) - (n - g'(i) - |\sigma| + k) = g'(i) - i.$$

Similarly, if $i > g'(i)$ we have that $|B_i| = i - g'(i)$ and $|A_i| = 0$. Therefore:

$$|A_i| + |B_i| = |g'(i) - i|.$$

It follows that:

$$\operatorname{sgn}(g') = (-1)^{\sum_{i\in\sigma}|g'(i)-i|} = \prod_{i\in\sigma}(-1)^{|g'(i)-i|} = \prod_{i\in\sigma}(-1)^{g'(i)-i}.$$

□

Let 2^V be the full simplex with vertex set V.

Lemma 2.4. *Let K be a simplicial complex with ground set V of size n. Then*

$$\widetilde{H}_{i+1}(2^V, K) \simeq \widetilde{H}^{n-i-3}(K^*).$$

Furthermore, if we consider the following representations of the group G on the homology spaces of the pair $(2^V, K)$ and K^*:

$$\alpha_{i+1} : \ G \longrightarrow \mathrm{GL}(\widetilde{H}_{i+1}(2^V, K), \mathbb{C})) \qquad \textit{and}$$

$$\rho^{*^{n-i-3}} : \ G \longrightarrow \mathrm{GL}(\widetilde{H}^{n-i-3}(K^*, \mathbb{C})) \ ,$$

we have that

$$\alpha_{i+1} \simeq \rho^{*^{n-i-3}} \otimes \mathrm{sgn}.$$

Or equivalently

$$\widetilde{H}_{i+1}(2^V, K) \simeq_G \widetilde{H}^{n-i-3}(K^*) \otimes \mathrm{sgn}.$$

Proof. The chain complex for reduced homology of the pair $(2^V, K)$ is the complex:

$$\cdots \mathcal{R}_{i+1} \xrightarrow{d_{i+1}} \mathcal{R}_i \xrightarrow{d_i} \mathcal{R}_{i-1} \xrightarrow{d_{i-1}} \cdots , \qquad i \in \mathbb{Z}$$

where $\mathcal{R}_i = \langle e_\sigma \mid \sigma \subseteq V,\ \sigma \notin K,\ \dim(\sigma) = i\rangle$, and the d_i's are the unique homomorphisms satisfying:

$$d_i(e_\sigma) = \sum_{\substack{k\in\sigma \\ \sigma\smallsetminus k\notin K}} \mathrm{sgn}(k, \sigma)\ e_{\sigma\smallsetminus k}.$$

The cochain complex for reduced cohomology of K^* is the complex:

$$\cdots \xrightarrow{\delta_{i-1}} C^{i-1} \xrightarrow{\delta_i} C^i \xrightarrow{\delta_{i+1}} \cdots , \qquad i \in \mathbb{Z}$$

where

$$\begin{aligned} C^i &= \langle e^*_\sigma \mid \sigma \subseteq V,\ \dim(\sigma) = i,\ \sigma \in K^*\rangle \\ &= \langle e^*_\sigma \mid \sigma \subseteq V,\ \dim(\overline{\sigma}) = n-i-2,\ \overline{\sigma} \notin K\rangle \end{aligned}$$

and the δ_i's are the unique homomorphisms satisfying:

$$\delta_i(e^*_\sigma) = \sum_{\substack{k\notin\sigma \\ \sigma\cup k\in K^*}} \mathrm{sgn}(k, \sigma\cup k) e^*_{\sigma\cup k} = \sum_{\substack{k\in\overline{\sigma} \\ \overline{\sigma}\smallsetminus k\notin K}} \mathrm{sgn}(k, \sigma\cup k) e^*_{\overline{\overline{\sigma}\smallsetminus k}}.$$

Let ϕ_i be the following isomorphism:

$$\begin{array}{rccc} \phi_i : & \mathcal{R}_i & \longrightarrow & C^{n-i-2} \\ & e_\sigma & \longmapsto & p(\sigma)\ e^*_{\overline{\sigma}} \end{array} \qquad \text{for } \sigma \notin K \text{ with } \dim(\sigma) = i. \tag{2.3}$$

We then have the following diagram:

$$
\begin{array}{ccccccc}
\xrightarrow{d_{i+1}} & \mathcal{R}_i & \xrightarrow{d_i} & \mathcal{R}_{i-1} & \xrightarrow{d_{i-1}} \\
 & \phi_i \downarrow & & \phi_{i-1} \downarrow & \\
\xrightarrow{\delta_{n-i-2}} & C^{n-i-2} & \xrightarrow{\delta_{n-i-1}} & C^{n-i-1} & \xrightarrow{\delta_{n-i}}
\end{array}
$$

We know from the proof of [BT09, Lemma 4.2] that

$$\phi_{i-1} \circ d_i = \delta_{n-i-1} \circ \phi_i. \tag{2.4}$$

Thus, we have that

$$\widetilde{H}_{i+1}(2^V, K) \simeq \widetilde{H}^{n-i-3}(K^*).$$

We now study the following two representations:

$$
\begin{array}{rccc}
\rho_1 : & G & \longrightarrow & \mathrm{GL}(\mathcal{R}_i) \\
 & g & \longmapsto & \rho^1_g
\end{array}
\qquad
\begin{array}{rccc}
\rho^1_g : & \mathcal{R}_i & \longrightarrow & \mathcal{R}_i \\
 & e_\sigma & \longmapsto & g.e_\sigma
\end{array}
$$

$$
\begin{array}{rccc}
\rho_2 : & G & \longrightarrow & \mathrm{GL}(C^{n-i-2} \otimes \mathbb{C}) \\
 & g & \longmapsto & \rho^2_g
\end{array}
\qquad
\begin{array}{rccc}
\rho^2_g : & C^{n-i-2} \otimes \mathbb{C} & \longrightarrow & C^{n-i-2} \otimes \mathbb{C} \\
 & e^*_{\bar\sigma} \otimes 1 & \longmapsto & g.e^*_{\bar\sigma} \otimes \mathrm{sgn}(g)
\end{array}
$$

for $\sigma \notin K$ with $\dim(\sigma) = i$. We want to show that these two representations are isomorphic. We extend the isomorphism (2.3):

$$
\begin{array}{rccc}
\tilde{\phi}_i : & \mathcal{R}_i & \longrightarrow & C^{n-i-2} \otimes \mathbb{C} \\
 & e_\sigma & \longmapsto & p(\sigma)\ e^*_{\bar\sigma} \otimes 1
\end{array}
\qquad \text{for } \sigma \notin K \text{ with } \dim(\sigma) = i.
$$

To prove that $\rho_1 \simeq \rho_2$ we have to show that the following diagram commutes for every $g \in G$:

$$
\begin{array}{ccc}
\mathcal{R}_i & \xrightarrow{\rho^1_g} & \mathcal{R}_{i-1} \\
\downarrow \tilde{\phi}_i & & \downarrow \tilde{\phi}_i \\
C^{n-i-2} \otimes \mathbb{C} & \xrightarrow{\rho^2_g} & C^{n-i-1} \otimes \mathbb{C}
\end{array}
$$

We have to prove that the following equation holds:

$$\rho^2_g \circ \tilde{\phi}_i = \tilde{\phi}_i \circ \rho^1_g. \tag{2.5}$$

$$(\rho_g^2 \circ \tilde{\phi}_i)(e_\sigma) = \rho_g^2(p(\sigma)\, e^*_{\overline{\sigma}} \otimes 1) = p(\sigma)\, g.e^*_{\overline{\sigma}} \otimes \mathrm{sgn}(g) = p(\sigma)\, \mathrm{sgn}(g) c(g, \overline{\sigma}) e^*_{g.\overline{\sigma}} \otimes 1$$

$$(\tilde{\phi}_i \circ \rho_g^1)(e_\sigma) = \tilde{\phi}_i(g.e_\sigma) = \tilde{\phi}_i(c(g,\sigma)e_{g.\sigma}) = p(g.\sigma)c(g,\sigma)e^*_{\overline{g.\sigma}} \otimes 1.$$

By applying Lemma 2.3, since $g.\overline{\sigma} = \overline{g.\sigma}$, we have that Equation (2.5) holds.

We consider now the following diagram:

$$\begin{array}{ccccccc}
\xrightarrow{d_{i+1}} & \mathcal{R}_i & \xrightarrow{d_i} & \mathcal{R}_{i-1} & \xrightarrow{d_{i-1}} \\
 & \downarrow{\scriptstyle \tilde{\phi}_i} & & \downarrow{\scriptstyle \tilde{\phi}_{i-1}} & \\
\xrightarrow{\tilde{\delta}_{n-i-2}} & C^{n-i-2} \otimes \mathbb{C} & \xrightarrow{\tilde{\delta}_{n-i-1}} & C^{n-i-1} \otimes \mathbb{C} & \xrightarrow{\tilde{\delta}_{n-i}}
\end{array}$$

And we define the $\tilde{\delta}_i$'s as an extension of the homomorphisms δ_i:

$$\tilde{\delta}_i(e^*_\sigma \otimes 1) = \sum_{\substack{k \notin \sigma \\ \sigma \cup k \in K^*}} \mathrm{sgn}(k, \sigma \cup k) e^*_{\sigma \cup k} \otimes 1.$$

From Equation (2.4) it follows that:

$$\tilde{\phi}_{i-1} \circ d_i = \tilde{\delta}_{n-i-1} \circ \tilde{\phi}_i.$$

Thus, we have that:

$$\alpha_{i+1} \simeq \rho^{*^{n-i-3}} \otimes \ \mathrm{sgn}.$$

$\square$

Lemma 2.5. *Let K be a simplicial complex with ground set V of size n. Then:*

$$\widetilde{H}_i(K) \simeq \widetilde{H}_{i+1}((2^V, K), \mathbb{C}).$$

Furthermore if we consider the representations of the group G on the reduced homology spaces of K and $(2^V, K)$

$$\rho_i : \ G \ \longrightarrow \ \mathrm{GL}(\widetilde{H}_i(K, \mathbb{C})) \qquad \alpha_{i+1} : \ G \ \longrightarrow \ \mathrm{GL}(\widetilde{H}_{i+1}((2^V, K), \mathbb{C}))$$

we have that

$$\rho_i \simeq \alpha_{i+1}.$$

Proof. The isomorphism follows from [Mun84, Theorem 23.3]: we have the long exact sequence of the pair $(2^V, K)$:

$$\cdots \to \widetilde{H}_{i+1}(2^V) \to \widetilde{H}_{i+1}(2^V, K) \to \widetilde{H}_i(K) \to \widetilde{H}_i(2^V) \to \cdots$$

Since 2^V is the full simplex the spaces $\widetilde{H}_{i+1}(2^V)$ and $\widetilde{H}_i(2^V)$ are zero. Hence, the sequence becomes:

$$\cdots \to 0 \to \widetilde{H}_{i+1}(2^V, K) \to \widetilde{H}_i(K) \to 0 \to \cdots$$

It follows that the groups $\widetilde{H}_{i+1}(2^V, K)$ and $\widetilde{H}_i(K)$ are isomorphic.

We now consider the following diagram:

$$\begin{array}{ccc} \tilde{H}_{i+1}(2^V, K) & \xrightarrow{\partial^*} & \tilde{H}_i(K) \\ {\scriptstyle \alpha_{i+1,g}}\downarrow & & {\scriptstyle \rho_{i,g}}\downarrow \\ \tilde{H}_{i+1}(2^V, K) & \xrightarrow{\partial^*} & \tilde{H}_i(K) \end{array}$$

where ∂^* is the *homology boundary isomorphism* (see [Mun84, Lemma 24.1]):

$$\begin{array}{ccccc} & & C_{i+1}(2^V) & \xrightarrow{\pi_\#} & \mathcal{R}_{i+1}(2^V, K) \\ & & \downarrow{\scriptstyle \partial^V_{i+1}} & & \\ C_i(K) & \xrightarrow{i_\#} & C_i(2^V) & & \end{array}$$

The isomorphism ∂^* is defined by a certain *zig-zag* process: pull back via $\pi_\#$, apply ∂^V_{i+1}, and pull back via $i_\#$. For each $g \in G$ we consider the action on the chain groups of the full simplex, of K and of $(2^V, K)$:

$$\begin{array}{cccc} \tilde{g}^V_{\#,i}: & C_i(2^V) & \longrightarrow & C_i(2^V) \\ & [j_1, \ldots, j_{i+1}] & \longmapsto & [g(j_1), \ldots, g(j_{i+1})], \\ & & & \\ \tilde{g}_{\#,i}: & C_i(K) & \longrightarrow & C_i(K) \\ & [j_1, \ldots, j_{i+1}] & \longmapsto & [g(j_1), \ldots, g(j_{i+1})], \\ & & & \\ \tilde{g}^{V,K}_{\#,i}: & \mathcal{R}_i(2^V, K) & \longrightarrow & \mathcal{R}_i(2^V, K). \\ & [j_1, \ldots, j_{i+1}] & \longmapsto & [g(j_1), \ldots, g(j_{i+1})]. \end{array}$$

We have that:

$$\tilde{g}^V_{\#,i}\big|_{C_i(K)} = \tilde{g}_{\#,i}, \qquad \tilde{g}^V_{\#,i}\big|_{\mathcal{R}_i(2^V,K)} = \tilde{g}^{V,K}_{\#,i}.$$

We also know that each boundary operator commutes with $\tilde{g}_{\#,i}$, $\tilde{g}^V_{\#,i}$ and $\tilde{g}^{V,K}_{\#,i+1}$ from [Mun84, Lemma 12.1]. Let $b \in \tilde{H}_{i+1}(2^V, K)$. There exists an $a \in \mathcal{R}_{i+1}(2^V, K)$ such that $b = a + \mathrm{Im}(d_{i+2})$. Therefore:

$$\begin{aligned}\rho_{i,g}(\partial^*(b)) &= \rho_{i,g}(\partial^V_{i+1}(a) + \mathrm{Im}(d_{i+2})) = \tilde{g}_{\#,i}(\partial^V_{i+1}(a)) + \mathrm{Im}(d_{i+2}) \\ &= \tilde{g}^V_{\#,i}(\partial^V_{i+1}(a)) + \mathrm{Im}(d_{i+2}) = \partial^V_{i+1}(\tilde{g}^V_{\#,i+1}(a)) + \mathrm{Im}(d_{i+2}),\end{aligned}$$

$$\begin{aligned}\partial^*(\alpha_{i+1,g}(b)) &= \partial^*(\tilde{g}^{V,K}_{\#,i+1}(a) + \mathrm{Im}(d_{i+2})) = \partial^V_{i+1}(\tilde{g}^{V,K}_{\#,i+1}(a)) + \mathrm{Im}(d_{i+2}) \\ &= \partial^V_{i+1}(\tilde{g}^V_{\#,i+1}(a)) + \mathrm{Im}(d_{i+2}).\end{aligned}$$

Thus, we have that

$$\partial^* \circ \alpha_{i+1,g} = \rho_{i,g} \circ \partial^* \quad \text{for every } g \in G,$$

and this implies that $\rho_i \simeq \alpha_{i+1}$. □

Combining the results of Lemma 2.4 and Lemma 2.5 we obtain the proof of Theorem 2.2.

Remark 3. From Alexander duality we know that for every simplicial complex K on vertex set V such that $V \notin K$, with $n = |V|$:

$$\widetilde{H}_i(K) \simeq \widetilde{H}^{n-3-i}(K^*).$$

In fact, working with complex coefficients the reduced cohomology group $\widetilde{H}^j(K)$ is the dual vector space of the reduced homology group $\widetilde{H}_j(K)$, so that $\widetilde{H}_j(K) \simeq \widetilde{H}^j(K)$. Combining the two results we obtain:

$$\widetilde{H}_i(K) \simeq \widetilde{H}_{n-3-i}(K^*). \tag{2.6}$$

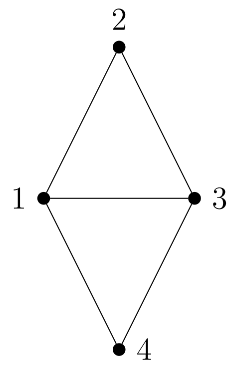


Figure 2.1: K

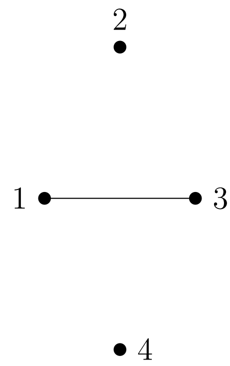


Figure 2.2: K^*

2.1.1 Example

Let $V = \{1, 2, 3, 4\}$ be the vertex set of the following simplicial complex

$$K = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1\}, \{2\}, \{3\}, \{4\}\}$$

shown in Figure 2.1. His Alexander dual is the following simplicial complex

$$K^* = \{\{1, 3\}, \{1\}, \{2\}, \{3\}, \{4\}\}$$

shown in Figure 2.2. The group G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$; seen as a subgroup of the symmetric group $\mathfrak{S}_4$ on V is the following

$$G = \{(), (13), (24), (13)(24)\}.$$

Therefore we can study the following two representations

$$\rho : \ G \longrightarrow \mathrm{GL}(\widetilde{H}_1(K, \mathbb{C})) \qquad \rho^* : \ G \longrightarrow \mathrm{GL}(\widetilde{H}_0(K^*, \mathbb{C})).$$

First we calculate the reduced homology space $\widetilde{H}_1(K, \mathbb{C})$, we have that

$$C_1(K) = \{[1, 2], [1, 3], [1, 4], [2, 3], [3, 4]\} \quad C_0(K) = \{1, 2, 3, 4\}$$

$$\begin{array}{rccc} \partial_1 : & C_1(K) & \longrightarrow & C_0(K) \\ & [1, 2] & \longmapsto & [2] - [1] \\ & [1, 3] & \longmapsto & [3] - [1] \\ & [1, 4] & \longmapsto & [4] - [1] \\ & [2, 3] & \longmapsto & [3] - [2] \\ & [3, 4] & \longmapsto & [4] - [3] \end{array}$$

$$\widetilde{H}_1(K,\mathbb{C}) = \ker\partial_1 = \langle \overset{d_1}{[1,3]-[1,4]+[3,4]}, \overset{d_2}{[1,2]-[1,3]+[2,3]} \rangle$$

Since G is a commutative group, we have four conjugacy classes, so we have to study the character of the representation on each class.

$$\begin{array}{rccc} \rho: & G & \longrightarrow & \mathrm{GL}(\widetilde{H}_1(K,\mathbb{C})) \\ & \mathrm{id} & \longmapsto & \rho_{()} \\ & (13) & \longmapsto & \rho_{(13)} \\ & (24) & \longmapsto & \rho_{(24)} \\ & (13)(24) & \longmapsto & \rho_{(13)(24)} \end{array}$$

$$\begin{array}{rccc} \rho_{(13)}: & \widetilde{H}_1(K,\mathbb{C}) & \longrightarrow & \widetilde{H}_1(K,\mathbb{C}) \\ & d_1 & \longmapsto & -d_1 \\ & d_2 & \longmapsto & -d_2 \end{array} \qquad \begin{array}{rccc} \rho_{(24)}: & \widetilde{H}_1(K,\mathbb{C}) & \longrightarrow & \widetilde{H}_1(K,\mathbb{C}) \\ & d_1 & \longmapsto & -d_2 \\ & d_2 & \longmapsto & -d_1 \end{array}$$

$$\begin{array}{rccc} \rho_{(13)(24)}: & \widetilde{H}_1(K,\mathbb{C}) & \longrightarrow & \widetilde{H}_1(K,\mathbb{C}) \\ & d_1 & \longmapsto & d_2 \\ & d_2 & \longmapsto & d_1 \end{array}$$

Thus, the character of the representation ρ is the following:

$$\begin{array}{rccc} \chi_\rho: & G & \longrightarrow & \mathbb{C} \\ & \mathrm{id} & \longmapsto & 2 \\ & (13) & \longmapsto & -2 \\ & (24) & \longmapsto & 0 \\ & (13)(24) & \longmapsto & 0 \end{array}$$

Now we can calculate the reduced homology group $\widetilde{H}_0(K^*,\mathbb{C})$; we consider the following sequence of chain spaces:

$$C_1(K^*) = \{[1,3]\} \quad C_0(K^*) = \{1,2,3,4\}$$

$$C_1(K^*) \xrightarrow{\partial_1} C_0(K^*) \xrightarrow{\epsilon} \mathbb{C}$$

$$\begin{array}{rccc} \partial_1: & C_1(K^*) & \longrightarrow & C_0(K^*) \\ & [1,3] & \longmapsto & 3-1 \end{array} \qquad \begin{array}{rccc} \epsilon: & C_0(K^*) & \longrightarrow & \mathbb{C} \\ & i & \longmapsto & 1 \end{array}$$

$$\ker\epsilon = \langle 2-1, 3-1, 4-1 \rangle \qquad \mathrm{Im}\,\partial_1 = \langle 3-1 \rangle$$

$$\widetilde{H}_0(K^*, \mathbb{C}) = \ker \epsilon / \operatorname{Im} \partial_1 = \langle 2 - 1 \overset{f_1}{+} \operatorname{Im} \partial_1, 4 - 1 \overset{f_2}{+} \operatorname{Im} \partial_1 \rangle.$$

Now we can study the representation ρ^*:

$$\begin{array}{rccc} \rho^* : & G & \longrightarrow & \mathrm{GL}(\widetilde{H}_0(K^*, \mathbb{C})) \\ & \mathrm{id} & \longmapsto & \rho_{()} \\ & (13) & \longmapsto & \rho_{(13)} \\ & (24) & \longmapsto & \rho_{(24)} \\ & (13)(24) & \longmapsto & \rho_{(13)(24)} \end{array}$$

$$\begin{array}{rccc} \rho^*_{(13)} : & \widetilde{H}_0(K^*, \mathbb{C}) & \longrightarrow & \widetilde{H}_0(K^*, \mathbb{C}) \\ & f_1 & \longmapsto & f_1 \\ & f_2 & \longmapsto & f_2 \end{array} \qquad \begin{array}{rccc} \rho_{(24)} : & \widetilde{H}_0(K^*, \mathbb{C}) & \longrightarrow & \widetilde{H}_0(K^*, \mathbb{C}) \\ & f_1 & \longmapsto & f_2 \\ & f_2 & \longmapsto & f_1 \end{array}$$

$$\begin{array}{rccc} \rho^*_{(13)(24)} : & \widetilde{H}_0(K^*, \mathbb{C}) & \longrightarrow & \widetilde{H}_0(K^*, \mathbb{C}) \\ & f_1 & \longmapsto & f_2 \\ & f_2 & \longmapsto & f_1 \end{array}$$

Thus, the character of the representation ρ^* is the following:

$$\begin{array}{rccc} \chi_{\rho^*} : & G & \longrightarrow & \mathbb{C} \\ & \mathrm{id} & \longmapsto & 2 \\ & (13) & \longmapsto & 2 \\ & (24) & \longmapsto & 0 \\ & (13)(24) & \longmapsto & 0 \end{array}$$

Therefore we have the following isomorphism of $\mathbb{C}[G]$-modules:

$$\widetilde{H}_1(K, \mathbb{C}) \simeq_G \widetilde{H}_0(K^*, \mathbb{C}) \otimes \mathrm{sgn}.$$

2.2 Equivariant cross-cut theory

Let L be a lattice with maximal and minimal elements $\widehat{1}$ and $\widehat{0}$ respectively. We recall the following definition from [Fol66]:

Definition 2.6. If L is a lattice with $\widehat{0}$ and $\widehat{1}$, a *cross-cut* of L is a set $C \subseteq L$ such that:

i) $\widehat{0}$, $\widehat{1} \notin C$.

ii) If $x, y \in C$ then $x \not< y$ and $y \not< x$. (x *and* y *are incomparable*)

iii) Any finite chain $x_1 < x_2 < \cdots < x_n$ in L can be extended to a chain which contains an element of C.

In particular, axiom iii) implies that every maximal chain contains an element of C.

Let L be a lattice with $\widehat{0}$ and $\widehat{1}$ and let C be a *cross-cut* of L.

Definition 2.7. A finite subset $\{x_1, \ldots, x_n\} \subseteq C$ *'spans'* if and only if

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n = \widehat{0} \qquad \text{and} \qquad x_1 \vee x_2 \vee \cdots \vee x_n = \widehat{1}.$$

Here $x \wedge y$ denotes the largest element $\leq x$ and $\leq y$, and $x \vee y$ denotes the smallest element $\geqslant x$ and $\geqslant y$.
Let $K(C)$ be the abstract simplicial complex whose vertices are the elements of C and whose simplices are all finite subsets of C which do not *'span'*. We denote $\widetilde{H}_i(C) = \widetilde{H}_i(K(C))$. Let $K(L)$ be the order complex of the lattice L and denote $\widetilde{H}_i(L) = \widetilde{H}_i(K(L))$. The following result was proved in [Fol66, Theorem 3.1]:

Theorem 2.8. *Let L be a lattice and let C be a cross-cut of L, then:*

$$\widetilde{H}_i(C) \simeq \widetilde{H}_i(L).$$

In order to see that the previous isomorphism is also a $\mathbb{C}[G]$-module isomorphism (Theorem 2.11) we need the following result:

Lemma 2.9. *Let K be an abstract simplicial complex and let K' be its first barycentric subdivision. Let also G be a finite group of automorphisms of the face poset of K. Then we have the following isomorphism of $\mathbb{C}[G]$-modules:*

$$\widetilde{H}_i(K) \simeq_G \widetilde{H}_i(K').$$

Proof. First, we need to describe the action of G on K'. Let $\mathcal{L}(K)$ be the face poset of K; it is clear that the order complex of $\mathcal{L}(K)$ is the barycentric subdivision of K. Thus, we have a straightforward G-action on the order complex of $\mathcal{L}(K)$ and its homology spaces. We have to show that the following two representations are isomorphic:

$$\begin{array}{rccc} \tilde{\rho}_i : & G & \longrightarrow & \mathrm{GL}(H_i(K)) \\ & g & \longmapsto & \tilde{\rho}_{i,g} \end{array} \quad \text{and} \quad \begin{array}{rccc} \tilde{\rho}'_i : & G & \longrightarrow & \mathrm{GL}(H_i(K')). \\ & g & \longmapsto & \tilde{\rho}'_{i,g} \end{array}$$

Let $w * K$ be a cone. If $e_\sigma = [a_0, \ldots, a_i]$ is an oriented simplex of K, let

$$\Big[w, e_\sigma\Big] = [w, a_0, \ldots, a_i]$$

denote an oriented simplex of $w * K$. This operation is well defined and is called the *bracket operation* (see [Mun84, Section §8]).

If $\sigma = \{a_0, \ldots, a_i\}$ is a simplex, let $\hat{\sigma}$ denote the barycenter of σ. The complex K' equals the collection of all simplices of the form

$$[\hat{\sigma}_1, \ldots, \hat{\sigma}_n] \quad \text{where } \sigma_1 \supset \cdots \supset \sigma_n.$$

We know from [Mun84, Section §17] that there is a unique augmentation-preserving chain map $\mathrm{sd} : \; C_i(K) \longrightarrow C_i(K')$ called the *barycentric subdivision operator* that induces an isomorphism of homology spaces. There is an inductive formula for the operator sd. It is the following:

$$\begin{aligned} \mathrm{sd}(v) &= \hat{v} = v \qquad \forall v \in K^0 \\ \mathrm{sd}(e_\sigma) &= \Big[\hat{\sigma}, \mathrm{sd}(\partial_i(e_\sigma))\Big] \qquad \text{for } \sigma \in K \text{ with } \dim(\sigma) = i. \end{aligned}$$

Now we consider the following two representations:

$$\begin{array}{rccc} \rho_i : & G & \longrightarrow & \mathrm{GL}(C_i(K)) \\ & g & \longmapsto & \rho_{i,g} \end{array} \qquad \begin{array}{rccc} \rho'_i : & G & \longrightarrow & \mathrm{GL}(C_i(K')) \\ & g & \longmapsto & \rho'_{i,g} \end{array}$$

We want to show that the following diagram

$$\begin{tikzcd} C_i(K) \arrow[r, "\mathrm{sd}"] \arrow[d, "\rho_{i,g}"'] & C_i(K') \arrow[d, "\rho'_{i,g}"'] \\ C_i(K) \arrow[r, "\mathrm{sd}"] & C_i(K') \end{tikzcd} \tag{2.7}$$

commutes for every $g \in G$. We proceed by induction on i:

- Suppose $i = 0$. It follows from the action of G on the vertices of K and K' that $\rho_{0,g}(v) = \rho'_{0,g}(v)$ for every $v \in K^0$. Thus:

$$\rho'_{0,g}(\mathrm{sd}(v)) = \rho'_{0,g}(v) = \rho_{0,g}(v) = \mathrm{sd}(\rho_{0,g}(v)).$$

- We now suppose the diagram commutes for $i = n$ and we prove it for $i = n + 1$. Let $g.\sigma = \tau$, thus:

$$\begin{aligned}
\rho'_{i+1,g}(\mathrm{sd}(e_\sigma)) &= \rho'_{i+1,g}(\left[\hat{\sigma}, \mathrm{sd}(\partial_{i+1}(e_\sigma))\right]) = \left[\hat{\tau}, \rho'_{i,g}(\mathrm{sd}(\partial_{i+1}(e_\sigma)))\right] \\
&=^i \left[\hat{\tau}, \mathrm{sd}(\rho_{i,g}(\partial_{i+1}(e_\sigma)))\right] = \left[\hat{\tau}, \mathrm{sd}(\partial_{i+1}(\rho_{i+1,g}(e_\sigma)))\right] \\
&= \left[\hat{\tau}, \mathrm{sd}(\partial_{i+1}(g.e_\sigma))\right] = \mathrm{sd}(g.e_\sigma) = \mathrm{sd}(\rho_{i+1,g}(e_\sigma)),
\end{aligned}$$

where the symbol $=^i$ stands for the application of the inductive hypothesis.

Since the diagram (2.7) commutes and both sd, $\rho_{i,g}$, $\rho'_{i,g}$ commute with the border operator ∂ we have that the following diagram commutes and consequently the lemma is proved:

$$\begin{tikzcd}
\widetilde{H}_i(K) \arrow[r, "\mathrm{sd}^*"] \arrow[d, "\tilde{\rho}_{i,g}"'] & \widetilde{H}_i(K') \arrow[d, "\tilde{\rho}'_{i,g}"'] \\
\widetilde{H}_i(K) \arrow[r, "\mathrm{sd}^*"] & \widetilde{H}_i(K').
\end{tikzcd}$$

□

Definition 2.10. Let L be a lattice and G a group of automorphism of L. A cross-cut C of L is *G-stable* if $G.C = C$, i.e., if C is the union of G-orbits.

Theorem 2.11. *Let L be a lattice and G a group of automorphism of L. Let C be a G-stable cross-cut of L. Then we have the following $\mathbb{C}[G]$-module isomorphism:*

$$\widetilde{H}_i(L) \simeq_G \widetilde{H}_i(C).$$

Proof. We briefly recall Folkman's argument. Let $K = K(L)$ be the order complex of L and let $C = \{\alpha_1, \ldots, \alpha_n\}$ be a cross-cut of L fixed by G.

For each $\alpha \in C$ let L_α be the subcomplex of K consisting of all simplices $\{y_1, \ldots, y_t\}$ such that the set $\{y_1, \ldots, y_t, \alpha\}$ is totally ordered. By the third property of a cross-cut, the family $\{L_\alpha\}_{\alpha \in C}$ is a covering of K. In the proof of [Fol66, Theorem 3.1] Folkman shows that $L_{\alpha_1} \bigcap \cdots \bigcap L_{\alpha_n}$ has the homology of a point or is empty and shows also that

$$K(C) = \mathcal{N}(\{L_\alpha\}_{\alpha \in C}) \tag{2.8}$$

where $K(C)$ is the simplicial complex associated to the cross-cut C and $\mathcal{N} = \mathcal{N}(\{L_\alpha\}_{\alpha \in C})$ is the nerve of the covering $\{L_\alpha\}_{\alpha \in C}$. Thus, we can apply a nerve theorem. We follow the construction made by Björner in [Bjö95, Theorem 10.6]. Let $P(K)$ and $P(\mathcal{N})$ be the face lattice associated to K and $\mathcal{N}$, respectively. Björner defines the following order-reversing map of posets:

$$\begin{array}{rccc} \tilde{f}: & P(K) & \longrightarrow & P(\mathcal{N}) \\ & \sigma & \longmapsto & \{\alpha \in C \mid \sigma \in L_\alpha\}. \end{array}$$

This map $\tilde{f}$ induces a simplicial map f between the respective order complex of $P(K)$ and $P(\mathcal{N})$ which are the first barycentric subdivision of K and $\mathcal{N}$:

$$\begin{array}{rccc} f: & K' & \longrightarrow & \mathcal{N}' \\ & \{\sigma_0, \ldots, \sigma_i\} & \longmapsto & \{\tilde{f}(\sigma_0), \ldots, \tilde{f}(\sigma_i)\} \end{array}$$

where $\{\sigma_0, \ldots, \sigma_i\}$ is a simplex of K', then we have $\sigma_0 \supseteq \cdots \supseteq \sigma_i$ with σ_j simplex of K. Applying [Bjö95, Theorem 10.6] we get, in particular, that f induces a chain map $f_\#$ between $C_i(K')$ and $C_i(\mathcal{N}')$ in the following manner:

$$f_\#([\sigma_0, \ldots, \sigma_i]) = \begin{cases} [\tilde{f}(\sigma_0), \ldots, \tilde{f}(\sigma_i)], & \text{if } \tilde{f}(v_0), \ldots, \tilde{f}(v_i) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases}$$

and moreover an isomorphism f_* on homology spaces:

$$\widetilde{H}_i(K') \simeq \widetilde{H}_i(\mathcal{N}').$$

We need to describe the action of G on K' and $\mathcal{N}'$: the G-action on L induces an action on K and therefore on K' (in the sense of 2.9). Since C is

G-stable, every $g \in G$ acts on C permuting its elements. Furthermore, since g is an order automorphism of L, it acts on the covering $\{L_\alpha\}_{\alpha \in C}$ respecting the intersection relations. Therefore G yields an action on the nerve $\mathcal{N}$ and therefore on $\mathcal{N}'$. We want to show that the following two representations are isomorphic:

$$\begin{array}{rccc} \tilde{\rho}_1 : & G & \longrightarrow & \mathrm{GL}(\tilde{H}_i(K')), \\ & g & \longmapsto & \tilde{\rho}_{1,g} \end{array} \qquad \begin{array}{rccc} \tilde{\rho}_2 : & G & \longrightarrow & \mathrm{GL}(\tilde{H}_i(\mathcal{N}')). \\ & g & \longmapsto & \tilde{\rho}_{2,g} \end{array}$$

Let

$$\begin{array}{rccc} \rho_1 : & G & \longrightarrow & \mathrm{GL}(C_i(K')), \\ & g & \longmapsto & \rho_{1,g} \end{array} \qquad \begin{array}{rccc} \rho_2 : & G & \longrightarrow & \mathrm{GL}(C_i(\mathcal{N}')). \\ & g & \longmapsto & \rho_{2,g} \end{array}$$

be the representations on the chain spaces. Since $g \in G$ is an order automorphism of L we have the following: if $\tilde{f}(\sigma) = \{\alpha_{j_0}, \ldots, \alpha_{j_t}\} = \beta$ then $\tilde{f}(g.\sigma) = \{g.\alpha_{j_0}, \ldots, g.\alpha_{j_t}\} = g.\beta$. We explicitly describe the maps induced by $g \in G$ on the chain spaces:

$$\begin{array}{rccc} \rho_{1,g} : & C_i(K') & \longrightarrow & C_i(K') \\ & [\sigma_0, \ldots, \sigma_i] & \longmapsto & [g.\sigma_0, \ldots, g.\sigma_i], \end{array}$$

$$\begin{array}{rccc} \rho_{2,g} : & C_i(\mathcal{N}') & \longrightarrow & C_i(\mathcal{N}'). \\ & [\beta_{j_0}, \ldots, \beta_{j_i}] & \longmapsto & [g.\beta_{j_0}, \ldots, g.\beta_{j_i}] \end{array}$$

where β_j are simplices of $\mathcal{N}$ satisfying $\beta_{j_0} \subseteq \cdots \subseteq \beta_{j_i}$. We want to show that the following diagram commutes:

$$\begin{tikzcd} C_i(K') \arrow[r, "f_\#"] \arrow[d, "\rho_{1,g}"'] & C_i(\mathcal{N}') \arrow[d, "\rho_{2,g}"'] \\ C_i(K') \arrow[r, "f_\#"] & C_i(\mathcal{N}'), \end{tikzcd}$$

i.e., $\rho_{2,g}(f_\#([\sigma_0, \ldots, \sigma_i])) = [g.\beta_{j_0}, \ldots, g.\beta_{j_i}] = (f_\#(\rho_{1,g}([\sigma_0, \ldots, \sigma_i]))$.

Therefore the diagram commutes and since $f_\#$ is a chain map we have that $\tilde{\rho}_1 \simeq \tilde{\rho}_2$, i.e., $\widetilde{H}_i(K') \simeq_G \widetilde{H}_i(\mathcal{N}')$. Using the results of Lemma 2.9 and Equation (2.8) we have the following $\mathbb{C}[G]$-module isomorphism

$$\widetilde{H}_i(K) \simeq_G \widetilde{H}_i((K(C)) = \widetilde{H}_i(C).$$

□

Remark 4. In [Lak72] Lasker proved that $K(L)$ and $K(C)$ are homotopy equivalent. It could be shown that this homotopy equivalence is G-equivariant, which would imply another proof of Theorem 2.11.

2.3 Applications to matroids

We now specialize the results of the previous two sections to matroids. Let $M = (E, I)$ be a matroid with ground set E and a collection of independent sets I, which forms an abstract simplicial complex. Let $M^* = (E, I^*)$ be its dual. We recall that the rank of $A \subseteq E$ is the maximal cardinality of an element of I contained in A. We say that $A \subseteq E$ is *non-spanning* in M if $\mathrm{rk}(A) < \mathrm{rk}(E)$, i.e., A does not contain any basis of M. Let

$$NS(M) = \{A \subseteq E \mid A \text{ is non-spanning in } M\}.$$

It is easy to see that $NS(M)$ is an abstract simplicial complex.

Proposition 2.12. *$A \subseteq E$ is non-spanning in M^* if and only if A^c is dependent in M.*

Proof. If $A \subseteq E$ is non-spanning in M^* we have:

$$\mathrm{rk}^*(A) < \mathrm{rk}^*(E).$$

This is equivalent to:

$$\mathrm{rk}(A^c) + |A| - \mathrm{rk}(E) < \mathrm{rk}^*(E)$$

and therefore to

$$\mathrm{rk}(A^c) < -|A| + \mathrm{rk}(E) + \mathrm{rk}^*(E) = |E| - |A| = |A^c| \Longleftrightarrow A^c \notin I.$$

For every $A \subseteq E$, we have $\mathrm{rk}(A) \leqslant |A|$, thus A is independent if and only if $\mathrm{rk}(A) = |A|$. $\square$

Proposition 2.13. *Let $IN(M) = I$ be the abstract simplicial complex associated with the independent sets of the matroid $M = (E, I)$ and let I^* be its Alexander dual, then:*

$$I^* = NS(M^*).$$

Proof. Using the result shown in Proposition 2.12 we claim that:

$$\begin{aligned} I^* &= \{A \subseteq E : A^c \notin I\} \\ &= \{A \subseteq E : A^c \text{ is dependent in } M = (E, I)\} \\ &= \{A \subseteq E : A \text{ is not spanning of } M^*\} = \mathrm{NS}(M^*). \end{aligned}$$

□

The previous result, together with Equation (2.6), implies the following:

$$\widetilde{H}_i(\mathrm{NS}(M)) \simeq \widetilde{H}_{n-3-i}(\mathrm{IN}(M^*)).$$

This is an isomorphism not only of vector spaces, but also of representations, up to a sign. Indeed, by applying Theorem 2.2, we obtain:

Theorem 2.14. *Let G be the automorphism group of a matroid. Then we have the following $\mathbb{C}[G]$-module isomorphism:*

$$\widetilde{H}_i(NS(M)) \simeq_G \widetilde{H}_{n-3-i}(IN(M^*)) \otimes \mathrm{sgn},$$

where n is the cardinality of the ground set of M.

Similarly, we can specialize the results from Section 2.2 to the case of matroids. Let $M = (E, I)$ be a simple matroid with $E = \{a_1, \ldots a_m\}$. Let $\mathcal{L}(M)$ be the lattice of *flats* of M ordered by inclusion. Since M is simple, each singleton of E is a flat. Thus $\{a_1\}, \{a_2\}, \ldots, \{a_m\} \in \mathcal{L}(M)$ and each corresponds to an atom of the poset $(\mathcal{L}(M), \subseteq)$. We now consider a set C defined as

$$C = \{\{a_1\}, \{a_2\}, \ldots, \{a_m\}\} \subseteq \mathcal{L}(M).$$

Since C satisfies the three axioms of Definition 2.6, the set C is a *cross-cut* of $\mathcal{L}$. We want to prove that:

$$K(C) = NS(M).$$

In the following proposition we perform a slight abuse of notation by identifying:

$$C = \{\{a_1\}, \{a_2\}, \dots, \{a_m\}\} = \{a_1, a_2, \dots, a_m\}.$$

Proposition 2.15. *$A \subseteq C$ does not 'span' (in the sense of Definition 2.7) if and only if A is a non-spanning set in $M = (E, I)$.*

Proof.

$\Longrightarrow$) In $\mathcal{L}(M)$ we have:

$$\widehat{0} = \emptyset \qquad \text{and} \qquad \widehat{1} = E.$$

Let $A = \{a_{i_1}, a_{i_2}, \dots, a_{i_n}\}$ be a subset of C. If $A \subseteq C$ does not *'span'*:

$$a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_n} = D \neq \widehat{1} \tag{2.9}$$

$D \in \mathcal{L}(M)$ and $D \neq \widehat{1}$ implies that D is a non-spanning subset of E because the only spanning subset in $\mathcal{L}(M)$ is $E = \widehat{1}$.

It follows from (2.9) that $A \subseteq D$; since D is a non-spanning subset of E therefore A is a non-spanning subset of E.

$\Longleftarrow$) In $NS(M)$ the bases are the maximal non-spanning subsets of E, (i.e., the subsets of E, such that if we add an element they become spanning set) so they are flats, in particular they correspond to the co-atoms of $(\mathcal{L}(M), \subseteq)$.

Let $A = \{a_{i_1}, a_{i_2}, \dots, a_{i_n}\}$ be a non-spanning subset of E, there exist a basis $\mathcal{B}$ of $NS(M)$ such that:

$$\text{if } A \subseteq \mathcal{B} \text{ and } \mathcal{B} \text{ is a flat, then } \mathcal{B} \in \mathcal{L}(M).$$

This implies:

$$a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_n} \subseteq \mathcal{B} \neq \widehat{1}$$

therefore A does not *'span'*.

□

Using the result of Proposition 2.15, we obtain:

$$K(C) = NS(M).$$

Since C is a cross-cut fixed by G, we can apply Theorem 2.11 to $\mathcal{L}(M)$ and C itself:

Theorem 2.16. *Let G be the group of automorphism of the simple matroid M. Then we have the following $\mathbb{C}[G]$-module isomorphism:*

$$\widetilde{H}_i(\mathcal{L}(M)) \simeq_G \widetilde{H}_i(C) = \widetilde{H}_i(NS(M))$$

where C is the cross-cut of $\mathcal{L}(M)$ composed of its atoms.

By combining Theorem 2.14 and Theorem 2.16, we get the following theorem:

Theorem 2.17. *Let G be the group of automorphism of the simple matroid M. Then we have the following $\mathbb{C}[G]$-module isomorphism:*

$$\widetilde{H}_{n-3-i}(IN(M^*)) \simeq_G \widetilde{H}_i(\mathcal{L}(M)) \otimes \operatorname{sgn}$$

where n is the cardinality of the ground set of M.

2.4 Top cohomology of hyperplane arrangements

Let $\mathcal{A}$ be a central arrangement of hyperplanes in $\mathbb{C}^r$ and let $\mathcal{L}(\mathcal{A})$ be its intersection lattice. Let $M_{\mathcal{A}}$ be the matroid associated with $\mathcal{A}$ (see Definition 1.30); then the lattice of flats $\mathcal{L}(M_{\mathcal{A}})$ of $M_{\mathcal{A}}$ is isomorphic to $\mathcal{L}(\mathcal{A})$. We can assume that the arrangement is essential: then the rank of the matroid is r. We define the complement of the arrangement:

$$\mathcal{M}(\mathcal{A}) = \mathbb{C}^r \setminus \bigcup_{H \in \mathcal{A}} H.$$

Let G be a subgroup of $GL(\mathbb{C}^r)$ that permutes the elements of $\mathcal{A}$; it is easy to see that G is also a group of automorphism of the matroid $M_{\mathcal{A}}$. Let $\mathfrak{A}$ be the Orlik-Solomon algebra associated to $\mathcal{L}(\mathcal{A})$, and let $\mathfrak{B}$ be the *algebra defined by shuffle* defined respectively in Section 2 and Section 3 of [OS80]. These algebras are $\mathbb{Z}-$graded: we denote by $\mathfrak{A}_r$ and $\mathfrak{B}_r$ the direct summands corresponding to the top degree r. In Theorem 3.7 of the same paper, Orlik and Solomon provide a G-isomorphism:

$$\theta : \ \mathfrak{A} \ \longrightarrow \ \mathfrak{B}.$$

Furthermore, we state [OS80, Theorem 4.3]:

Theorem 2.18. *Let L be a finite geometric lattice of rank $r > 1$. Then $\mathfrak{B}_r$ and $H_{r-2}(L)$ are isomorphic $\mathbb{C}[G]$-modules.*

Combining the previous results we get the following $\mathbb{C}[G]$-module isomorphism:

$$H^r(\mathcal{M}(\mathcal{A})) \simeq_G \mathfrak{A}_r \simeq_G \mathfrak{B}_r \simeq_G H_{r-2}(\mathcal{L}(\mathcal{A})). \tag{2.10}$$

Applying Theorem 2.17 we obtain the following:

Theorem 2.19. *Let $\mathcal{A}$ be a central essential hyperplane arrangement of dimension r and let $M_{\mathcal{A}}$ be the associated matroid with ground set of cardinality n. Then we have the following $\mathbb{C}[G]$-module isomorphism:*

$$H^r(\mathcal{M}(\mathcal{A})) \simeq_G H_{n-r-1}(IN(M^*_{\mathcal{A}})) \otimes \mathrm{sgn}.$$

In [LS86] Lehrer and Solomon conjecture that if W is a Coxeter group and $\mathcal{A}_W$ is the hyperplane arrangement associated to W then there is a $\mathbb{C}[G]$-module isomorphism

$$H^p(\mathcal{M}(\mathcal{A}_W)) \simeq_W \bigoplus_c \mathrm{Ind}^W_{Z(c)}(\xi_c) \qquad p = 0, \dots, \mathrm{rank}(W)$$

where c runs over a set of representatives for the conjugacy classes of W such that the dimension of the image of c (viewed as an element of $GL(V)$) is equal to p and ξ_c is a suitable character of the centralizer $Z(c)$ of c in W.

They proved the conjecture for group of rank 2 and for $W = \mathfrak{S}_r$. In the case of the symmetric group $\mathfrak{S}_r$ the arrangement $\mathcal{A}_{\mathfrak{S}_r}$ is the braid arrangement and the intersection lattice $\mathcal{L}(\mathcal{A})$ is the *partition lattice* Π_r, that is, the family of all partitions of the set $\{1, \dots, r\}$ partially ordered by refinement. Stanley studied the representations on the homology of the partition lattice in [Sta82]. By Equation (2.10), his result agrees with the conjecture of Lehrer and Solomon.

We remark that Theorem 2.19 allows us to rewrite Lehrer and Solomon' conjecture in the top cohomology case in the language of matroids:

$$H_{n-r-1}(IN(M^*_{\mathcal{A}_W})) \simeq_W \bigoplus_c \mathrm{Ind}^W_{Z(c)}(\xi_c) \otimes \mathrm{sgn}.$$

2.5 Coned graphs and complete bipartite graphs

In [Koo07], Woong Kook studied the homology of the independence complex $IN(M(\widehat{\Gamma}))$ of the matroid associated to a *coned graph* $\widehat{\Gamma}$, i.e. the graph obtained by adding a new vertex p to a graph Γ and joining each vertex of Γ to p by a simple edge. We recall the following definition from [Koo07, Section 2]:

Definition 2.20. An *edge-rooted forest* $(F, \mathbf{e})$ in Γ is a spanning forest F that contains at least one edge for each connected component of Γ, together with the datum $\mathbf{e}$ of one edge for each component (called *edge root*).

The rank of the only non zero homology group of $IN(M(\widehat{\Gamma}))$ is shown to be equal to the cardinality of the set of *edge-rooted forests* $\mathcal{F}_e(\Gamma)$ in Γ.

In [Koo07, Section 3], Kook constructs a basis $\{z_{F,\mathbf{e}} \ : \ (F, \mathbf{e}) \in \mathcal{F}_e(\Gamma)\}$ for $\widetilde{H}_{n-1}(IN(M(\widehat{\Gamma})))$ (where n is the number of vertices of Γ). This basis is indexed by the elements $(F, \mathbf{e}) \in \mathcal{F}_e(\Gamma)$. In the same Section, Kook describes the action of the automorphism group $G = \mathrm{Aut}(\Gamma)$ on $\widetilde{H}_{n-1}(IN(M(\widehat{\Gamma})))$ for a finite simple graph Γ, showing that this action is isomorphic to the permutation action on $\mathcal{F}_e(\Gamma)$ tensored with the sign representation:

$$g(z_{F,\mathbf{e}}) = \mathrm{sgn}(g)\ z_{g(F,\mathbf{e})}$$

where $g(F, \mathbf{e}) = (g(F), g(\mathbf{e}))$ (see [Koo07, Theorem 6]). Extending by linearity those two G-actions, we obtain two representations of G, respectively on $\widetilde{H}_{n-1}(IN(M(\widehat{\Gamma})), \mathbb{C})$ and on the vector space $\boldsymbol{\mathcal{F}_e(\Gamma)}$ of formal $\mathbb{C}$-linear combinations of elements of $\mathcal{F}_e(\Gamma)$, which are isomorphic up to a sign:

$$\widetilde{H}_{n-1}(IN(M(\widehat{\Gamma}))) \simeq_G \boldsymbol{\mathcal{F}_e(\Gamma)} \otimes \mathrm{sgn}.$$

Applying Theorem 2.17 we obtain the following $\mathbb{C}[G]$-module isomorphism:

$$\widetilde{H}_{l-n-2}(\mathcal{L}(M^*(\widehat{\Gamma}))) \simeq_G \boldsymbol{\mathcal{F}_e(\Gamma)} \tag{2.11}$$

where l is the number of edges of $\widehat{\Gamma}$.

Furthermore, in [KL18] Woong Kook and Kang-Ju Lee studied the homology of the independence complex $IN(M(K_{m+1,n+1}))$ of the matroid associated to the complete bipartite graph $K_{m+1,n+1}$. We need the following definition:

Definition 2.21. A *B-edge-rooted forest* $(F, \mathbf{b}, \mathbf{e})$ in a complete bipartite graph $K_{m,n}(m, n \geq 1)$ is a spanning forest F in $K_{m,n}$ composed of two kinds of connected components such that

- exactly one component is *bi-rooted*, i.e., has one vertex-root in each bipartite set;
- each of the remaining components is edge-rooted, i.e., has one edge marked as edge-root (see [KL18, Definition 3.3]).

The rank of the only non zero homology group of $IN(M(K_{m+1,n+1}))$ is shown to be equal to the cardinality of the set of the *B-edge-rooted forests* $\mathcal{F}_e^B(K_{m,n})$ in $K_{m,n}$.

In Section 5, the authors construct a basis $\{z_{F,\mathbf{b},\mathbf{e}} : (F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B(K_{m,n})\}$ for $\widetilde{H}_{m+n}(IN(M(K_{m+1,n+1})))$. This basis is indexed by the elements $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B(K_{m,n})$. In the same section, they proved the following theorem:

Theorem 2.22. *The action of $\mathfrak{S}_m \times \mathfrak{S}_n$ as a subgroup of $\mathfrak{S}_{m+1} \times \mathfrak{S}_{n+1}$ on $\widetilde{H}_{m+n}(IN(M(K_{m+1,n+1})))$ is isomorphic to the action on $\mathcal{F}_e^B(K_{m,n})$ tensored* *with the sign representation:*

$$\sigma(z_{F,\boldsymbol{b},\boldsymbol{e}}) = \operatorname{sgn}(\sigma) z_{\sigma(F,\boldsymbol{b},\boldsymbol{e})}.$$

Now we consider the representations of the group $\mathfrak{S}_m \times \mathfrak{S}_n$ that extend by linearity the two $\mathfrak{S}_m \times \mathfrak{S}_n$- actions, respectively on the vector space $\widetilde{H}_{m+n}(IN(M(K_{m+1,n+1})))$ and on the vector space $\boldsymbol{\mathcal{F}_e^B(K_{m,n})}$ of formal $\mathbb{C}$-linear combinations of elements of $\mathcal{F}_e^B(K_{m,n})$. Clearly we have the following $\mathbb{C}[\mathfrak{S}_m \times \mathfrak{S}_n]$-module isomorphism:

$$\widetilde{H}_{m+n}(IN(M(K_{m+1,n+1}))) \simeq_{\mathfrak{S}_m \times \mathfrak{S}_n} \boldsymbol{\mathcal{F}_e^B(K_{m,n})} \otimes \operatorname{sgn}.$$

Applying Theorem 2.17 we obtain the following $\mathbb{C}[\mathfrak{S}_m \times \mathfrak{S}_n]$-module isomorphism:

$$\widetilde{H}_{l-m-n-3}(\mathcal{L}(M^*(K_{m+1,n+1}))) \simeq_{\mathfrak{S}_m \times \mathfrak{S}_n} \boldsymbol{\mathcal{F}_e^B(K_{m,n})} \tag{2.12}$$

where l is the number of edges of $K_{m+1,n+1}$.

2.6 The dual matroid of the complete graph

We now consider the matroid $M(K_m)$ of the complete graph K_m, which has rank $r = m-1$ and ground set of cardinality $n = \binom{m}{2}$. This matroid is isomorphic to the matroid $M(\Phi^+_{A_{m-1}})$ associated with the positive roots of the root system of type A_{m-1}. In fact, this is the case of interest in [dCHM21].

We recall that the lattice of flats of this matroid is isomorphic to the partition lattice Π_m. In this case, Theorem 2.17 specializes to the following:

Theorem 2.23. *$\widetilde{H}_{n-3-i}(IN(M^*(K_m)))$ and $\widetilde{H}_i(\Pi_m) \otimes \operatorname{sgn}$ are isomorphic as $\mathfrak{S}_m$-modules for every $i \geq 0$.*

Remark 5. In [dCHM21], de Cataldo, Heinloth and Migliorini apply this result to the computation of the supports of the perverse cohomology sheaves of the Hitchin fibration for GL_m over the locus of reduced spectral curves.

Rephrased in terms of root system of type A_{m-1}, the theorem above yields the following $\mathbb{C}[G]$-module isomorphism:

$$\widetilde{H}_{n-3-i}(IN(M^*(\Phi^+_{A_{m-1}}))) \simeq_{\mathfrak{S}_m} \widetilde{H}_i(\Pi_m) \otimes \mathrm{sgn} \tag{2.13}$$

where

$$n = |E(M^*(\Phi^+_{A_{m-1}}))| = |\Phi^+(A_{m-1})| = \binom{m}{2} = \frac{m(m-1)}{2}.$$

Remark 6. We can make a dimensional calculation to better understand the dimensional shift. The matroid $M(\Phi^+_{A_{m-1}}, I)$ has rank equal to $m-1$, i.e. each basis has $m-1$ elements. Therefore, the matroid $M^*(\Phi^+_{A_{m-1}}, I)$ has rank equal to:

$$n-(m-1) = \frac{m(m-1)}{2} - (m-1) = \frac{(m-1)(m-2)}{2}.$$

Thus, the dimension of the top homology of $IN(M^*(\Phi^+_{A_{m-1}})$ is one less than the number of the elements of a basis of $M^*(\Phi^+_{A_{m-1}}, I)$:

$$\frac{(m-1)(m-2)}{2} - 1.$$

By Equation (2.13) we have the following isomorphism of $\mathbb{C}$-vector spaces:

$$\widetilde{H}_{n-3-i}(IN(M^*(\Phi^+_{A_{m-1}}))) \simeq \widetilde{H}_i(\Pi_m).$$

We impose

$$n-3-i = \frac{(m-1)(m-2)}{2} - 1$$

then we have $i = m-3$ from $n = m(m-1)/2$. Indeed, $H_{m-3}(\Pi_m)$ is the only nonzero homology group of Π_m.

By Theorem 2.23 these two representations

$$\rho_{n-m}: \quad \mathfrak{S}_m \quad \longrightarrow \quad \mathrm{GL}(\widetilde{H}_{n-m}(IN(M^*(\Phi^+_{A_{m-1}}))))$$

and

$$\gamma_{m-3}: \quad \mathfrak{S}_m \quad \longrightarrow \quad \mathrm{GL}(\widetilde{H}_{m-3}(\Pi_m) \otimes \mathrm{sgn})$$

are isomorphic. From a result due to Stanley ([Sta82, Theorem 7.3]) we know that the representations on the top homology of the partition lattice

$$\tilde{\gamma}_{m-3}: \quad \mathfrak{S}_m \quad \longrightarrow \quad \mathrm{GL}(\widetilde{H}_{m-3}(\Pi_m))$$

are the following

$$\tilde{\gamma}_{m-3} \simeq \mathrm{sgn} \otimes \mathrm{ind}_{C_m}^{\mathfrak{S}_m}(e^{2\pi i/m}).$$

Thus, we get

$$\rho_{n-m} \simeq \mathrm{ind}_{C_m}^{\mathfrak{S}_m}(e^{2\pi i/m})$$

or as $\mathbb{C}[\mathfrak{S}_m]$-modules:

$$\widetilde{H}_{n-m}(IN(M^*(\Phi^+_{A_{m-1}}))) \simeq_{\mathfrak{S}_m} \mathrm{ind}_{C_m}^{\mathfrak{S}_m}(e^{2\pi i/m}).$$

Chapter 3

Hodge theory for polymatroids

In this Chapter we construct a Leray model for a discrete polymatroid (see Definition 3.4) and we prove a generalized Goresky-MacPherson formula (see Theorem 3.24). We prove Poincaré duality, Hard Lefschetz, and Hodge-Riemann theorems for the Chow ring of a polymatroid (see Theorem 3.31 and Theorem 3.43). Furthermore, we provide a relative Lefschetz decomposition with respect to the deletion of an element (see Theorem 3.46). The last section contains an example that illustrate our definitions and properties (see Section 3.6).

3.1 The Leray model

First, we extend Definition 1.65 and 1.68 to an arbitrary polymatroid, not necessarily realizable.

Definition 3.1 (Combinatorial building set)**.** Let $P = (E, \mathrm{cd})$ be a polymatroid and let L be its lattice of flats. A subset $\mathcal{G}$ in $L \setminus \{\hat{0}\}$ is called a *combinatorial building set* if for any $x \in L$ the morphism of lattices:

$$\varphi_x \colon \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y] \to [\hat{0}, x]$$

induced by the inclusions is an isomorphism and the equality

$$\mathrm{cd}(x) = \sum_{y \in \max(\mathcal{G}_{\leq x})} \mathrm{cd}(y)$$

holds.

We define $F(P, \mathcal{G}, x) = \max(\mathcal{G}_{\leq x})$ the set of $\mathcal{G}$-*factors* of x.

Definition 3.2 ($\mathcal{G}$-nested set complex)**.** A subset S of $\mathcal{G}$ is called $\mathcal{G}$-nested if, for any set of incomparable elements $x_1, \ldots, x_t$ in S of cardinality at least two, the join $x_1 \vee \cdots \vee x_t$ is not contained in $\mathcal{G}$. The $\mathcal{G}$-nested sets form an abstract simplicial complex $n(P, \mathcal{G})$.

We suggest to visualize a (realizable) polymatroid (E, cd) as a collection of linear subspaces S_e for $e \in E$ in a fixed complex vector space V. For each $A \subseteq E$, the codimension $\mathrm{cd}(A)$ is the codimension of the corresponding flat $\cap_{a \in A} S_a$. The (combinatorial) building set $\mathcal{G}$ is a good choice of some flats to blow up, in order to obtain a wonderful model $Y_{\mathcal{G}}$ with some exceptional divisors $D_g \subset Y_{\mathcal{G}}$, $g \in \mathcal{G}$ indexed by $\mathcal{G}$. A subset S of $\mathcal{G}$ is $\mathcal{G}$-nested if and only if the corresponding divisors $\{D_W\}_{W \in S}$ have nonempty intersection.

The following proposition summarizes the main properties of building and nested sets.

Proposition 3.3. *Let P be a polymatroid with poset of flats L and $\mathcal{G}$ be a building set. Then:*

1. *For each $g \in \mathcal{G}$, $x \in L$ with $x \geq g$, there exists a unique $\mathcal{G}$-factor p of x such that $p \geq g$.*

2. *If $g, h \in \mathcal{G}$ and $g \wedge h > \hat{0}$, then $g \vee h \in \mathcal{G}$.*

3. *If S is a $\mathcal{G}$-nested set, then the $\mathcal{G}$-factors of $\bigvee S$ are the maximal elements in S (i.e. $F(P, \mathcal{G}, \bigvee S) = \max(S)$).*

4. *Let S be a $\mathcal{G}$-nested set, the Hasse diagram of S (as subset of L) is a forest.*

Proof. For (1) see [FK04] Proposition 2.5(1). For (2) see [BDF20] Proposition 2.5.3(b). For (3) see [FK04] Proposition 2.8. In order to prove (4) we suppose that the Hasse diagram Γ_S of S is not a forest. Thus there exist two incomparable elements $g, h \in S$ and $t \in S$ such that $g \wedge h \geq t \in S$; in particular $g \wedge h > \hat{0}$. By part (2) we get that $g \vee h \in \mathcal{G}$ but this contradicts the definition of nested set. Therefore Γ_S is a forest. $\square$

Let $P = (E, \mathrm{cd})$ be a polymatroid, L be its poset of flats, and $\mathcal{G}$ be a building set in L. Let $\mathcal{R}(\mathcal{G}) = \mathbb{Q}[e_g, x_g \mid g \in \mathcal{G}]$ be the bigraded commutative algebra with exterior generators e_g in bidegree $(0, 1)$ and commutative generators x_g in bidegree $(2, 0)$.

This algebra is equipped with a differential d of bidegree $(2, -1)$ defined on generators by $d(e_g) = x_g$, $d(x_g) = 0$. Fix a linear extension $\succ$ of the order on $\mathcal{G}$, this gives a reverse order among the e-variables and among the x-variables, i.e. $x_h \prec x_g$ and $e_h \prec e_g$ if and only if $h \succ g$. We also set $x_g \prec e_h$ for each g, h. The algebra $\mathcal{R}(\mathcal{G})$ has a monomial basis given by:

$$e_T x_S^b := e_{g_1} \cdots e_{g_t} x_{h_1}^{b_1} \cdots x_{h_s}^{b_s}$$

where $T = \{g_1, \ldots, g_t\}$ with $g_i \in \mathcal{G}$ satisfying $g_1 \prec g_2 \prec \cdots \prec g_t$, $S = \{h_1, \ldots, h_s\}$ with $h_i \in \mathcal{G}$ and $b = (b_1, \ldots, b_s)$ is a s-tuple of positive integers. We define the element:

$$c_g = \sum_{\substack{h \in \mathcal{G} \\ h \geq g}} x_h.$$

Definition 3.4 (The Leray model of a polymatroid). Let $I(\mathcal{G})$ be the ideal of $\mathcal{R}(\mathcal{G})$ generated by

(i) $e_T x_S$ whenever $S \cup T \notin n(P, \mathcal{G})$,

(ii) $e_T x_S c_g^b$ whenever $S, T \subseteq \mathcal{G}$, $g \in \mathcal{G}$ and $b \geq \mathrm{cd}(g) - \mathrm{cd}(\bigvee (S \cup T)_{<g})$.

The ideal $I(\mathcal{G})$ is preserved by the differential d, so the quotient

$$B(P, \mathcal{G}) = {}^{\mathcal{R}(\mathcal{G})}\!\big/_{I(\mathcal{G})}$$

is a bigraded differential algebra, called the *Leray model* of the polymatroid.

In the realizable case, the Leray model $B(P, \mathcal{P})$ is the second page of the Leray spectral sequence for the natural inclusion

$$V \setminus \cup_{e\in E} S_e \cong Y_{\mathcal{G}} \setminus \cup_{g\in\mathcal{G}} D_g \hookrightarrow Y_{\mathcal{G}}.$$

This spectral sequence collapses at the third page, hence it becomes a differential bigraded algebra also known as the Morgan algebra [Mor78].

Remark 7. Let $e_T x_S c_g^b$ be a monomial of type (ii) and let

$$S' = S \cap \max(S \cup T)_{<g} \quad \text{and} \quad T' = \max(S \cup T)_{<g} \setminus S.$$

The monomial $e_{T'} x_{S'} c_g^b$ divides the monomial $e_T x_S c_g^b$. Thus, when we consider a monomial of type (ii) we can always assume that $S \cap T = \emptyset$, $S \cup T$ is an antichain and $\bigvee(S \cup T) < g$.

Theorem 3.5. *The generators of type (i) and (ii) of the ideal $I(\mathcal{G})$ of Definition 3.4 form a Gröbner basis with respect to the deg-lex order.*

Proof. We adapt the method used in [FY04, Theorem 2] and in [BDF20, Theorem 5.3.1]. We are fixing a linear extension of the order on $\mathcal{G}$ with $x_g \prec e_h$ for each g, h. With this we consider the deg-lex monomial order on $\mathcal{R}(\mathcal{G})$. We explicitly compute S-polynomials.

Case (i)-(i) Since relations type (i) are monomials the S-polynomials will be zero.

Case (i)-(ii) Now we consider $f_1 = e_T x_S$ of type (i) and $f_2 = e_A x_B c_g^b$ of type (ii). We can assume that $\bigvee(A \cup B) < g$ (see Remark 7). Let $U = T \cup A$, $V = B \cup S \setminus \{g\}$, therefore the S-polynomial is

$$S(f_1, f_2) = e_U x_V x_g^b - e_U x_V c_g^b = e_U x_V (x_g^b - c_g^b).$$

If $g \notin S$, we have that $S \subseteq V$ and therefore

$$S(f_1, f_2) = \pm e_{A\setminus T} e_T x_S x_{V\setminus S}(x_g^b - c_g^b)$$

is divisible by $e_T x_S$.

Then assume $g \in S$, since $S \cup T$ is not nested we have that $U \cup V \cup \{g\}$ is not nested. If $U \cup V$ is not nested, then $S(f_1, f_2)$ would be divisible by $e_U x_V$.

So assume $U \cup V$ is nested, (thus we have that $g \notin U \cup V$) since $U \cup V \cup \{g\}$ is not nested the S-polynomial modulo $e_U x_V x_g$ became

$$S(f_1, f_2) \equiv e_U x_V \Big(\sum_{f>g} x_f \Big)^b.$$

The set $U \cup V \cup \{g\}$ contains a non trivial antichain Y whose join $\bigvee Y = y$ is in $\mathcal{G}$ and Y must contain g since $U \cup V$ is nested; let $y' = \bigvee (Y \smallsetminus \{g\})$. We have

$$\begin{aligned} b &= \operatorname{cd}(g) - \operatorname{cd}\Big(\bigvee_{\substack{l \in A \cup B \\ l<g}} l \Big) \\ &\geq \operatorname{cd}(g \vee y') - \operatorname{cd}\Big(\bigvee_{\substack{l \in A \cup B \\ l<g}} l \vee y' \Big) \\ &\geq \operatorname{cd}(y) - \operatorname{cd}\Big(\bigvee_{\substack{l \in U \cup V \\ l<y}} l \Big) = b' \end{aligned}$$

and so $e_U x_V c_y^b$ is a relation of type (ii). We claim that modulo relations of type (i)

$$S(f_1, f_2) \equiv e_U x_V c_y^b.$$

To obtain this, we will show that if $f \in \mathcal{G}$ with $f > g$ and $f \ngeq y$, then $U \cup V \cup \{f\}$ is not nested. Suppose that $U \cup V \cup \{f\}$ is nested and consider the following antichain $Y' = \max(Y \smallsetminus \{g\} \cup \{f\}) \subseteq U \cup V \cup \{f\}$. The set Y' is nested and by Proposition 3.3 the $\mathcal{G}$-factors of $\bigvee (Y \smallsetminus \{g\} \cup \{f\})$ are exactly the elements of $Y' = \{y_1, \dots, y_k, f\}$. We have

$$\bigvee (Y \smallsetminus \{g\} \cup \{f\}) = y' \vee f \geq y' \vee g = y.$$

By definition of $\mathcal{G}$-factor we have two cases:

- $y \leq y_i$ for a certain i. But this is impossible since $y_i < y$;

- $y \leq f$ contrary to the assumption $f \not\geq y$.

Thus, $U \cup V \cup \{f\}$ is not nested and $S(f_1, f_2)$ reduces to zero.

Case (ii)-(ii) Let $f_1 = e_T x_S c_g^d$ and $f_2 = e_A x_B c_h^f$ be two relations of type (ii). We may assume $\bigvee(S \cup T) < g$ and $\bigvee(A \cup B) < h$ (see Remark 7). We have the following cases:

First $g = h$ and $d \leq f$, then the S-polynomial is

$$S(f_1, f_2) = e_{T \cup A} x_{S \cup B} c_g^d (x_g^{f-d} - c_g^{f-d});$$

which is divisible by $e_T x_S c_g^d$.

Second $g \neq h$, $g \notin B$, $h \notin S$, we also assume that $h \succ g$. The S-polynomial is

$$S(f_1, f_2) = e_{S \cup A} x_{T \cup B} (x_h^f c_g^d - x_g^d c_h^f).$$

Let $y = e_{T \cup A} x_{S \cup B} c_g^d (c_h^f - x_h^f)$, which is divisible by $e_T x_S c_g^d$ and has a leading term smaller or equal to that of $S(f_1, f_2)$. The remainder is

$$S(f_1, f_2) + y = e_{T \cup A} x_{S \cup B} (c_g^d - x_g^d) c_h^f,$$

it is divisible by $e_A x_B c_h^f$, and reduces to zero.

Finally, assume $g \neq h$ and $g \in B$, by Remark 7 we must have $g \prec h$ and $h \notin S$. Let $U = T \cup A$ and $V = S \cup B \setminus \{g\}$, the S-polynomial is

$$S(f_1, f_2) = e_U x_V (x_h^f c_g^d - x_g^d c_h^f).$$

Let $y = e_U x_V c_g^d (c_h^f - x_h^f)$, which is divisible by $e_T x_S c_g^d$ and has a leading term smaller or equal to that of S. It remains to verify that

$$S(f_1, f_2) + y = e_U x_V (c_g^d - x_g^d) c_h^f$$

reduces to zero. First, through division by $e_A x_B c_h^f$, since $g \in B$ we have

$$S(f_1, f_2) + y \equiv e_U x_V \Big(\sum_{k > g} x_k \Big)^d c_h^f. \tag{3.1}$$

We claim that for any $k > g$, $k \ngeq h$ we have

$$e_U x_V x_k c_h^f \equiv e_U x_V x_k c_{h\vee k}^f \equiv 0$$

modulo relations of type (i) and (ii). For the first claim, if $p \geq h$ but $p \ngeq h \vee k$ then $\{p, k\}$ is not nested by Proposition 3.3 and we can divide by the relations of type (i) $x_h x_p$. The last claim follows since $h \vee k \in \mathcal{G}$ by Proposition 3.3 and

$$\begin{aligned} f &\geq \mathrm{cd}(h) - \mathrm{cd}(\bigvee(A \cup B)) \\ &\geq \mathrm{cd}(h \vee k) - \mathrm{cd}(\bigvee(U \cup V \cup \{k\})). \end{aligned}$$

Therefore, the element in eq. (3.1) reduces to

$$S(f_1, f_2) + y \equiv e_U x_V c_h^{d+f}.$$

Then since $d + f \geq \mathrm{cd}(h) - \mathrm{cd}(\bigvee(U \cup V \cup \{k\}))$ we may divide by $e_U x_V c_h^{d+f}$ and reduce to zero.

This completes the proof. □

Corollary 3.6. *The algebra $B(P, \mathcal{G})$ has an additive basis given by the monomials $e_T x_S^b$ for which $S \cup T \in n(P, \mathcal{G})$ and for each $s \in S$ we have that $0 < b(s) < \mathrm{cd}(s) - \mathrm{cd}(\bigvee(S \cup T)_{<s})$.*

Proof. An additive basis for the algebra $B(P, \mathcal{G})$ is given by all the monomials which are not divisible by the initial monomials of the Gröbner basis. The proof follows immediately. □

We now provide a second presentation for the algebra $B(P, \mathcal{G})$ using a different set of generators (τ_g and σ_g for $g \in \mathcal{G}$) and different relations.

Theorem 3.7. *The morphism*

$$\varphi \colon \Lambda[\tau_g \mid g \in \mathcal{G}] \otimes \mathbb{Q}[\sigma_g \mid g \in \mathcal{G}] \to B(P, \mathcal{G})$$

defined by $\varphi(\tau_g) = \sum_{h \geq g} e_h$ and by $\varphi(\sigma_g) = \sum_{h \geq g} x_h$, is surjective with kernel generated by:

(i) $\prod_{t\in T}(\tau_t - \tau_g)\prod_{s\in S}(\sigma_s - \sigma_g)$ *for* $S \cup T$ *a non-trivial antichain and* $g = \bigvee(S\cup T) \in \mathcal{G}$,

(ii) $\prod_{t\in T}(\tau_t - \tau_g)\prod_{s\in S}(\sigma_s - \sigma_g)\sigma_g^b$ *for* $g \in \mathcal{G}$ *and* $b = \mathrm{cd}(g) - \mathrm{cd}(\bigvee(S\cup T)_{<g})$.

We will identify the elements τ_g, σ_g with their images in $B(P, \mathcal{G})$. In the realizable case the element σ_g is the fundamental class of D_g, the total transform of the flat g. Analogously, τ_g is the sum of irreducible components of the total transform of the flat g. The elements σ_g can be also seen in the following way: consider the inclusion $Y_\mathcal{G} \hookrightarrow \prod_{h\in\mathcal{G}} \mathbb{P}^{\mathrm{cd}(h)-1}$ of [DCP95], σ_g is the pullback of the hyperplane class of the factor $\mathbb{P}^{\mathrm{cd}(g)-1}$.

Before the proof of Theorem 3.7 we need a couple of technical lemmas.

Lemma 3.8. *Let* $g \in \mathcal{G}$ *and* $S = \{s_1, \dots s_n\} \subset \mathcal{G}$ *such that* $\bigvee S \leq g$, *set* $b = \mathrm{cd}(g) - \mathrm{cd}(\bigvee S)$. *Consider a set* $A = \{a_1, \dots a_n\} \subset \mathcal{G}$ *such that* $a_i \geq s_i$ *and* $a_i \not\geq g$ *for all* $i = 1, \dots, n$. *Then*

$$y_A c_g^b = 0,$$

where y_{a_i} *is equal to* e_{a_i} *or* x_{a_i} *and* $y_A = y_{a_1} \cdots y_{a_n}$.

Proof. Define the element $h = \bigvee A \vee g$, we first prove the equality $y_A\sigma_g^b = y_A\sigma_h^b$ and then $y_A\sigma_h^b = 0$.

We want to show that $h \in \mathcal{G}$. Let $h' \in \mathcal{G}$ be the unique $\mathcal{G}$-factor of h such that $h' \geq g$. For each a_i we have $h' \wedge a_i \geq s_i$ and so $a_i \vee h' \in \mathcal{G}$. By maximality of h' we have $a_i \leq h'$ for all i. Therefore $h = h' \in \mathcal{G}$.

Firstly, let $g' \in \mathcal{G}$ be any element such that $g' \geq g$ and $g' \not\geq h$. Suppose that $A \cup \{g'\}$ is a $\mathcal{G}$-nested set. Then the $\mathcal{G}$-factors of $h \vee g'$ are the maximal elements of $A \cup \{g'\}$ by Proposition 3.3. So there exists an element in $A \cup \{g'\}$ bigger or equal to h, this is impossible since $g' \not\geq h$ and $a_i \not\geq g$. It follows that $A \cup \{g'\}$ is not $\mathcal{G}$-nested and $y_A x_{g'} = 0$.

Finally, we show that $y_A\sigma_h^b = 0$. Indeed, $b \geq \mathrm{cd}(g) - \mathrm{cd}(g \wedge \bigvee A)$ which is bigger than $\mathrm{cd}(h) - \mathrm{cd}(\bigvee A)$ by submodularity of cd. Applying the relations of type *(ii)* in Definition 3.4 we complete the proof. $\square$

Lemma 3.9. *The elements* $\prod_{t\in T}(\tau_t - \tau_g)\prod_{s\in S}(\sigma_s - \sigma_g)\sigma_g^b$ *for* $g \in \mathcal{G}$, *and* $b = \mathrm{cd}(g) - \mathrm{cd}(\bigvee(S \cup T)_{<g})$ *belong to the kernel of* φ.

Proof. From the argument of Remark 7 we may assume that $S \cap T = \emptyset$, $S \sqcup T$ is an antichain, and $\bigvee(S \cup T) \leq g$.

We have

$$\varphi\Big(\prod_{t\in T}(\tau_t - \tau_g)\prod_{s\in S}(\sigma_s - \sigma_g)\sigma_g^b\Big) = \sum_{A,B} e_A x_B \Big(\sum_{l\geq g} x_l\Big)^b,$$

where the sum is taken over the sets $A = (a_i)_i$ and $B = (b_j)_j$ such that $a_i \geq t_i$, $b_j \geq s_j$, $a_i \not\geq g$, and $b_j \not\geq g$. Each term $e_A x_B \Big(\sum_{l\geq g} x_l\Big)^b$ is zero by Lemma 3.8. □

Proof of Theorem 3.7. Let $\prec$ be a reverse linear extension of the order on $\mathcal{G}$ with $x_g \prec e_h$ and $\sigma_g \prec \tau_h$ for each g, h. Now we consider the basis formed respectively by the σ_g, τ_h and by the x_g,e_h ordered with $\prec$; with respect of these two basis the matrix associated to the morphism φ is upper unitriangular and therefore invertible. It follows that the map φ is surjective.

We want to prove that $\ker\varphi$ is generated by relations of type (i) and (ii) of Theorem 3.7. From Lemma 3.9 we know that elements of the form (ii) belong to $\ker\varphi$. The relations (i) are a particular case of relations (ii) with $b = 0$. Let J be the ideal generated by relation of type (i) and (ii), we denote also by $\mathrm{in}(J)$ the initial ideal of J. It suffices to prove that

$$\dim {}^{\mathcal{C}}\!/_{\mathrm{in}(J)} \leq \dim {}^{\mathcal{R}(\mathcal{G})}\!/_{\mathrm{in}(I(\mathcal{G}))}$$

where $\mathcal{C} = \Lambda[\tau_g \mid g \in \mathcal{G}] \otimes \mathbb{Q}[\sigma_g \mid g \in \mathcal{G}]$. Let $K \subseteq \mathrm{in}(I)$ be the ideal generated by the leading monomial of relation of type (i) and (ii), since

$$\dim {}^{\mathcal{C}}\!/_{K} \geq \dim {}^{\mathcal{C}}\!/_{\mathrm{in}(J)}$$

it suffices to check that

$$\dim {}^{\mathcal{C}}\!/_{K} = \dim {}^{\mathcal{R}(\mathcal{G})}\!/_{\mathrm{in}(I(\mathcal{G}))}. \tag{3.2}$$

The leading monomials of relation type (i) are of the form $\tau_T\sigma_S$ where $S\cup T$ is not $\mathcal{G}$-nested; the leading monomials of relation type (ii) are of the form $\tau_T\sigma_S\sigma_g^b$ whenever $S,T\subseteq\mathcal{G}$, $g\in\mathcal{G}$ and $b=\mathrm{cd}(g)-\mathrm{cd}(\bigvee(S\cup T)_{<g})$. The monomials in $\mathcal{C}$, which are not divisible by the these two type of leading monomials, are of the form $\tau_T\sigma_S^m$ with $S\cup T\in n(P,\mathcal{G})$ and and for each $s\in S$ we have that $0<m(s)<\mathrm{cd}(s)-\mathrm{cd}(\bigvee(S\cup T)_{<s})$. Hence eq. (3.2) follows. Since the map φ is surjective it is also injective; and the initial ideal $\mathrm{in}(J)$ is equal to K. Therefore, relation type (i) and (ii) form a Gröbner basis for $\ker\varphi$. □

From the proof of Theorem 3.7 we obtain also the following corollary:

Corollary 3.10. *The set of monomials $\tau_T\sigma_S^m$ with $S\cup T\in n(P,\mathcal{G})$ and, for each $s\in S$, $0<m(s)<\mathrm{cd}(s)-\mathrm{cd}(\bigvee(S\cup T)_{<s})$ is an additive basis of $B(P,\mathcal{G})$.* □

3.2 Generalized Goresky-MacPherson formula

In this section we generalize the Goresky-MacPherson formula (see [GM88]) to the non-realizable case and to arbitrary building sets. The choice of the minimal building set yields a significantly smaller nested set complex and it can be useful in practical computations. Other generalizations of this formula can be found in [BLZ15, Des18, MP20].

3.2.1 Critical monomials

Definition 3.11. A *standard monomial* e_Tx_S (resp. $\tau_T\sigma_S$) is a monomial that appears in the basis given by Corollary 3.6 (resp. by Corollary 3.10).

For the sake of notation for any standard monomial $\tau_T\sigma_S^b$ we extend the function b by setting $b(g)=0$ for $g\notin S$.

Definition 3.12. Let $\tau_T\sigma_S^b$ be a standard monomial. An element $g\in\mathcal{G}$ is called *critical* with respect to the monomial $\tau_T\sigma_S^b$ if $g\in T$ and $b(g)=$

$\mathrm{cd}(g) - \mathrm{cd}(\bigvee(S \cup T)_{<g}) - 1$. If every element of $S \cup T$ is critical with respect to $\tau_T \sigma_S^b$ then the monomial $\tau_T \sigma_S^b$ is called *critical*.

Notice that if the monomial $\tau_T \sigma_S^b$ is critical, then $S \subseteq T$ and so the critical monomial is uniquely determined by T.

Definition 3.13. The *critical monomial* associated with $T \in n(P, \mathcal{G})$ is

$$c\mu(T) = \tau_T \sigma_S^b,$$

where $S = \{t \in T \mid \mathrm{cd}(t) - \mathrm{cd}(\bigvee(S_{<t})) > 1\}$ and $b(s) = \mathrm{cd}(s) - \mathrm{cd}(\bigvee(T_{<s})) - 1$ for all $s \in S$.

In Theorem 3.23 we will prove that the linear span of critical monomials form a subcomplex (indeed a subalgebra) of $B(P, \mathcal{G})$. Moreover, we will show that this subalgebra is quasi-isomorphic to the Leray model. This first lemma implies that the span of critical monomials is a sub-complex.

Lemma 3.14. *For every critical monomial $c\mu(T)$ we have*

$$\mathrm{d}(c\mu(T)) = \sum_{t \in T \setminus \max(T)} (-1)^{|T_{\prec t}|} c\mu(T \setminus \{t\}).$$

Proof. Let $c\mu(T) = \tau_T \sigma_S^b$, we have

$$\begin{aligned} \mathrm{d}(c\mu(T)) &= \sum_{t \in T} (-1)^{|T_{\prec t}|} \tau_{T \setminus \{t\}} \sigma_S^b \sigma_t \\ &= \sum_{t \in T \setminus \min(T)} (-1)^{|T_{\prec t}|} \tau_{T \setminus \{t\}} \sigma_S^b \sigma_t, \end{aligned}$$

because if $t \in \min(T)$ then $b(t) = \mathrm{cd}(t) - 1$ and so $\sigma_t^{\mathrm{cd}(t)} = 0$.

Fix $t \in T \setminus \min(T)$, the set $R = \max(T_{<t})$ is nonempty. By using relation (ii) of Theorem 3.7 and the fact that $\tau_t^2 = 0$, we have

$$\begin{aligned} \tau_R \sigma_R^b \sigma_t^{b(t)+1} &= \sum_{r \in R} (-1)^{|R_{\prec r}|} \tau_{R \setminus \{r\}} \tau_t \sigma_R^b \sigma_t^{b(t)+1} \\ &= \sum_{r \in R} (-1)^{|R_{\prec r}|} \tau_{R \setminus \{r\}} \tau_t \sigma_{R \setminus \{r\}}^b \sigma_t^{b(t)+b(r)+1}, \end{aligned}$$

where in the last equality we used

$$0 = \tau_t(\sigma_r - \sigma_t)\sigma_t^{b(t)+1} \prod_{l \neq r} (\tau_l - \tau_t) = \tau_t \tau_{R \setminus \{r\}} (\sigma_r - \sigma_t)\sigma_t^{b(t)+1}.$$

Notice that $T \in n(P, \mathcal{G})$ implies $\operatorname{cd}(\bigvee R) = \operatorname{cd}(\bigvee (R \setminus \{r\})) + \operatorname{cd}(r)$ and $\operatorname{cd}(\bigvee (R \setminus \{r\}) \vee \bigvee (T_{<r})) = \operatorname{cd}(\bigvee (R \setminus \{r\})) + \operatorname{cd}(\bigvee (T_{<r}))$; so $b_{T \setminus \{r\}}(t) = b_T(t) + b_T(r) + 1$. Therefore

$$\tau_R \sigma_R^b \sigma_t^{b(t)+1} = \sum_{r \in R} (-1)^{|R_{\prec r}|} c\mu((R \setminus \{r\}) \cup \{t\})$$

and finally:

$$\begin{aligned} \mathrm{d}(c\mu(T)) &= \sum_{t \in T \setminus \min(T)} \sum_{r \in \max(T_{<t})} (-1)^{|T_{\prec r}|} \mu(T \setminus \{r\}) \\ &= \sum_{r \in T \setminus \max(T)} (-1)^{|T_{\prec r}|} \mu(T \setminus \{r\}), \end{aligned}$$

because T is a forest by Proposition 3.3. This conclude the proof. $\square$

We want to apply algebraic Morse theory to the complex $B(P, \mathcal{G})$. We refer to [JW09] for basic definitions and properties of algebraic Morse theory.

We define the following *matching* $\mathcal{M}$: for each non-critical monomial $\tau_T \sigma_S^b$ let $g \in S \cup T$ be the smallest (with respect to $\prec$) non-critical element. If g belongs to T, then the pair $(\tau_T \sigma_S^b, \tau_{T \smallsetminus \{g\}} \sigma_S^b \sigma_g)$ is in $\mathcal{M}$.

The algebraic Morse theory, together with Lemma 3.15 and Proposition 3.18, implies that the complex of critical monomials is quasi-isomorphic to the Leray model.

Lemma 3.15. *The set $\mathcal{M}$ is a matching. Moreover a monomial is critical if and only if it is critical for the matching $\mathcal{M}$.*

Proof. We need to check that each non-critical monomial appears exactly once in $\mathcal{M}$ and that all monomials in $\mathcal{M}$ are non-critical.

By definition if the monomial $\tau_T \sigma_S^b$ appears in the first position in $\mathcal{M}$, it is non-critical. Moreover $\tau_{T \smallsetminus \{g\}} \sigma_S^b \sigma_g$ is non-critical because $S \cup \{g\} \not\subseteq T \smallsetminus \{g\}$. So every monomial in the matching is non-critical.

Vice versa, if $\tau_T\sigma_S^b$ is a non-critical monomial, let g be the minimal non-critical element in $S \cup T$. If $g \in T$ then $\tau_T\sigma_S^b$ appears in the matching (in the first position). Otherwise, $g \in S \setminus T$ so the monomial $\tau_T\tau_g\frac{\sigma_S^b}{\sigma_g}$ is basic and non-critical. Notice that an element $f \in \mathcal{G}$ is critical for $\tau_T\sigma_S^b$ if and only if is critical for $\tau_T\tau_g\frac{\sigma_S^b}{\sigma_g}$. Therefore the pair $(\tau_T\tau_g\frac{\sigma_S^b}{\sigma_g}, \tau_T\sigma_S^b)$ is in $\mathcal{M}$. □

Definition 3.16. Given a standard monomial $\tau_T\sigma_S^b$ we define $m(T,S,b)$ as the multiset $\{g^{a(g)} \mid g \in \mathcal{G}\}$ where $a(g)$ is the sum of the exponents of τ_g and σ_g in the monomial $\tau_T\sigma_S^b$. Moreover, we order these multisets lexicographically using the reverse order on $\mathcal{G}$.

As an example, if $h < g$ then $h \prec g$ and $\{h^2\} \succ \{h, g\}$.

Definition 3.17. Let G be the directed graph whose vertices are the standard monomials with a directed edge from $\tau_T\sigma_S^b$ to $\tau_{T'}\sigma_{S'}^{b'}$ if the monomial $\tau_{T'}\sigma_{S'}^{b'}$ appears with nonzero coefficient in $\mathrm{d}(\tau_T\sigma_S^b)$.

Let $G_{\mathcal{M}}$ be the directed graph G with all directed edges in $\mathcal{M}$ reversed.

Proposition 3.18. *The matching $\mathcal{M}$ is a Morse matching.*

Proof. We need to show that the graph $G_{\mathcal{M}}$ is acyclic.

Although m is not a term order (because $m(\tau_g) = m(\sigma_g)$) it has the property that for any relation of Theorem 3.7

$$\prod_{t\in T}(\tau_t - \tau_g)\prod_{s\in S}(\sigma_s - \sigma_g)\sigma_g^b \tag{3.3}$$

with $\bigvee(S \cup T) \leq g$ the monomial $\tau_T\sigma_S\sigma_g^b$ has $m(T,S,b)$ strictly bigger than any other monomial in the expansion of eq. (3.3). Moreover m is multiplicative.

First notice that:

$$\begin{aligned}
d(\tau_T\sigma_S^b) &= \sum_{g\in T}(-1)^{|T_{\prec g}|}\tau_{T\setminus g}\sigma_S^b\sigma_g \\
&= \sum_{\substack{g\in T\\ g \text{ non-critical}}} (-1)^{|T_{\prec g}|}\tau_{T\setminus g}\sigma_S^b\sigma_g + \sum_{\substack{g\in T\\ g \text{ critical}}} (-1)^{|T_{\prec g}|}\tau_{T\setminus g}\sigma_S^b\sigma_g \\
&= \sum_{\substack{g\in T\\ g \text{ non-critical}}} (-1)^{|T_{\prec g}|}\tau_{T\setminus g}\sigma_S^b\sigma_g + \sum_{\substack{\text{some } T',S',b'\\ m(T',S',b')\prec m(T,S,b)}} \alpha_{T',S',b'}\tau_{T'}\sigma_{S'}^{b'},
\end{aligned}$$

where $\alpha_{T',S',b'}$ are some coefficients. In the last equality we used the relations of Theorem 3.7 in order to write the non-standard monomials $\tau_{T\setminus g}\sigma_S^b\sigma_g$ as linear combination of standard ones. Notice also that if the pair $(\tau_T\sigma_S^b, \tau_{T\setminus g}\sigma_{S\cup g}^{b'})$ is in $\mathcal{M}$ then $m(T,S,b) = m(T\setminus g, S\cup\{g\}, b')$. This implies that the function m is weakly decreasing on every direct path in $G_{\mathcal{M}}$, so it is constant on every directed cycle.

It is a classical fact that it is enough to prove that there are no alternating directed cycles, i.e. cycles such that for every pair of consecutive edges exactly one is in $\mathcal{M}$. Suppose that there exists a directed cycle and consider two consecutive edges. We can assume that the first one is in $\mathcal{M}$ and the second one is not. The first edge is $(\tau_T\sigma_S^b, \tau_T\tau_g\frac{\sigma_S^b}{\sigma_g})$ for some non-critical monomial $\tau_T\sigma_S^b$ with g the smaller non-critical element and $g \in S\setminus T$. The second edge is $(\tau_T\tau_g\frac{\sigma_S^b}{\sigma_g}, \tau_{T'}\sigma_{S'}^{b'})$ for some standard monomial $\tau_{T'}\sigma_{S'}^{b'}$. Since the value of m is constant on the cycle we have that $\tau_{T'}\sigma_{S'}^{b'} = \tau_{T\setminus\{f\}}\tau_g\frac{\sigma_S^b}{\sigma_g}\sigma_f$ for some $f \in T$ non-critical for the monomial $\tau_T\tau_g\frac{\sigma_S^b}{\sigma_g}$. These two edges are shown below.

$$\begin{tikzcd}
& \tau_T\tau_g\frac{\sigma_S^b}{\sigma_g} \arrow[dr] & \\
\tau_T\sigma_S^b \arrow[ur] & & \tau_{T\setminus\{f\}}\tau_g\frac{\sigma_S^b}{\sigma_g}\sigma_f
\end{tikzcd}$$

The sets of critical elements for $\tau_T\sigma_S^b$ and for $\tau_T\tau_g\frac{\sigma_S^b}{\sigma_g}$ coincide, so both g and f are non-critical for $\tau_T\tau_g\frac{\sigma_S^b}{\sigma_g}$. By minimality of g we have $g \prec f$, so $T \prec (T\setminus\{f\})\cup\{g\} = T'$.

We have proved that in every alternating path after two steps the set indexing the variable τ strictly increases. Therefore there are no alternating cycles. $\square$

3.2.2 Multiplicative structure

We want to describe the product of two critical monomials in $B(P,\mathcal{G})$.

Let $(g_1, g_2, \ldots, g_k)$ be a list of elements in $\mathcal{G}$ and recall that $\prec$ is a linear extension of the order on $\mathcal{G}$. Define

$$\tilde{\lambda}(g_1, g_2, \dots, g_k) = (f_1, f_2, \dots, f_k)$$

where f_i is the unique $\mathcal{G}$-factor of $g_1 \vee g_2 \vee \cdots \vee g_i$ bigger than g_i guaranteed by Proposition 3.3(1). Define $\lambda(g_1, g_2, \dots, g_k) = \tilde{\lambda}(g_1, g_2, \dots, g_k)$ if $(f_1, f_2, \dots, f_k)$ form a $\mathcal{G}$-nested set and $f_i \prec f_{i+1}$ for $i = 1, \dots, k-1$. Set $\lambda(g_1, g_2, \dots, g_k) = 0$ otherwise. We will use the convention that $c\mu(0) = 0$ and $c\mu(\emptyset) = 1$. Let $\pi \in \mathfrak{S}_k$ be a permutation, we write $\pi(g_1, g_2, \dots, g_k)$ for the list $(g_{\pi(1)}, g_{\pi(2)}, \dots, g_{\pi(k)})$ and we denote the concatenation of two lists T_1 and T_2 by $T_1 \cup T_2$.

Remark 8. If $\tilde{\lambda}(g_1, g_2, \dots, g_k) = (f_1, f_2, \dots, f_k)$, then $g_1 \vee g_2 \vee \cdots \vee g_j = f_1 \vee f_2 \vee \cdots \vee f_j$. Moreover, $\lambda(g_1, g_2, \dots, g_k) = 0$ if there exist $i < j$ such that $g_j \leq g_i$. Indeed, $f_j \leq f_1 \vee f_2 \vee \cdots \vee f_{j-1}$ and $\{f_1, \dots, f_j\}$ is $\mathcal{G}$-nested, hence $f_j = f_c$ for some $c < j$ contradicting $f_c \prec f_j$.

Let T_1 and T_2 be two lists of elements in $\mathcal{G}$ and $\pi \in \mathfrak{S}_{|T_1 \cup T_2|}$ be a permutation. If $\lambda(\pi(T_1 \cup T_2)) \neq 0$ then the last element of $\pi(T_1 \cup T_2)$ belongs to $\max(T_1 \cup T_2)$.

In the particular case when $\mathcal{G}$ is the maximal building set and T_1, T_2 are chains in $\mathcal{G}$, $\lambda(\pi(T_1 \cup T_2))$ is zero if π is not a $(|T_1|, |T_2|)$-shuffles.

The following proposition describes the multiplication of critical monomials using shuffles.

Proposition 3.19. *Let T_1 and T_2 be $\mathcal{G}$-nested sets. If $\operatorname{cd}(\bigvee(T_1 \cup T_2)) < \operatorname{cd}(\bigvee T_1) + \operatorname{cd}(\bigvee T_2)$, then $c\mu(T_1)c\mu(T_2) = 0$. Otherwise*

$$c\mu(T_1)c\mu(T_2) = \sum_{\pi \in \mathfrak{S}_{|T_1 \cup T_2|}} \operatorname{sgn}(\pi) c\mu(\lambda\pi(T_1 \cup T_2)).$$

Before the proof of Proposition 3.19 we need two technical lemmas.

Lemma 3.20. *Let T_1 and T_2 be nested sets such that $\operatorname{cd}(\bigvee T_1) + \operatorname{cd}(\bigvee T_2) = \operatorname{cd}(\bigvee(T_1 \cup T_2))$ and $\bigvee(T_1 \cup T_2) \notin \mathcal{G}$. Then $T_1 \cup T_2$ is $\mathcal{G}$-nested and*

$$c\mu(T_1)c\mu(T_2) = (-1)^s c\mu(T_1 \cup T_2),$$

where s is the length of the permutation that reorder T_1 and T_2. Moreover:

$$c\mu(T_1)c\mu(T_2) = \sum_{\pi\in\mathfrak{S}_{|T_1\cup T_2|}} \operatorname{sgn}(\pi)c\mu(\lambda\pi(T_1\cup T_2)).$$

Proof. Notice that $[\hat{0}, \bigvee(T_1\cup T_2)] = [\hat{0}, \bigvee T_1] \times [\hat{0}, \bigvee T_2]$ with the same codimension, therefore

$$c\mu(T_1)c\mu(T_2) = (-1)^s c\mu(T_1\cup T_2).$$

Since each subset of $T_1 \cup T_2$ is $\mathcal{G}$-nested, for each $\pi \in \mathfrak{S}_{|T_1\cup T_2|}$ we have $\tilde{\lambda}\pi(T_1\cup T_2) = \pi(T_1\cup T_2)$ by (3) of Proposition 3.3. Hence $\lambda\pi(T_1\cup T_2)$ is zero for all permutations π except for the unique permutation that reorders T_1 and T_2. □

Lemma 3.21. *Suppose that T is a $\mathcal{G}$-nested set and $g\in\mathcal{G}$ such that $\operatorname{cd}(g\vee \bigvee T) - \operatorname{cd}(\bigvee T) = \operatorname{cd}(g) - \operatorname{cd}(\bigvee T_{<g})$. Set $b = \operatorname{cd}(g) - \operatorname{cd}(\bigvee T_{<g}) - 1$, then*

$$c\mu(T)\tau_g\sigma_g^b = \sum_{\pi\in\mathfrak{S}_{|T|+1}} \operatorname{sgn}(\pi)c\mu(\lambda\pi(T\cup\{g\})), \tag{3.4}$$

where the sum is taken over all permutations of $T\cup\{g\}$.

Proof. We prove the statement by induction on $|T|$.

If $T\cup\{g\}$ is nested then both side of eq. (3.4) agree with $c\mu(T\cup\{g\})$. Let $f = g\vee\bigvee T$, if $f\notin\mathcal{G}$ then there exist nonempty $\mathcal{G}$-nested sets T' and T'' such that $T\cup\{g\} = T'\sqcup T''$ and $[\hat{0}, f] = [\hat{0}, \bigvee T'] \times [\hat{0}, \bigvee T'']$. Assume that $g\in T'$ and set $t' = |T'|$ and $t'' = |T''|$. By using the inductive hypothesis and Lemma 3.20, we have

$$\begin{aligned}
c\mu(T)\tau_g\sigma_g^b &= (-1)^s c\mu(T'')c\mu(T'\setminus\{g\})\tau_g\sigma_g^b \\
&= (-1)^s c\mu(T'') \sum_{\alpha\in\mathfrak{S}_{t''}} \operatorname{sgn}(\alpha)c\mu(\lambda\alpha(T')) \\
&= \sum_{\alpha\in\mathfrak{S}_{t''}} (-1)^{s+s_\alpha} \operatorname{sgn}(\alpha)c\mu(T''\sqcup\lambda\alpha(T')) \\
&= \sum_{\pi\in\mathfrak{S}_{t'+t''}} \operatorname{sgn}(\pi)c\mu(\lambda\pi(T\cup\{g\})),
\end{aligned}$$

where s corresponds to the permutation that reorder T'' and $T' \setminus \{g\}$, s_α to the permutation that reorders T'' and $\alpha(T')$.

Now suppose that $f \in \mathcal{G}$ and let $Y = \{g_1, g_2, \dots, g_k\} = \max(g \vee \bigvee T)$ numbered such that $g = g_k$. For the sake of notation set $m(g_k) = m$. We assume $g \succ t$ for all $t \in T$, the general case differs only by a sign. We have $\prod_{i=1}^k (\tau_{g_i} - \tau_f) = 0$ and so

$$\tau_Y = \sum_{i=1}^{k} (-1)^{k-i} \tau_{Y \setminus \{g_i\} \cup \{f\}}.$$

Set $b(g_i) = \mathrm{cd}(g_i) - \mathrm{cd}(\bigvee T_{<g_i}) - 1$, for all $i \leq k$ we have

$$(\sigma_{g_i}^{b(g_i)} - \sigma_f^{b(g_i)})\tau_f \prod_{j \neq i} (\tau_{g_j} - \tau_f) = 0,$$

so $\tau_{Y \setminus \{g_i\} \cup \{f\}} \sigma_{g_i}^{b(g_i)} = \tau_{Y \setminus \{g_i\} \cup \{f\}} \sigma_f^{b(g_i)}$. Therefore we have

$$\begin{aligned} c\mu(T)\tau_g \sigma_g^b &= (-1)^s c\mu(T \setminus Y) \prod_{i=1}^{k} \tau_{g_i} \sigma_{g_i}^{b(g_i)} \\ &= (-1)^s c\mu(T \setminus Y) \sum_{i=1}^{k} (-1)^{k-i} \prod_{j \neq i} \tau_{g_j} \sigma_{g_j}^{b(g_j)} \tau_f \sigma_f^{b(g_i)} \\ &= \sum_{i=1}^{k-1} (-1)^{t_i+1} c\mu(T \setminus \{g_i\}) \tau_g \sigma_g^b \tau_f \sigma_f^{b(g_i)} + c\mu(T \cup \{f\}), \end{aligned}$$

where s (and t_i) is the length of the permutation that reorders $T \setminus Y$ and $Y \setminus \{g\}$ (respectively $T \setminus \{g_i\}$ and $\{g_i\}$). The last summand corresponds to the identity permutation. Apply the inductive hypothesis on the terms $c\mu(T \setminus \{g_i\})\tau_g \sigma_g^b$ so that

$$(-1)^{t_i+1} c\mu(T \setminus \{g_i\}) \tau_g \sigma_g^b \tau_f \sigma_f^{b(g_i)} = \sum_{\pi} \mathrm{sgn}(\pi) c\mu(\lambda\pi(T \cup \{g\}))$$

where the sum is taken over all permutations π in $\mathfrak{S}_{|T|+1}$ that sends the element g_i in the last position. Since every π such that $\lambda\pi(T \cup \{g\}) \neq 0$ has in the last position an element of $\max(T \cup \{g\})$, the result follows. □

Proof of Proposition 3.19. For the first part notice that $c\mu(T_i)$ is in bidegree $(2(\mathrm{cd}(\bigvee T_i) - |T_i|), |T_i|)$ for $i = 1, 2$. Let $f = \bigvee(T_1 \cup T_2)$, the product $c\mu(T_1)c\mu(T_2)$ can be rewritten as sum of standard monomials using only relations of type

$$\prod_{s\in S}(\tau_s - \tau_g)\prod_{t\in T}(\sigma_t - \sigma_g)\sigma_g^b$$

for $\bigvee(S \cup T) \leq g \leq f$. The standard monomials $\tau_S\sigma_T^b$ with $\bigvee(S\cup T) \leq f$ have bidegree at most $(2(\mathrm{cd}(f) - |S|), |S|)$. Therefore, if $\mathrm{cd}(f) < \mathrm{cd}(\bigvee T_1) + \mathrm{cd}(\bigvee T_2)$ we have $c\mu(T_1)c\mu(T_2) = 0$ by degree argument.

We prove the second statement by induction on $|T_2|$. The base case $T_2 = \emptyset$ is trivial. If $\bigvee T_2 \notin \mathcal{G}$ then there exist T_3 and T_4 nonempty $\mathcal{G}$-nested sets such that $T_2 = T_3 \sqcup T_4$ and $[\hat{0}, \bigvee T_2] = [\hat{0}, \bigvee T_3] \times [\hat{0}, \bigvee T_4]$. Applying Lemma 3.20 and the inductive step we have

$$\begin{aligned}
c\mu(T_1)c\mu(T_2) &= (-1)^s c\mu(T_1)c\mu(T_3)c\mu(T_4)\\
&= (-1)^s \sum_{\alpha\in\mathfrak{S}_{t_1+t_3}} \mathrm{sgn}(\alpha)c\mu(\lambda\alpha(T_1\cup T_3))c\mu(T_4)\\
&= (-1)^s \sum_{\alpha\in\mathfrak{S}_{t_1+t_3}} \mathrm{sgn}(\alpha) \sum_{\beta\in\mathfrak{S}_{t_1+t_2}} \mathrm{sgn}(\beta)c\mu(\lambda\beta(\alpha(T_1\cup T_3), T_4))\\
&= \sum_{\pi\in\mathfrak{S}_{t_1+t_2}} \mathrm{sgn}(\pi)c\mu(\lambda\pi(T_1\cup T_2)),
\end{aligned}$$

where $t_i = |T_i|$.

Now we deal with the case $\bigvee T_2 \in \mathcal{G}$. Let $g = \max T_2 \in \mathcal{G}$, $T_2' = T_2 \setminus \{g\}$, and $m = \mathrm{cd}(g) - \mathrm{cd}(T_2') - 1$. We have

$$\begin{aligned}
c\mu(T_1)c\mu(T_2) &= c\mu(T_1)c\mu(T_2')\tau_g\sigma_g^m\\
&= \sum_{\alpha\in\mathfrak{S}_{t_1+t_2-1}} \mathrm{sgn}(\alpha)c\mu(\lambda\alpha(T_1\cup T_2'))\tau_g\sigma_g^m\\
&= \sum_{\alpha\in\mathfrak{S}_{t_1+t_2-1}} \mathrm{sgn}(\alpha) \sum_{\beta\in\mathfrak{S}_{t_1+t_2}} \mathrm{sgn}(\beta)c\mu(\lambda\beta(\alpha(T_1\cup T_2')\cup\{g\}))\\
&= \sum_{\pi\in\mathfrak{S}_{t_1+t_2}} \mathrm{sgn}(\pi)c\mu(\lambda\pi(T_1\cup T_2)),
\end{aligned}$$

where we used the inductive hypothesis on T_1 and T_2' and Lemma 3.21 on $\alpha(T_1 \cup T_2')$ and $\{g\}$. $\square$

We define the algebra of critical monomials abstractly, by generators and relations.

Definition 3.22. Let $\mathrm{CM}(P, \mathcal{G})$ be the $\mathbb{Q}$-vector space generated by all the nested sets $T \in n(P, \mathcal{G})$ with bidegree $(2(\mathrm{cd}(\bigvee T) - |T|), |T|)$. The differential is defined on the base by

$$\mathrm{d}(T) = \sum_{t \in T \setminus \max(T)} (-1)^{|T_{\prec t}|} (T \setminus \{t\})$$

and the product by $T \cdot S = 0$ if $\mathrm{cd}(\bigvee (T \cup S)) < \mathrm{cd}(\bigvee T) + \mathrm{cd}(\bigvee S)$ and

$$T \cdot S = \sum_{\pi \in \mathfrak{S}_{|T|+|S|}} \mathrm{sgn}(\pi) \lambda(\pi(T \cup S))$$

otherwise. This structure make $\mathrm{CM}(P, \mathcal{G})$ a differential bigraded algebra.

We summarize all the previous results of this section in the following theorem.

Theorem 3.23. *The morphism $\xi \colon \mathrm{CM}(P, \mathcal{G}) \to B(P, \mathcal{G})$ defined by $\xi(T) = c\mu(T)$ is an inclusion of differential algebras and a quasi-isomorphism.*

Proof. The map ξ is well defined as a morphism of $\mathbb{Q}$-vector spaces. It is an inclusion since the monomials $c\mu(T)$ for $T \in n(P, \mathcal{G})$ are standard monomials and are linearly independent by Corollary 3.10. The equality $\mathrm{d}\,\xi = \xi\,\mathrm{d}$ follows from Lemma 3.14 and the equality $\xi(S \cdot T) = \xi(S)\xi(T)$ from Proposition 3.19. This also proves that $\mathrm{CM}(P, \mathcal{G})$ is a differential bigraded algebra.

Finally, the algebraic Morse theory applied to $B(P, \mathcal{G})$ and the matching $\mathcal{M}$ ensures that there exists a subcomplex $N_{\mathcal{M}}$ such that the projection

$$B(P, \mathcal{G}) \twoheadrightarrow {}^{B(P, \mathcal{G})}\!/_{N_{\mathcal{M}}}$$

is a quasi-isomorphism and the quotient is freely generated by critical monomials. The composition of ξ with the projection gives an isomorphism of chain complexes. Therefore ξ is a quasi-isomorphism. $\square$

Let $n((\hat{0},g),\mathcal{G})$ be the full subcomplex of $n(P,\mathcal{G})$ on the set of vertices $\{h \in \mathcal{G} \mid h < g\}$.

All the homology groups are taken with rational coefficients. We use the standard convention for the reduced homology that $\tilde{H}_{-1}(\emptyset) = \mathbb{Q}$.

This final theorem provides an explicit description of the cohomology of the Leray model in term of cohomology of very small simplicial complexes.

Theorem 3.24. *Let P be a polymatroid and $\mathcal{G}$ be a building set. Then*

$$H^\bullet(B(P,\mathcal{G}),\mathrm{d}) \cong H^\bullet(\mathrm{CM}(P,\mathcal{G}),\mathrm{d}) \cong \bigoplus_{f\in L}\bigotimes_{g\in F} \tilde{H}_{2\,\mathrm{cd}(g)-2-\bullet}\Big(n((\hat{0},g),\mathcal{G})\Big),$$

where $F = F(P,\mathcal{G},f)$ is the set of $\mathcal{G}$-factors of f.

In particular the summand $\tilde{H}_i(n((\hat{0},g),\mathcal{G}))$ contributes in bidegree $(2(\mathrm{cd}(g)-2-i), 2+i)$.

Proof. We use Theorem 3.23 to obtain

$$H(B(P,\mathcal{G}),\mathrm{d}) \cong H(\mathrm{CM}(P,\mathcal{G}),\mathrm{d}).$$

For each flat f let CM_f be the subcomplex of $\mathrm{CM}(P,\mathcal{G})$ generated by all nested sets T such that $\max(T) = F(P,\mathcal{G},f)$. Moreover for each $g \in \mathcal{G}$ set $\mathrm{CM}(g)$ to be the subcomplex of $\mathrm{CM}(P,\mathcal{G})$ generated by all nested sets T such that $\{g\} = \max(T)$. We have

$$\mathrm{CM}(P,\mathcal{G}) = \bigoplus_{f\in L} \mathrm{CM}_f$$

and

$$\mathrm{CM}_f = \bigotimes_{g\in F(P,\mathcal{G},f)} \mathrm{CM}(g)$$

as complexes. It is enough to prove that

$$H^\bullet(\mathrm{CM}(g),\mathrm{d}) = \tilde{H}_{2\,\mathrm{cd}(g)-2-\bullet}\Big(n((\hat{0},g),\mathcal{G})\Big).$$

Indeed $\mathrm{CM}(g)$ coincides with the reduced simplicial chain complex for $n((\hat{0},g),\mathcal{G})$, under the correspondence $T \mapsto T \setminus \{g\}$. Notice that the bidegree of $T \in \mathrm{CM}(g)$ is $(2(\mathrm{cd}(g)-|T|), |T|)$ and the degree of $T\setminus\{g\}$ in the reduced chain complex is $|T\setminus\{g\}|-1 = |T|-2$. $\square$

Definition 3.22 has a straightforward generalization to integer coefficients, we left open the following question.

Conjecture 3.25. *Does Theorem 3.24 generalizes to integer coefficients?*

The analogous statement in the realizable case with the maximal building set was proven in [DGM00, dLS01].

3.3 Kähler package

Let $\mathrm{DP}^\bullet(P,\mathcal{G})$ be the graded algebra $B^{2\bullet,0}(P,\mathcal{G})$. This algebra, in the realizable case, is the Chow ring of the De Concini Procesi wonderful model for the subspace arrangement. A presentation of $\mathrm{DP}(P,\mathcal{G})$ is given by the generators x_g for $g \in \mathcal{G}$ with relations

$$x_S c_g^b$$

where $S \subseteq \mathcal{G}$, $g \in \mathcal{G}$ and $b \geq \mathrm{cd}(g) - \mathrm{cd}(\bigvee S_{<g})$. From Corollary 3.6 the algebra $\mathrm{DP}(P,\mathcal{G})$ has an additive basis given by

$$x_S^b$$

where $S \in n(P,\mathcal{G})$ and for each $s \in S$ we have that $0 < b(s) < \mathrm{cd}(s) - \mathrm{cd}(\bigvee(S)_{<s})$.

A second presentation is given by the generators σ_g for $g \in \mathcal{G}$ with relations

$$\sigma_g^b \prod_{s\in S}(\sigma_s - \sigma_g)$$

where $\bigvee S \leq g$ and $b = \mathrm{cd}(g) - \mathrm{cd}(\bigvee S)$, see Theorem 3.7. From Corollary 3.10 the algebra $\mathrm{DP}(P,\mathcal{G})$ has an additive basis given by

$$\sigma_S^b$$

where $S \in n(P,\mathcal{G})$ and for each $s \in S$ we have that $0 < b(s) < \mathrm{cd}(s) - \mathrm{cd}(\bigvee S_{<s})$.

Remark 9. If $\hat{1} \notin \mathcal{G}$ then the polymatroid P is direct sum of other polymatroids. Indeed, let $a_1, \dots, a_k$ be the $\mathcal{G}$-factors of $\hat{1}$, the poset L is a product $\prod_{i=1}^{k}[\hat{0}, a_k]$. There exist polymatroids P^{a_i} (defined in the following, see Lemma 3.35) such that $P = \oplus_{i=1}^{k} P^{a_i}$ and building sets $\mathcal{G}^{a_i} = \mathcal{G} \cap [\hat{0}, a_i]$. Moreover, $\mathrm{DP}(P, \mathcal{G}) = \otimes_{i=1}^{k} \mathrm{DP}(P^{a_i}, \mathcal{G}^{a_i})$ and the dimension of $\mathrm{DP}(P, \mathcal{G})$ is $\mathrm{cd}(\hat{1}) - |F(P, \mathcal{G}, \hat{1})|$ (where $k = |F(P, \mathcal{G}, \hat{1})|$).

For the clarity of exposition, we assume $\hat{1} \in \mathcal{G}$ in this section. Consider the isomorphism $\deg\colon \mathrm{DP}^{\mathrm{cd}(\hat{1})-1}(P, \mathcal{G}) \to \mathbb{Q}$ defined by

$$\deg(x_{\hat{1}}^{\mathrm{cd}(\hat{1})-1}) = (-1)^{\mathrm{cd}(\hat{1})-1}.$$

Definition 3.26. Let A be a graded algebra with top degree n and $\deg\colon A^n \to \mathbb{Q}$ an isomorphism. We say that

- the algebra A satisfies *Poincaré duality* if the bilinear pairing
$$A^k \times A^{n-k} \to \mathbb{Q}$$
defined by $(a, b) \mapsto \deg(ab)$ is non-degenerate.
- the element $\ell \in A^1$ satisfies the *Hard Lefschetz property* if the multiplication map
$$\cdot \ell^{n-2k}\colon A^k \to A^{n-k}$$
is an isomorphism for all $k \leq \frac{n}{2}$.
- the element $\ell \in A^1$ satisfies the *Hodge-Riemann relations* if the bilinear form
$$Q_\ell^k\colon A^k \times A^k \to \mathbb{Q}$$
defined by $Q_\ell^k(a, b) = (-1)^k \deg(a\ell^{n-2k}b)$ (for $k \leq \frac{n}{2}$) is positive definite on the subspace
$$P_k = \ker(\cdot \ell^{n-2k+1}\colon A^k \to A^{n-k+1}).$$

We will abbreviate these properties with PD_A, $\mathrm{HL}_A(\ell)$, and $\mathrm{HR}_A(\ell)$ respectively.

3.3.1 Poincaré duality

In this subsection we give a direct proof of the Poincaré duality property for $\mathrm{DP}(P, \mathcal{G})$.

Definition 3.27. Suppose that $\hat{1} \in \mathcal{G}$ and let x_S^b be a standard monomial. The element $\epsilon(x_S^b)$ is

$$\epsilon(x_S^b) = (-1)^{|S \setminus \{\hat{1}\}|} x_{S^+}^c,$$

where $S^+ = S \cup \{\hat{1}\}$, $c(\hat{1}) = \mathrm{cd}(\hat{1}) - \mathrm{cd}(\bigvee S_{<\hat{1}}) - b(\hat{1}) - 1$, and $c(g) = \mathrm{cd}(g) - \mathrm{cd}(\bigvee S_{<g}) - b(g)$ for $g \in S \setminus \{\hat{1}\}$.

We will write c_S instead of c when we want to stress the dependency on S and b.

Recall the chosen monomial order with the property that if $h > g$ then $h \succ g$ and $x_h \prec x_g$. We fix the basis of DP^k consisting in all standard monomials x_S^b of degree k ordered with the aforementioned monomial order. In complementary degree $\mathrm{DP}^{\mathrm{cd}(\hat{1})-k}$, we consider the basis given by $\epsilon(x_S^b)$ ordered using the monomial order on x_S^b. In order to prove Poincaré duality we will show that the matrix with entries $\deg(x_S^b \epsilon(x_T^c))$ is non-degenerate. Lemma 3.28 proves that the matrix has values ± 1 on the diagonal and Lemma 3.30 shows that the matrix is upper triangular.

Lemma 3.28. *If $\hat{1} \in \mathcal{G}$ then for all standard monomials we have*

$$x_S^b \epsilon(x_S^b) = x_{\hat{1}}^{\mathrm{cd}(\hat{1})-1}.$$

Proof. We prove the statement by induction on $|S \setminus \{\hat{1}\}|$. The base case $S = \{\hat{1}\}$ is trivial. For the inductive step we choose $g \in \max(S_{<\hat{1}})$ and set $T = S \setminus \{g, \hat{1}\}$. For the sake of notation, let $n(h) = b(h) + c_S(h)$ for all $h \in S^+$ (where $c_S(h)$ is introduced in Definition 3.27). Notice that $x_T x_f x_{\hat{1}}^{n(\hat{1})} = 0$ for all $f \in (g, \hat{1}) \cap \mathcal{G}$, because $f \vee \bigvee T > g \vee \bigvee T$. Since $x_T^n \sigma_g^{n(g)} = 0$ by relation (ii), we have

$$\begin{aligned}0 &= x_T^n \sigma_g^{n(g)} x_{\hat{1}}^{n(\hat{1})}\\ &= x_T^n (x_g + x_{\hat{1}})^{n(g)} x_{\hat{1}}^{n(\hat{1})}\\ &= x_T^n (x_g^{n(g)} + x_{\hat{1}}^{n(g)}) x_{\hat{1}}^{n(\hat{1})},\end{aligned}$$

where in the last equality we used $x_T x_g x_{\hat{1}}^{n(\hat{1})+1} = 0$. Therefore,

$$\begin{aligned}x_S^b \epsilon(x_S^b) &= (-1)^{|S\setminus\{\hat{1}\}|} x_T^n x_g^{n(g)} x_{\hat{1}}^{n(\hat{1})}\\ &= (-1)^{|S\setminus\{\hat{1}\}|-1} x_T^n x_{\hat{1}}^{n(g)} x_{\hat{1}}^{n(\hat{1})}\\ &= (-1)^{|T\setminus\{\hat{1}\}|} x_T^n x_{\hat{1}}^{n(g)+n(\hat{1})}\\ &= x_T^b \epsilon(x_T^b) = x_{\hat{1}}^{\mathrm{cd}(\hat{1})-1},\end{aligned}$$

by the inductive hypotheses on T. □

We set d_S be the function defined by $d_S(\hat{1}) = \mathrm{cd}(\hat{1}) - \mathrm{cd}(\bigvee S_{<\hat{1}}) - 1$ and by $d_S(g) = \mathrm{cd}(g) - \mathrm{cd}(\bigvee S_{<g})$ for $g \neq \hat{1}$.

Lemma 3.29. *Let S be a nested set, $g \in S$ and x_S^b be a monomial such that for all $h > g$ we have $b(h) \geq d_S(h)$ and $b(g) > d_S(h)$. Then $x_S^b = 0$.* □

The proof of the lemma is the same of [BDF20, Lemma 5.4.1 (b)]. Recall the chosen monomial order with the property that if $h > g$ then $h \succ g$ and $x_h \prec x_g$. We need the following statement.

Lemma 3.30. *Let x_S^b and x_T^c be two standard monomials in $\mathrm{DP}^k(P, \mathcal{G})$ such that $x_S^b \prec_{revlex} x_T^c$. Then $x_S^b \epsilon(x_T^c) = 0$.*

Proof. Consider T' and c' such that $x_{T'}^{c'} = \epsilon(x_T^c)$ and notice that $T' \setminus \{\hat{1}\} = T \setminus \{\hat{1}\}$. Define $g = \max_{\prec}\{h \mid b(h) \neq c(h)\}$ and, by hypothesis, $b(g) > c(g)$. If $S \cup T'$ is not $\mathcal{G}$-nested then we have $x_S^b \epsilon(x_T^c) = 0$. Otherwise set $A = (S \cup T')_{\geq g}$, by (4) of Proposition 3.3 we have that A is a chain $(a_1 < a_2 < \cdots < a_l)$ with $a_1 = g$. For $a_i \neq g, \hat{1}$ we have

$$b(a_i) + c'(a_i) = b(a_i) + \mathrm{cd}(a_i) - \mathrm{cd}(\bigvee T_{<a_i}) - c(a_i) \tag{3.5}$$

$$= \mathrm{cd}(a_i) - \mathrm{cd}(\bigvee T'_{<a_i}) \tag{3.6}$$

$$\geq \mathrm{cd}(a_i) - \mathrm{cd}(\bigvee (S \cup T')_{<a_i}) = d_{S\cup T'}(a_i). \tag{3.7}$$

The same holds for $\hat{1}$ (the proof has a minus one in the mid steps). For a_1 we have $b(g)+c'(g) > d_{S\cup T'}(g)$ because $b(g) > c(g)$. Therefore the monomial $x_S^b \epsilon(x_T^c) = x_{S\cup T'}^{b+c'}$ satisfies the hypothesis of Lemma 3.29 and we obtain the claimed result $x_S^b \epsilon(x_T^c) = 0$. □

Finally we can prove the Poincaré duality property:

Theorem 3.31 (Poincaré duality)**.** *If $\hat{1} \in \mathcal{G}$ then the algebra $\mathrm{DP}(P,\mathcal{G})$ is a Poincaré duality algebra of dimension $\mathrm{cd}(\hat{1})-1$.*

More generally, $\mathrm{DP}(P,\mathcal{G})$ is a Poincaré duality algebra of dimension $\mathrm{cd}(\hat{1}) - |F(P,\mathcal{G},\hat{1})|$.

Proof. The function ϵ has the property $\epsilon^2 = \mathrm{Id}$, and gives a bijection between standard monomials in degree k and in degree $r-k$. This, together with Corollary 3.6, ensures that $\dim \mathrm{DP}^k(P,\mathcal{G}) = \dim \mathrm{DP}^{r-k}(P,\mathcal{G})$. We consider on standard monomials the reverse lexicographical order. Lemma 3.30 ensures that the matrix of the Poincaré pairing (in the chosen basis) is upper triangular. From Lemma 3.28 we obtain that the entries on the diagonal are ± 1 and so the Poincaré pairing is non degenerate. The last statement follows from the first one together with Remark 9. □

We remark that the bases of standard monomials $\{x_S^b\}$ and $\{(-1)^r \epsilon(x_S^b)\}$ are not dual bases.

3.3.2 Tensor decomposition

This technical section is devoted to computing the annihilator $\mathrm{Ann}(\sigma_g)$ and $\mathrm{Ann}(x_g)$ for $g \in \mathcal{G}$. We describe it using the Chow ring of different polymatroids: $\mathrm{tr}_g P$, P^g and P_g. In the case of matroids this operation are known as truncation, restriction, and contraction.

The following proposition is needed for the proof of the main result of this section.

Proposition 3.32. *Let A and B be Poincaré duality algebra of the same dimension n, then:*

- *for each* $x \in A^k$, $x \neq 0$, *the ring* $A/\operatorname{Ann}(x)$ *is a Poincaré duality algebra of dimension* $n-k$,
- *each surjective homomorphism* $f\colon A \to B$ *is an isomorphism.*

The proof of the above proposition can be found, for example, in [AHK18, Proposition 7.2, Proposition 7.13].

Let $P = (E, \mathrm{cd})$ be a polymatroid with building set $\mathcal{G}$. Consider $g \in \mathcal{G}$ such that $\mathrm{cd}(g) > 1$. Let $\mathrm{tr}_g \mathrm{cd}\colon 2^E \to \mathbb{N}$ be the function defined by:

$$\mathrm{tr}_g \mathrm{cd}(h) = \begin{cases} \mathrm{cd}(h) - 1 & \text{if } \mathrm{cd}(h) = \mathrm{cd}(h \cup g), \\ \mathrm{cd}(h) & \text{otherwise.} \end{cases}$$

We denote by $\mathrm{tr}_g L$ the poset of flats of $\mathrm{tr}_g \mathrm{cd}$. Finally, define

$$\mathrm{tr}_g \mathcal{G} = \{\overline{h} \in \mathrm{tr}_g L \mid h \in \mathcal{G}\},$$

where $\overline{h}$ is the closure with respect to $\mathrm{tr}_g \mathrm{cd}$ of the flat h. Notice that $\mathrm{tr}_g L$ is a subposet of L but with a different codimension function.

Lemma 3.33. *For all* $g \in \mathcal{G}$ *with* $\mathrm{cd}(g) > 1$, *the pair* $\mathrm{tr}_g P = (E, \mathrm{tr}_g \mathrm{cd})$ *is a polymatroid and* $\mathrm{tr}_g \mathcal{G}$ *is a building set for the poset of flats* $\mathrm{tr}_g L$.

Proof. It is easy to see that $(E, \mathrm{tr}_g \mathrm{cd})$ is a polymatroid. Let $x \in \mathrm{tr}_g L$ and notice that for all $h \in \mathcal{G}$ we have $h \leq x$ in L if and only if $\overline{h} \leq x$ in $\mathrm{tr}_g L$. Thus, we have that $\max \mathrm{tr}_g \mathcal{G}_{\leq x} = \max \mathcal{G}_{\leq x}$ and it follows that

$$[\hat{0}, x] \simeq \prod_{y \in \max(\mathcal{G}_{\leq x})} [\hat{0}, y] \simeq \prod_{y \in \max(\mathrm{tr}_g \mathcal{G}_{\leq x})} [\hat{0}, y].$$

For the second part of the definition of a building set we have two cases. Let $\{y_1, \ldots, y_n\} = \max \mathcal{G}_{\leq x}$ and assume $g \not\leq x$, which implies $g \not\leq y_i$ for every i:

$$\mathrm{tr}_g \mathrm{cd}(x) = \mathrm{cd}(x) = \sum_{y \in \max \mathcal{G}_{\leq x}} \mathrm{cd}(y) = \sum_{y \in \max \mathrm{tr}_g \mathcal{G}_{\leq x}} \mathrm{tr}_g \mathrm{cd}(y).$$

Finally, let $g \leq x$ then by Proposition 3.3 there exists only one h_i such that $g \leq h_i$. Thus, we have the following:

$$\mathrm{tr}_g \mathrm{cd}(x) = \mathrm{cd}(x) - 1 = \Big(\sum_{y \in \max \mathcal{G}_{\leq x}} \mathrm{cd}(y) \Big) - 1 = \sum_{y \in \max \mathrm{tr}_g \mathcal{G}_{\leq x}} \mathrm{tr}_g \mathrm{cd}(y).$$

This concludes the proof. $\square$

Define the map

$$\zeta_g \colon \operatorname{DP}(\operatorname{tr}_g P, \operatorname{tr}_g \mathcal{G}) \to \operatorname{DP}(P, \mathcal{G}) / \operatorname{Ann}(\sigma_g)$$

by $\zeta_g(\sigma_k) = \sigma_h$ where h is any element in $\mathcal{G}$ such that $\overline{h} = k$.

Remark 10. In the realizable case, this construction can be viewed geometrically: consider a generic hyperplane H containing the flat g. The intersection of the subspace arrangement with H describes a subspace arrangement in H whose poset of intersection is $\operatorname{tr}_g L$. Moreover, the natural closed inclusion between the two wonderful compactification induces a surjective map $\operatorname{DP}(P, \mathcal{G}) \to \operatorname{DP}(\operatorname{tr}_g P, \operatorname{tr}_g \mathcal{G})$ with kernel $\operatorname{Ann}(\sigma_g)$. The map ζ_g is its pseudo-inverse.

Lemma 3.34. *For $g \in \mathcal{G}$ with $\operatorname{cd}(g) > 1$, the map ζ_g is well defined and an isomorphism. Moreover $\deg(\alpha) = \deg(-\sigma_g \zeta_g(\alpha))$ for all $\alpha \in \operatorname{DP}(\operatorname{tr}_g P, \operatorname{tr}_g \mathcal{G})$.*

Proof. We show that the map ζ_g does not depend on the choice of h: suppose that exist $h, f \in \mathcal{G}$ such that $\overline{h} = \overline{f}$. By symmetry we may assume $h \not\geq f$. Since $g \vee h = \overline{h} = \overline{f} = g \vee f$, we have $\overline{h} \in \mathcal{G}$, so replacing f with $g \vee f$ we assume $f > h$. Notice that $\operatorname{cd}(f) = \operatorname{cd}(h) + 1$ and $f = g \vee h$ so

$$\sigma_g(\sigma_h - \sigma_f) = \sigma_f(\sigma_h - \sigma_f) = 0.$$

We verify that the relations (i) and (ii) of Theorem 3.7 are send to zero. Consider an antichain $A \subset \operatorname{tr}_g \mathcal{G}$ and $k \in \operatorname{tr}_g \mathcal{G}$ such that $k \geq \bigvee A$, set $n = \operatorname{tr}_g \operatorname{cd}(k) - \operatorname{tr}_g \operatorname{cd}(\bigvee A)$. Let $h \in \mathcal{G}$ such that $\overline{h} = k$ and $B \subset \mathcal{G}$ such that $\overline{b_i} = a_i$ for all i. We have

$$\sigma_g \zeta_g\Big(\sigma_k^n \prod_{a \in A} (\sigma_a - \sigma_k)\Big) = \sigma_g \sigma_h^n \prod_{b \in B} (\sigma_b - \sigma_h).$$

Notice that $\operatorname{cd}(h) - \operatorname{cd}(\bigvee B) = n$ unless $h \geq g$ and $\bigvee B \not\geq g$ which is $\operatorname{cd}(h) - \operatorname{cd}(\bigvee B) = n + 1$. The non trivial case is the latter. Notice also that

$h = g \vee \bigvee B$. We use the relations to obtain:

$$\begin{aligned} \sigma_g \zeta_g \Big(\sigma_k^n \prod_{a \in A} (\sigma_a - \sigma_k) \Big) &= \sigma_g \sigma_h^n \prod_{b \in B} (\sigma_b - \sigma_h) \\ &= \sigma_h^{n+1} \prod_{b \in B} (\sigma_b - \sigma_h) \\ &= 0. \end{aligned}$$

We have proven that ζ_g is well defined.

The map is surjective because for each $h \in \mathcal{G}$ we have $\zeta_g(\sigma_{\overline{h}}) = \sigma_h$. Finally applying Proposition 3.32 we obtain the sought isomorphism.

For the last statement it is enough to notice that $\sigma_g \zeta_g (x_{\hat{1}}^{r-1}) = x_{\hat{1}}^r$. □

Let $P = (E, \mathrm{cd})$ be a polymatroid, $\mathcal{G}$ be a building set and $g \in \mathcal{G}$ any element. The restriction of the polymatroid to the flat g is $P^g = (E^g, \mathrm{cd}^g)$ where $E^g = \{h \in E \mid h \leq g\}$. The contraction of $P = (E, \mathrm{cd})$ to the flat g is $P_g = (E_g, \mathrm{cd}_g)$ where $E_g = E \setminus E^g$ and $\mathrm{cd}_g(h) = \mathrm{cd}(h \vee g) - \mathrm{cd}(g)$.

Define $L^g = [\hat{0}, g]$, $\mathcal{G}^g = \mathcal{G} \cap L^g$, $L_g = [g, \hat{1}]$, and

$$\mathcal{G}_g = \{h \vee g \mid h \in \mathcal{G} \setminus [\hat{0}, g]\}.$$

The proof of the following lemma is analogous to the one of Lemma 3.33, so we omit it.

Lemma 3.35. *The restriction and the contraction at $g \in \mathcal{G}$ are polymatroids with poset of flats L^g (respectively L_g) and building set $\mathcal{G}^g$ (resp. $\mathcal{G}_g$).*

Remark 11. In the case of matroids M, we have for every $e \in E$ that $M_e = \mathrm{tr}_e M$ is the contraction of the matroid.

Define the map

$$\psi_g \colon \mathrm{DP}(P^g, \mathcal{G}^g) \otimes \mathrm{DP}(P_g, \mathcal{G}_g) \to \mathrm{DP}(P, \mathcal{G}) \big/ \mathrm{Ann}(x_g)$$

by $\psi_g(\sigma_h \otimes 1) = \sigma_h$ and $\psi_g(1 \otimes \sigma_{g \vee h}) = \sigma_h$.

Lemma 3.36. *For all $g \in \mathcal{G} \setminus \{\hat{1}\}$ the map ψ_g is well defined and an isomorphism. Moreover* $\deg(\alpha)\deg(\beta) = \deg(x_g \psi_g(\alpha \otimes \beta))$ *for all* $\alpha \in \mathrm{DP}(P^g, \mathcal{G}^g)$ *and* $\beta \in \mathrm{DP}(P_g, \mathcal{G}_g)$.

Proof. We verify that $\psi_g(1 \otimes \sigma_{g\vee h})$ does not depend on the choice of the element h. Suppose that there exist $h, f \in \mathcal{G}$ such that $g \vee h = g \vee f$ and $h, f \not\leq g$. By symmetry we may assume $h \ngeq f$. Replacing f with $g \vee f$ we assume $f > h$, then

$$x_g(\sigma_h - \sigma_f) = x_g \sum_{\substack{l \geq h \\ l \ngeq g}} x_l = 0,$$

because $\{g, l\}$ cannot be $\mathcal{G}$-nested since $g < f \leq g \vee l$ and $l \ngeq g$.

We verify that all relations in the domain are mapped to zero. The ones in $\mathrm{DP}(P^g, \mathcal{G}^g)$ hold also in $\mathrm{DP}(P, \mathcal{G})$ trivially. Consider $h \in \mathcal{G}$ and $S \subset \mathcal{G}$ an antichain such that $\bigvee S \leq h$ and $s \not\leq g$ for all $s \in S$. Set $n = \mathrm{cd}(g \vee h) - \mathrm{cd}(g \vee \bigvee S)$. There are two cases:

- if $g \vee h \notin \mathcal{G}$ then $n = \mathrm{cd}(h) - \mathrm{cd}(\bigvee S)$ and

$$x_g \psi_g\Big(1 \otimes \sigma_{g\vee h}^n \prod_{a \in S}(\sigma_{g\vee s} - \sigma_{g\vee h})\Big) = x_g \sigma_h^n \prod_{a \in S}(\sigma_s - \sigma_h) = 0,$$

- if $g \vee h \in \mathcal{G}$ then

$$\begin{aligned} x_g \psi_g\Big(1 \otimes \sigma_{g\vee h}^n \prod_{s \in S}(\sigma_{g\vee s} - \sigma_{g\vee h})\Big) &= x_g \sigma_{g\vee h}^n \prod_{s \in S}(\sigma_s - \sigma_{g\vee h}) \\ &= \sum_A x_g x_A \sigma_{g\vee h}^n, \end{aligned}$$

  where the sum is taken over all sets $A = \{a_1, \dots a_k\}$ such that $a_i \geq s_i$ and $a_i \ngeq g \vee h$. Applying Lemma 3.8 to $g \vee h$, $S \cup \{g\}$ and $A \cup \{g\}$ we obtain that each term $x_g x_A \sigma_{g\vee h}^n$ is zero.

The map ψ_g is surjective because either $h \in \mathcal{G}^g$ or $g \vee h \in \mathcal{G}_g$ for all $h \in \mathcal{G}$. We apply Proposition 3.32, $\mathrm{DP}(L, \mathcal{G})/\operatorname{Ann}(x_g)$ is a Poincaré duality algebra of dimension $\mathrm{cd}(\hat{1}) - 2$. The algebra $\mathrm{DP}(P^g, \mathcal{G}^g) \otimes \mathrm{DP}(P_g, \mathcal{G}_g)$ is Poincaré duality of dimension $(\mathrm{cd}(g) - 1) + (\mathrm{cd}(\hat{1}) - \mathrm{cd}(g) - 1)$ (here is the only point were we use $g \neq \hat{1}$). Since ψ_g is surjective between Poincaré duality algebras of the same dimension, it is an isomorphism.

For the last statement we have

$$\begin{aligned} x_g\psi_g(x_g^{\mathrm{cd}(g)-1}\otimes x_{\hat{1}}^{\mathrm{cd}(\hat{1})-\mathrm{cd}(g)-1}) &= x_g\sigma_g^{\mathrm{cd}(g)-1}x_{\hat{1}}^{\mathrm{cd}(\hat{1})-\mathrm{cd}(g)-1} \\ &= (x_g-\sigma_g)\sigma_g^{\mathrm{cd}(g)-1}x_{\hat{1}}^{\mathrm{cd}(\hat{1})-\mathrm{cd}(g)-1} \\ &= -x_{\hat{1}}\sigma_g^{\mathrm{cd}(g)-1}x_{\hat{1}}^{\mathrm{cd}(\hat{1})-\mathrm{cd}(g)-1} \\ &= -x_{\hat{1}}^{\mathrm{cd}(\hat{1})-1}, \end{aligned}$$

so $\deg(x_g^{\mathrm{cd}(g)-1})\deg(x_{\hat{1}}^{\mathrm{cd}(\hat{1})-\mathrm{cd}(g)-1}) = (-1)^{\mathrm{cd}(\hat{1})} = \deg(-x_{\hat{1}}^{\mathrm{cd}(\hat{1})-1})$. □

3.3.3 Hard Lefschetz and Hodge-Riemann

We define a simplicial cone $\Sigma \subset \mathrm{DP}^1(P,\mathcal{G})$ and we will show that each element $\ell \in \Sigma$ satisfies Hard Lefschetz and Hodge-Riemann relations.

Definition 3.37. The *σ-cone* $\Sigma_{P,\mathcal{G}} \subset \mathrm{DP}^1(P,\mathcal{G})$ is the convex cone

$$\Sigma_{P,\mathcal{G}} = \Big\{-\sum_{g\in\mathcal{G}} d_g\sigma_g \mid d_g > 0\Big\}.$$

Let $a \in E$ be an atom in L, i.e. the interval $(\hat{0}, a)$ is empty. Consider the set

$$\{g \in \mathcal{G}\setminus\{a\} \mid g \neq \overline{S} \text{ for all } S \subseteq E\setminus\{a\}\}, \tag{3.8}$$

of all elements $g \in \mathcal{G}$ that cannot be written as the closure of some subset $S \subset E$ not containing a. Define $E(a)$ as the disjoint union of $E \setminus \{a\}$ and the minimal elements of the set in (3.8). Define the pair $P(a) = (E(a), \mathrm{cd})$, where with a slight abuse of notation

$$\mathrm{cd}(\{e_1,\dots,e_l,g_1,\dots,g_k\}) = \mathrm{cd}(\{e_1,\dots,e_l\}\cup g_1\cup\dots\cup g_k\}).$$

We also define $\mathcal{G}(a) = \mathcal{G}\setminus\{a\}$. The polymatroid $P(a)$ depends on $\mathcal{G}$ but we omit this dependency in our notation.

In the realizable case, this polymatroidal operation corresponds to removing only the subspace S_a from the building set $\mathcal{G}$ and from the arrangement $\mathcal{A}$. Now, there are subspaces in the lattice of flats $\mathcal{L}_\mathcal{A}$ that are not flats of

$\mathcal{A} \setminus S_a$. Among them we want to keep trace only of the ones blown up, i.e. belonging to $\mathcal{G}$; so we add to the deleted arrangement $\mathcal{A} \setminus S_a$ all the flats corresponding to elements in the set (3.8).

Lemma 3.38. *The pair $P(a) = (E(a), \mathrm{cd})$ is a polymatroid and $\mathcal{G}(a)$ is a building set for the poset of flats of $P(a)$.*

Proof. It is easy to see that $(E(a), \mathrm{cd})$ is a polymatroid and that the lattice of flats $L_{P(a)}$ of $P(a)$ is a subposet of the lattice of flats L of P. We verify that $\mathcal{G}(a)$ is a building set. We check the definition for all $x \in L_{P(a)}$: if a is not a $\mathcal{G}$-factor of x then $\max(\mathcal{G}_{\leq x}) = \max(\mathcal{G}(a)_{\leq x})$ and it follows from the properties of $\mathcal{G}$. Otherwise, a is a $\mathcal{G}$-factor of x and x cannot lie in the lattice $L_{P(a)}$ generated by $\mathcal{G} \setminus \{a\}$. $\square$

Lemma 3.39. *For an atom $a \in E$, $a \neq \hat{1}$, consider the element $\mu_0 = (x_a - \sigma_a)^{\mathrm{cd}(a)}$. There exists an isomorphism:*

$$p_a \colon \mathrm{DP}(P_a, \mathcal{G}_a) \to {\mathrm{DP}(P(a), \mathcal{G}(a))}\big/{\mathrm{Ann}(\mu_0)}.$$

Moreover $\deg(\alpha) = \deg(\mu_0 p_a(\alpha))$ *for all* $\alpha \in \mathrm{DP}(P_a, \mathcal{G}_a)$.

Proof. Notice that $\mu_0 = (x_a - \sigma_a)^{\mathrm{cd}(a)}$ is a multiple of x_a because $\sigma_a^{\mathrm{cd}(a)} = 0$, hence $\mathrm{Ann}(x_a) \subseteq \mathrm{Ann}(\mu_0)$. Define the morphism p_a as the composition

$$\mathrm{DP}(P_a, \mathcal{G}_a) \hookrightarrow \mathrm{DP}(P^a, \mathcal{G}^a) \otimes \mathrm{DP}(P_a, \mathcal{G}_a) \xrightarrow{\psi_a} \mathrm{DP}(P, \mathcal{G})/\operatorname{Ann}(x_a) \twoheadrightarrow \mathrm{DP}(P, \mathcal{G})/\operatorname{Ann}(\mu_0),$$

where the first map is the inclusion $x \mapsto 1 \otimes x$. Explicitly $p_a(\sigma_{a \vee h}) = [\sigma_h]$ for all $h \neq a$. Since $\mathcal{G}(a)$ is a subset of $\mathcal{G}$, $\mathrm{DP}(P(a), \mathcal{G}(a))$ is a subalgebra of $\mathrm{DP}(P, \mathcal{G})$. The range of the map p_a is equal to $\mathrm{DP}(P(a), \mathcal{G}(a))/\operatorname{Ann}(\mu_0)$, so the morphism in the statement is well defined and surjective. Since $a \neq \hat{1}$ we have $\mu_0 \neq 0$ and by Proposition 3.32 the map p_a is an isomorphism, because both algebras satisfy Poincaré duality of dimension $\mathrm{cd}(\hat{1}) - \mathrm{cd}(a) - 1$.

For the last statement we have $\mu_0 p_a(x_{\hat{1}}^{\mathrm{cd}(\hat{1}) - \mathrm{cd}(a) - 1}) = (-1)^{\mathrm{cd}(a)} x_{\hat{1}}^{\mathrm{cd}(\hat{1}) - 1}$ and so $\deg(x_{\hat{1}}^{\mathrm{cd}(\hat{1}) - \mathrm{cd}(a) - 1}) = (-1)^{\mathrm{cd}(\hat{1}) - \mathrm{cd}(a) - 1} = \deg((-1)^{\mathrm{cd}(a)} x_{\hat{1}}^{\mathrm{cd}(\hat{1}) - 1})$. $\square$

Lemma 3.40. *Let $a \in E$, $a \neq \hat{1}$, be an atom and $\mu_0 = (x_a - \sigma_a)^{\mathrm{cd}(a)}$. Consider the polynomial $p(x) = \sum_{i=0}^{\mathrm{cd}(a)} \binom{\mathrm{cd}(a)}{i} x^i (x_a - \sigma_a)^{\mathrm{cd}(a)-i}$, then*

$$\mathrm{DP}(P(a), \mathcal{G}(a))[x] \big/ (x \operatorname{Ann}(\mu_0), p(x)) \cong \mathrm{DP}(P, \mathcal{G}).$$

Proof. Define the morphism

$$\mathrm{DP}(P(a), \mathcal{G}(a))[x] \to \mathrm{DP}(P, \mathcal{G})$$

by $\sigma_g \mapsto \sigma_g$ and $x \mapsto -x_a$. By Lemmas 3.39 and 3.36 the elements of the form $x \operatorname{Ann}(\mu_0)$ are in the kernel. Also $p(x)$ is in the kernel because its image is $(-\sigma_a)^{\mathrm{cd}(a)} = 0$. Clearly, the map is surjective.

Notice that if A is a Poincaré duality algebra and $p(x) \in A[x]$ a monic polynomial with constant term μ_0 then $A[x]/(x \operatorname{Ann}(\mu_0), p(x))$ is a Poincaré duality algebra. Indeed, if a generic element $\sum_{i=0}^{j} a_i x^i$ (with $a_j \notin \operatorname{Ann}(\mu_0)$ and $j < \deg(p)$) of degree k is orthogonal to all elements of degree $n-k$, then $(\sum_{i=0}^{j} a_i x^i) a' = 0$ for all $a' \in A^{n-k}$. This implies $a_0 a' = 0$ and $a_0 = 0$. Moreover, $(\sum_{i=1}^{j} a_i x^i) a' x^{\deg(p)-j} = 0$ implies $a_j a' \mu_0 = 0$ and $a_j \mu_0 = 0$ by Poincaré duality in A, contradicting the fact $a_j \notin \operatorname{Ann}(\mu_0)$. In particular, $\mathrm{DP}(P(a), \mathcal{G}(a))[x]/(x \operatorname{Ann}(\mu_0), p(x))$ is a Poincaré duality algebra of dimension $\mathrm{cd}(\hat{1}) - 1$.

The map $\mathrm{DP}(P(a), \mathcal{G}(a))[x] \to \mathrm{DP}(P, \mathcal{G})$ is injective by Proposition 3.32 because domain and codomain are Poincaré duality algebras of the same dimension equal to $\mathrm{cd}(\hat{1}) - 1$. □

The following theorem provides an abstract procedure to prove the Hodge-Riemann relations inductively.

Theorem 3.41. *Let C be a Poincaré duality algebra and $p(x) = x^d + \mu_{d-1} x^{d-1} + \cdots + \mu_0 = 0 \in C[x]$ be a homogeneous polynomial with $\mu_0 \neq 0$. Let $B = C/\operatorname{Ann}(\mu_0)$ and $A = C[x]/(x \operatorname{Ann}(\mu_0), p(x))$. Let $\ell \in C^1$ be an element satisfying $\mathrm{HR}_C(\ell)$ and $\mathrm{HR}_B(\ell)$. Then $\mathrm{HR}_A(\ell + \epsilon x)$ holds for sufficiently small positive ϵ.*

In the above theorem the degree function on B is induced by μ_0, i.e. $\deg_B(\alpha) = \deg_C(\alpha\mu_0)$. Since the top degrees coincides $A^{top} = C^{top}$, we also implicitly assume that $\deg_A = \deg_C$.

The proof of Theorem 3.41 is the same of the proof of [AHK18, Proposition 8.2], so we omit it.

The following easy lemma shows that the maps introduced in Subsection 3.3.2 preserve the Σ-cone.

Lemma 3.42. *The following holds:*

1. *For all $g \in \mathcal{G}$, $g \neq \hat{1}$ the natural map*
$$\mathrm{DP}^1(P, \mathcal{G}) \to \mathrm{DP}^1(P^g, \mathcal{G}^g) \oplus \mathrm{DP}^1(P_g, \mathcal{G}_g)$$
induced by the quotient by $\mathrm{Ann}(x_g)$ composed with ψ_g^{-1}, maps $\Sigma_{P,\mathcal{G}}$ into $\Sigma_{P^g,\mathcal{G}^g} \times \Sigma_{P_g,\mathcal{G}_g}$.

2. *For all $g \in \mathcal{G}$ the morphism*
$$\mathrm{DP}^1(P, \mathcal{G}) \to \mathrm{DP}^1(\mathrm{tr}_g P, \mathrm{tr}_g \mathcal{G})$$
induced by the quotient by $\mathrm{Ann}(\sigma_g)$ composed with ζ_g^{-1}, maps $\Sigma_{P,\mathcal{G}}$ into $\Sigma_{\mathrm{tr}_g P, \mathrm{tr}_g \mathcal{G}}$.

3. *For an atom $a \in E, a \neq \hat{1}$ the natural map*
$$\mathrm{DP}^1(P(a), \mathcal{G}(a)) \to \mathrm{DP}^1(P_a, \mathcal{G}_a)$$
induced by the quotient by $\mathrm{Ann}(\mu_0)$ composed with p_a^{-1}, maps $\Sigma_{P(a),\mathcal{G}(a)}$ into $\Sigma_{P_a,\mathcal{G}_a}$.

Proof.

1. Let $l = -\sum_{h\in\mathcal{G}} d_h\sigma_h$ be an element of the σ-cone, we have that
$$\psi_g^{-1}([l]) = -\sum_{h\leq g} d_h\sigma_h \otimes 1 - \sum_{h\not\leq g} d_h \otimes \sigma_{g\vee h}.$$
It may occur that there are two different $h, h' \in \mathcal{G}$ such that $g\vee h = g\vee h'$ but, also in this case, the coefficient of $1\otimes\sigma_{g\vee h}$ is still negative. It follows that $\psi_g^{-1}([l]) \in \Sigma_{P^g,\mathcal{G}^g} \times \Sigma_{P_g,\mathcal{G}_g}$.

2. Let $l = -\sum_{h\in\mathcal{G}} d_h\sigma_h$ be an element of the σ-cone, we have that
$$\zeta_g^{-1}([l]) = -\sum_{h\in\mathcal{G}} d_h\sigma_{\bar{h}}.$$
It may occur that there are two different $h, h' \in \mathcal{G}$ such that $\bar{h} = \bar{h}'$ but, also in this case, the coefficient of $\sigma_{\bar{h}}$ is still negative. Thus, $\zeta_g^{-1}([l]) \in \Sigma_{\mathrm{tr}_g P, \mathrm{tr}_g \mathcal{G}}$.

3. Let $l = -\sum_{h\in\mathcal{G}} d_h\sigma_h$ be an element of the σ-cone, we have that
$$p_a^{-1}([l]) = -\sum_{h\in\mathcal{G}} d_h\sigma_{a\vee h}.$$
It may occur that there are two different $h, h' \in \mathcal{G}$ such that $a\vee h = a\vee h'$ but, also in this case, the coefficient of $\sigma_{a\vee h}$ is still negative. It follows that $p_a^{-1}([l]) \in \Sigma_{P_a, \mathcal{G}_a}$.

$\square$

Now we are ready to prove the main theorem.

Theorem 3.43. *For every element ℓ in the σ-cone $\Sigma_{P,\mathcal{G}}$ the conditions $\mathrm{HL}_{\mathrm{DP}(P,\mathcal{G})}(\ell)$ and $\mathrm{HR}_{\mathrm{DP}(P,\mathcal{G})}(\ell)$ hold.*

Proof. We prove the statement by induction on $|\mathcal{G}|$ and $\mathrm{cd}(\hat{1})$. The base case is $|\mathcal{G}| = 1$, so $\mathrm{DP}(P,\mathcal{G}) = \mathbb{Q}[x_{\hat{1}}]/(x_{\hat{1}}^{\mathrm{cd}(\hat{1})})$. In this case, it is known that $-\lambda x_{\hat{1}}$ satisfies Hard Lefschetz and Hodge-Riemann for all positive λ.

For the inductive step consider a polymatroid P, a building set $\mathcal{G}$, and an element $\ell \in \Sigma_{P,\mathcal{G}}$. Under the morphisms of Lemma 3.42 Item 2 ℓ is mapped in $\Sigma_{\mathrm{tr}_g P, \mathrm{tr}_g \mathcal{G}}$ for all $g \in \mathcal{G}$. Therefore by the inductive hypothesis the image of ℓ in $\mathrm{DP}(P,\mathcal{G})/\operatorname{Ann}(\sigma_g)$ satisfies Hodge-Riemann relations for all $g \in \mathcal{G}$. Notice also that ℓ is a sum of $-\sigma_g$ with positive coefficients. By [AHK18, Proposition 7.15], $\mathrm{HL}_{\mathrm{DP}(P,\mathcal{G})}(\ell)$ holds.

We want to prove that the Hodge-Riemann relations hold for all $\ell \in \Sigma_{P,\mathcal{G}}$. By [AHK18, Proposition 7.16] it is enough to prove $\mathrm{HR}_{\mathrm{DP}(P,\mathcal{G})}(\ell)$ for some $\ell \in \Sigma_{P,\mathcal{G}}$. We want to apply Theorem 3.41: consider any atom $a \in E$, since

$|\mathcal{G}| > 1$ then $a \neq \hat{1}$. Set $C = \mathrm{DP}(P(a), \mathcal{G}(a))$ and $p(x) = \sum_{i=0}^{\mathrm{cd}(a)} \binom{\mathrm{cd}(a)}{i} x^i (x_a - \sigma_a)^{\mathrm{cd}(a)-i}$; Lemma 3.39 ensures that $B = \mathrm{DP}(P_a, \mathcal{G}_a)$ and Lemma 3.40 that $A = \mathrm{DP}(P, \mathcal{G})$. Let $\ell \in \Sigma_{P(a),\mathcal{G}(a)}$, then under the morphism $C \to B$ (Lemma 3.42 Item 3) the class ℓ is mapped in $\Sigma_{P_a,\mathcal{G}_a}$. By the inductive hypothesis we have $\mathrm{HR}_{\mathrm{DP}(P(a),\mathcal{G}(a))}(\ell)$ and $\mathrm{HR}_{\mathrm{DP}(P_a,\mathcal{G}_a)}(\ell)$, hence by Theorem 3.41 $\mathrm{HR}_{\mathrm{DP}(P,\mathcal{G})}(\ell - \epsilon x_a)$ holds for sufficiently small $\epsilon > 0$.

Moreover if ϵ is small enough then $\ell - \epsilon x_a$ belongs to $\Sigma_{P,\mathcal{G}}$. Indeed using the Möbius inversion formula we have

$$x_a = \sum_{g \geq a} \mu_{\mathcal{G}}(a, g)\sigma_g$$

(where we consider $\mathcal{G}$ as a sub-poset of L). Let $\ell = -\sum_{g\in\mathcal{G}} d_g \sigma_g$, taking ϵ smaller than

$$\min_{g \geq a} \left\{ \left| \frac{d_g}{\mu_{\mathcal{G}}(a, g)} \right| \right\},$$

then $\ell - \epsilon x_a \in \Sigma_{P,\mathcal{G}}$. This concludes the proof. □

Remark 12. The ample cone depends on the geometric realization, however our σ-cone is contained in the ample cone of every realization. Indeed, consider 3 distinct lines in $\mathbb{C}^3$ and let P be the polymatroid realized by this subspace arrangement. The projective wonderful model is the blowup of $\mathbb{P}^2$ in 3 distinct points; there are two cases. If the three points are collinear the ample cone coincides with the σ-cone. Otherwise the three points are in general position and the ample cone is

$$\{-d_{\hat{1}}x_{\hat{1}} - d_a x_a - d_b x_b - d_c x_c \mid d_{\hat{1}} > d_a + d_b, d_{\hat{1}} > d_a + d_c, d_{\hat{1}} > d_b + d_c\}$$

which strictly contains the σ-cone.

Remark 13. If we restrict to the case of matroids with arbitrary building sets, the generator $x_{\hat{1}}$ can be eliminated using the relation $x_{\hat{1}} = -\sum_{g\geq e,\, g\neq\hat{1}} x_g$ for any $e \in E$. Thus the Hard Lefschetz theorem (and so the Hodge-Riemann relations) can be proven for the entire ample cone using as generators $\{x_g\}_{g\neq\hat{1}}$ instead of $\{\sigma_g\}_{g\in\mathcal{G}}$ and Lemma 3.36 instead of Lemma 3.34.

Remark 14. In [CHL+22] Crowley, Huh, Larson, Simpson and Wang introduce the *Bergman fan* $\Sigma_{P,\mathcal{G}}$ of a polymatroid P, a combinatorial model for the wonderful compactification of a subspace arrangement, and they prove that the Chow ring of the Bergman fan $A(\Sigma_{P,\mathcal{G}})$ is isomorphic to the Chow ring of the polymatroid $\mathrm{DP}(P,\mathcal{G})$ (See [CHL+22, Theorem 4.2]). Let P be a polymatroid and let $\Sigma_{P,\mathcal{G}}$ be the relative Bergman fan, in this work the authors show that there is a matroid M such that its Bergman fan Σ_M has the same support as $\Sigma_{P,\mathcal{G}}$. The Kähler package for $A(\Sigma_{P,\mathcal{G}})$, with respect to the cone of strictly convex piecewise linear function on $\Sigma_{P,\mathcal{G}}$, follows applying the results of [AHK18] to the matroid M and the general fact that the validity of the Kähler package for the Chow ring of a fan depends only on the support of the fan [ADH20]. In this way, they not only recover our result relative to the Chow ring of a polymatroid but they also managed to extend our σ-cone; in fact our σ-cone is always contained in the cone of strictly convex piecewise linear function on $\Sigma_{P,\mathcal{G}}$ (See [CHL+22, Remark 4.8]).

3.4 The relative Lefschetz decomposition

In this section we provide a decomposition of $\mathrm{DP}(P,\mathcal{G})$ as $\mathrm{DP}(P\setminus a,\mathcal{G}\setminus a)$-module. This is analogous to the semi-small decomposition of [BHM+20a], but in this more general setting the corresponding map is not always semi-small.

Indeed, consider an arrangement of hyperplanes $\mathcal{A}$ and the deleted arrangement $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ for some hyperplane $H \in \mathcal{A}$. There is a projection map between the wonderful models $Y_{\mathcal{A}} \to Y_{\mathcal{A}'}$ (constructed using the maximal building sets). This map is semi-small and induces the semi-small decomposition of the Chow ring.

In the case of subspace arrangements, the projection between the wonderful models exists but is not semi-small, because the dimension of the fiber of the blow up is too big. Therefore, the proof of the Kähler package done in [BHM+20a] for matroids cannot be adapted to polymatroids.

Recall that for a polymatroid $P = (E, \mathrm{cd})$ an *atom* $a \in E$ is an element such that the interval $(\hat{0}, \overline{a}) \subset L$ is empty (where $\overline{a}$ is the closure of a).

Definition 3.44. For an atom a define the polymatroid $P \setminus a$ on the ground set $E \setminus \{a\}$ with the restricted codimension function cd. The building set $\mathcal{G} \setminus a$ is the intersection of $\mathcal{G}$ with the poset of flats of $P \setminus a$.

Define a map

$$\theta_a \colon \mathrm{DP}(P \setminus a, \mathcal{G} \setminus a) \to \mathrm{DP}(P, \mathcal{G})$$

by $\theta_a(\sigma_h) = \sigma_{\overline{h}}$ where $\overline{h}$ is the closure of h in P. Define the subalgebra $\mathrm{DP}_{(a)} = \mathrm{Im}(\theta_a)$.

Lemma 3.45. *The map θ_a is injective.*

Proof. Consider a standard monomial $\sigma_S^b \in \mathrm{DP}(P \setminus a, \mathcal{G} \setminus a)$ and let $\overline{S} = \{\overline{h} \mid h \in S\}$. We have $\theta_a(\sigma_S^b) = \sigma_{\overline{S}}^b$ and it is enough to prove that $\sigma_{\overline{S}}^b$ is a standard monomial. Notice that $\overline{h} \vee \overline{g} = \overline{h \vee g}$ and the map between the two poset of flats is an inclusion. Therefore $\overline{S}$ is $\mathcal{G}$-nested. Since $\mathrm{cd}(h) = \mathrm{cd}(\overline{h})$, then $\sigma_{\overline{S}}^b$ is a standard monomial. $\square$

Let $S_a = \{g \in \mathcal{G} \mid a \in g \text{ and } g \setminus \{a\} \in L\}$ be the set of all flats such that a is a coloop for that flat.

Remark 15. Notice that $\theta_a(x_g) = x_g + x_{g \cup \{a\}}$, where we use the convention that $x_h = 0$ if h is not a flat of P. Moreover $\mathrm{DP}_{(a)}$ is generated as an algebra by σ_g with $g \notin S_a$ and as vector space by the monomials σ_S^b with $S \cap S_a = \emptyset$.

For $f \in S_a$ define DP_f as the $\mathrm{DP}_{(a)}$-submodule of $\mathrm{DP}(P, \mathcal{G})$ generated by $x_f, x_f^2, \dots x_f^{n_f}$, where

$$n_f = \mathrm{cd}(f) - \mathrm{cd}(f \setminus \{a\}) - 1 + |F(P, \mathcal{G}, f \setminus \{a\})|.$$

For a graded module $M = \oplus_i M^i$ we define $M[k]$ to be the graded module such that $(M[k])^i = M^{i+k}$.

Theorem 3.46. *Let a be an atom, then:*

$$x_f^k \operatorname{DP}_{(a)}[-k] \cong \operatorname{DP}((P \setminus a)^{f\setminus a}, (\mathcal{G} \setminus a)^{f\setminus a}) \otimes \operatorname{DP}(P_f, \mathcal{G}_f), \tag{3.9}$$

$$\operatorname{DP}_f = \bigoplus_{k=1}^{n_f} x_f^k \operatorname{DP}_{(a)}, \tag{3.10}$$

$$\operatorname{DP}(P, \mathcal{G}) = \operatorname{DP}_{(a)} \oplus \bigoplus_{f \in S_a} \operatorname{DP}_f. \tag{3.11}$$

as $\operatorname{DP}_{(a)}$*-modules. Moreover, the last decomposition is orthogonal with respect to the Poincaré pairing, with the exception of the summands* $\operatorname{DP}_{(a)}$ *and* $\operatorname{DP}_{\hat{1}}$ *(if a is a coloop).*

Before the proof of the above theorem we need some lemmas.

Lemma 3.47. *For all $f \in S_a$ and $k \leq n_f$ we have*

$$x_f \sigma_f^{k-1} \operatorname{DP}_{(a)}[-k] \cong \operatorname{DP}((P \setminus a)^{f\setminus a}, (\mathcal{G} \setminus a)^{f\setminus a}) \otimes \operatorname{DP}(P_f, \mathcal{G}_f), \tag{3.12}$$

and these modules are in direct sum in $\operatorname{DP}(P, \mathcal{G})$.

Proof. Notice that, for $k \leq n_f$ we have

$$\operatorname{DP}((\operatorname{tr}_f^{k-1}(P^f)) \setminus a, (\operatorname{tr}_f^{k-1}(\mathcal{G}^f)) \setminus a) = \operatorname{DP}((P \setminus a)^{f\setminus a}, (\mathcal{G} \setminus a)^{f\setminus a}).$$

Using Lemma 3.36 and Lemma 3.34 we obtain the isomorphism

$$x_f \sigma_f^{k-1} \operatorname{DP}(P, \mathcal{G})[-k] \simeq \operatorname{DP}(\operatorname{tr}_f^{k-1}(P^f), \operatorname{tr}_f^{k-1}(\mathcal{G}^f)) \otimes \operatorname{DP}(P_f, \mathcal{G}_f).$$

It is easy to check that the above isomorphism restricts to the one in eq. (3.12). For the second claim suppose that there exists a linear combination

$$\sum_{k=l}^{n_f} \sigma_f^{k-1} p_k = 0,$$

for some $p_k \in \operatorname{DP}((P \setminus a)^{f\setminus a}, (\mathcal{G} \setminus a)^{f\setminus a})$ with $p_l \neq 0$. The above equality implies

$$\sum_{k=l}^{n_f} \sigma_f^{k-l} p_k = 0$$

in $\mathrm{DP}(\mathrm{tr}_f^{l-1} P^f, \mathrm{tr}_f^{l-1} \mathcal{G}^f)$. Therefore p_l belongs to the ideal generated by σ_f (where f is the top element in the poset of flats of $\mathrm{tr}_f^{l-1} P^f$). The ideal (σ_f) is linearly generated by all monomials σ_T^b with $f \in T$. This yields a contradiction since p_l lies in $\mathrm{DP}((P \setminus a)^{f \setminus a}, (\mathcal{G} \setminus a)^{f \setminus a})$, which does not contain the generator σ_f. □

Lemma 3.48. *For all elements $f, g \in S_a$ such that $f \not\geq g$ we have $x_f \sigma_g = x_f \sigma_{g \setminus \{a\}}$. Moreover, we have*

$$\mathrm{DP}_f = \bigoplus_{k=1}^{n_f} x_f \sigma_f^{k-1} \, \mathrm{DP}_{(a)} .$$

Proof. Consider $h \in \mathcal{G}$ such that $h \geq g \setminus \{a\}$ and $h \not\geq g$, we need to prove that $x_f x_h = 0$. Notice that $\{f, h\}$ is an antichain, $f \vee g \in \mathcal{G}$ and so $(f \vee g) \vee h \in \mathcal{G}$ because $g \setminus \{a\} \neq \hat{0}$. Therefore

$$f \vee h = (f \vee a) \vee ((g \setminus \{a\}) \vee h) = f \vee g \vee h \in \mathcal{G},$$

and $\{f, h\}$ is not $\mathcal{G}$-nested that implies $x_f x_h = 0$.

For the second statement it is sufficient to prove that $x_f \sigma_f^{k-1} = x_f^k + z$ with some $z \in \sum_{j=1}^{k-1} x_f^j \, \mathrm{DP}_{(a)}$. We write $\sigma_f = x_f + \sum_{g > f} b_g \sigma_g$ for some coefficients $b_g \in \mathbb{Z}$. Therefore,

$$x_f \sigma_f = x_f^2 + x_f \sum_{\substack{g > f \\ g \notin S_a}} b_g \sigma_g + x_f \sum_{\substack{g > f \\ g \in S_a}} b_g \sigma_{g \setminus \{a\}},$$

and all the summand (except x_f^2) belongs to $x_f \, \mathrm{DP}_{(a)}$. An inductive argument on the exponent k concludes the proof. □

Lemma 3.49. *The submodules $\mathrm{DP}_{(a)}$ and DP_f for all $f \in S_a$ generate $\mathrm{DP}(P, \mathcal{G})$.*

Proof. We prove that each monomial σ_S^b belongs to the submodule $M := \mathrm{DP}_{(a)} + \sum_{F \in S_a} \mathrm{DP}_F$ by complete induction on $f = \min(S \cap S_a)$ and on $b(f)$.

The base case is $S \cap S_a = \emptyset$ and so $\sigma_S^b \in \mathrm{DP}_{(a)}$. For the inductive step notice that $S \cap S_a$ is $\mathcal{G}$-nested, so it is a chain. Call $f = \min(S \cap S_a)$ and suppose that all monomials $\sigma_S^{b'}$ with $b'(f) < b(f)$ and all monomials $\sigma_{S'}^{b'}$ with $\min(S' \cap S_a) > f$ lie in M.

Let $\{g_1, \dots, g_l\} = F(P, \mathcal{G}, f \setminus \{a\})$ be the set of $\mathcal{G}$-factors of $f \setminus \{a\}$. We have the relation

$$\sigma_f^{\mathrm{cd}(f) - \mathrm{cd}(f \setminus \{a\})} \prod_{i=1}^{k} (\sigma_{g_i} - \sigma_f) = 0,$$

so in the case $b(f) > n_f$ we can rewrite $\sigma_f^{b(f)} \sigma_{T \setminus \{f\}}$ as sum of monomials with $b'(f) < b(f)$ using the above relation and the fact that $g_i \notin S_a$.

In the case $b(f) \leq n_f$ we have

$$\sigma_S^b = x_f \sigma_f^{b(f)-1} \sigma_{S\setminus\{f\}}^b + (\sigma_f - x_f) \sigma_f^{b(f)-1} \sigma_{S\setminus\{f\}}^b.$$

Using the first assertion of Lemma 3.48, it follows that the element $x_f \sigma_f^{b(f)-1} \sigma_{S\setminus\{f\}}^b$ belongs to $x_f \sigma_f^{b(f)-1} \mathrm{DP}_{(a)} \subset M$. The second summand $(\sigma_f - x_f) \sigma_f^{b(f)-1} \sigma_{S\setminus\{f\}}^b$ is a linear combination of monomials $\sigma_h \sigma_f^{b(f)-1} \sigma_{S\setminus\{f\}}^b$ with $h > f$ and so belongs to M by the inductive hypothesis. □

Proof of Theorem 3.46. As in Remark 9, we may assume that $\hat{1} \in \mathcal{G}$. By Lemma 3.48 and Lemma 3.47, DP_f is a free $\mathrm{DP}((P \setminus a)^{f \setminus a}, (\mathcal{G} \setminus a)^{f \setminus a}) \otimes \mathrm{DP}(P_f, \mathcal{G}_f)$-module with basis $x_f \sigma_f^{k-1}$ for $k = 1, \dots, n_f$. The elements $\{x_f^k\}_k$ written in the basis $\{x_f \sigma_f^{k-1}\}_k$ for an upper triangular matrix with ones on the diagonal (the inverse of the one given in Lemma 3.48). Eq. (3.9) and eq. (3.10) follow.

In order to prove eq. (3.11) we first prove the orthogonality. Let $f \neq \hat{1}$; the elements DP_f and $\mathrm{DP}_{(a)}$ are orthogonal because the product is contained in DP_f which is zero in degree $\mathrm{cd}(\hat{1}) - 1$. Indeed from eq. (3.9) and eq. (3.10), it follows that the top degree of DP_f is $\mathrm{cd}(\hat{1}) - 2$.

Consider generic elements $x_f^b y \in \mathrm{DP}_f$ and $x_g^c z \in \mathrm{DP}_g$ in complementary degrees (with $y, z \in \mathrm{DP}_{(a)}$). The product is zero if f and g are incomparable. Otherwise, by symmetry we may assume $g > f$, hence

$$x_f x_g^c = x_f (x_g + x_{g \setminus \{a\}})^c.$$

Since $x_g + x_{g\setminus\{a\}} \in \mathrm{DP}_{(a)}$, we obtain that the product lie in DP_f. Again the top degree is zero since $f \neq \hat{1}$.

We prove that if a is a coloop then $\mathrm{DP}_{(a)} \cap \mathrm{DP}_{\hat{1}} = 0$. In that case $\mathrm{DP}_{\hat{1}}$ is the ideal generated by $\sigma_{\hat{1}}$. This ideal is linearly generated by all standard monomials σ_S^b with $\hat{1} \in S$. Since $\hat{1} \in S_a$ then $\mathrm{DP}_{(a)} \cap \mathrm{DP}_{\hat{1}} = 0$. The direct sum of eq. (3.11) follows from the orthogonality of all other summands together with Lemma 3.49 and Theorem 3.31. □

3.5 Characteristic polynomial

In this section we study the coefficients of the (reduced) characteristic polynomial.

We consider only maximal building sets, so we omit the building set from the notations. Moreover we suppose that the polymatroid is without loops, i.e. $\mathrm{cd}(\{e\}) > 0$ for all $e \in E$.

Let $\alpha = \alpha_P = -x_{\hat{1}}$ and $\beta = \beta_P = \sum_{g \in \mathcal{G}_{\max}} x_g$ be two elements in $\mathrm{DP}^1(P)$. We denote by $\mu_L(a,b)$ the Möbius function of L.

Lemma 3.50. *For any polymatroid P with $\mathrm{cd}(E) > 0$ and $r = \mathrm{cd}(E) - 1$ we have*

$$\deg(\beta_P^r) = (-1)^r + \sum_{g \in L \setminus \{\hat{0}, \hat{1}\}} (-1)^{\mathrm{cd}(g)-1} \deg(\beta_{P_g}^{r-\mathrm{cd}(g)}).$$

Proof. A flag with repetition is $\mathcal{F} = (F_1^{a_1} \subsetneq F_2^{a_2} \subsetneq \cdots \subsetneq F_l^{a_l})$ where $a_i > 0$ are the multiplicity of the flats $F_i \in L$. We also require that $\sum_{i=1}^l a_i = r$. Define $x_{\mathcal{F}} = \prod_{i=1}^{|\mathcal{F}|} x_{F_i}^{a_i}$, we will prove that $x_{\mathcal{F}} = 0$ if $\mathrm{cd}(F_1) > a_1$. More generally we have $x_{\mathcal{F}} = 0$ if $\mathrm{cd}(F_i) > \sum_{j=1}^i a_j$ for some i, but we prove and use the implication only for $i = 1$. From the isomorphism ψ_g of Lemma 3.36 we obtain

$$x_{\mathcal{F}} = x_{F_1} \psi_{F_1}((x_{F_1} \otimes 1 - 1 \otimes \beta_{P_g})^{a_1 - 1}(1 \otimes x_{\mathcal{F}'})),$$

where $\mathcal{F}' = (F_2^{a_2} \subsetneq \cdots \subsetneq F_l^{a_l})$. Notice that the degree of $x_{\mathcal{F}'}$ is $r - a_1$, which is greater than $r - \mathrm{cd}(F_1)$, the top degree of $\mathrm{DP}(P_g)$.

Let $\binom{r}{a}$ be the multinomial coefficient where $a = (a_1, \dots, a_l)$ and $\sum_{i=1}^{l} a_i = r$. Since $x_f x_g = 0$ if f and g are incomparable, we have

$$\begin{aligned}
\beta_P^r &= \sum_{\mathcal{F} \text{ flag of } P} \binom{r}{a} x_{\mathcal{F}} \\
&= \sum_{F \in L \setminus \{\hat{0}\}} \sum_{\substack{\mathcal{F} \text{ flag of } P \\ F_1 = F}} \binom{r}{a} x_{\mathcal{F}} \\
&= \sum_{F \in L \setminus \{\hat{0}\}} \sum_{k=\mathrm{cd}(F)}^{r} \binom{r}{k} x_F^k \sum_{\mathcal{F}' \text{ flag of } P_F} \binom{r-k}{a'} x_{\mathcal{F}'} \\
&= \sum_{F \in L \setminus \{\hat{0}\}} \sum_{k=\mathrm{cd}(F)}^{r} \binom{r}{k} x_F^k \beta_{P_F}^{r-k}.
\end{aligned}$$

The summand relative to $F = \hat{1}$ is exactly $x_{\hat{1}}^r$ and contributes $(-1)^r$ to $\deg(\beta_P^r)$. It is enough to prove that for every $g \in L \setminus \{\hat{0}, \hat{1}\}$

$$\deg\Big(\sum_{k=\mathrm{cd}(g)}^{r} \binom{r}{k} x_g^k \beta_{P_g}^{r-k} \Big) = \deg(\beta_{P_g}^{r-\mathrm{cd}(g)}).$$

We use Lemma 3.36 to obtain:

$$\begin{aligned}
&\deg\Big(\sum_{k=\mathrm{cd}(g)}^{r} \binom{r}{k} x_g^k \beta_{P_g}^{r-k} \Big) = \\
&= \sum_{k=\mathrm{cd}(g)}^{r} \binom{r}{k} \deg((x_g \otimes 1 - 1 \otimes \beta_{P_g})^{k-1} (1 \otimes \beta_{P_g}^{r-k})) \\
&= \sum_{k=\mathrm{cd}(g)}^{r} (-1)^{k-\mathrm{cd}(g)} \binom{r}{k} \binom{k-1}{\mathrm{cd}(g)-1} \deg(x_g^{\mathrm{cd}(g)-1} \otimes \beta_{P_g}^{r-\mathrm{cd}(g)}) \\
&= \sum_{k=\mathrm{cd}(g)}^{r} (-1)^{k-1} \binom{r}{k} \binom{k-1}{\mathrm{cd}(g)-1} \deg(\beta_{P_g}^{r-\mathrm{cd}(g)}) \\
&= (-1)^{\mathrm{cd}(g)-1} \deg(\beta_{P_g}^{r-\mathrm{cd}(g)}),
\end{aligned}$$

where in the last equality we used the identity

$$\sum_{k=\mathrm{cd}(g)}^{r} (-1)^k \binom{r}{k} \binom{k-1}{\mathrm{cd}(g)-1} = (-1)^{\mathrm{cd}(g)}$$

which follows from [GKP94, eq. 5.24] with $l = r$, $m = 0$, $n = \mathrm{cd}(g) - 1$, and $s = -1$. $\square$

Lemma 3.51. *For every polymatroid P with poset of flats L and $r = \mathrm{cd}(E) - 1$ with $\mathrm{cd}(E) > 0$ we have*

$$\deg(\beta_P^r) = (-1)^{\mathrm{cd}(E)} \mu_L(\hat{0}, \hat{1}).$$

Proof. It is known that $\mu_L(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(\hat{0}, \hat{1}))$, i.e. the Möbius function coincides with the reduced Euler characteristic of the order complex of the poset $L \setminus \{\hat{0}, \hat{1}\}$ (e.g. see [Rot64b]). Let L^{op} be the opposite (dual) lattice of L which is defined on the same set of L but with reversed order, i.e., $x \leq y$ in L^{op} if and only if $y \leq x$ in L. Since the order complexes of L and L^{op} are the same simplicial complex, we have

$$\mu_L(\hat{0}, \hat{1}) = \mu_{L^{\mathrm{op}}}(\hat{0}, \hat{1}).$$

Define $\deg(\beta_P^0) = 1$ for rank zero polymatroids P. Therefore, the functions $(-1)^{\mathrm{cd}(E)} \deg(\beta_P^r)$ and $\mu_{L^{\mathrm{op}}}(\hat{0}, \hat{1})$ satisfy the same recurrence relation. One is given by the definition of $\mu_{L^{\mathrm{op}}}(\hat{0}, \hat{1})$ and the other from Lemma 3.50. This concludes the proof. $\square$

Notice that if L is a geometric lattice (i.e. the poset of flats of a matroid), then the Möbius function has alternating sign, hence in this case $\deg(\beta^r) \in \mathbb{N}_0$.

Definition 3.52. The *characteristic polynomial* of a polymatroid P is

$$\chi_P(\lambda) = \sum_{g \in L} \mu_L(\hat{0}, g) \lambda^{\dim(g)},$$

where $\dim(g) = \mathrm{cd}(\hat{1}) - \mathrm{cd}(g)$. Since $\chi_P(1) = 0$ by the definition of Möbius function, we define the *reduced characteristic polynomial* as

$$\overline{\chi}_P(\lambda) = \frac{\chi_P(\lambda)}{\lambda - 1}$$

This definition of reduced characteristic polynomial coincides with the one stated in [Whi93].

Theorem 3.53. *For every polymatroid P, we define $r = \operatorname{cd}(E) - 1$. We have*

$$\overline{\chi}_P(\lambda) = \sum_{i=0}^{r} (-1)^i \deg_P(\alpha_P^i \beta_P^{r-i})\lambda^i.$$

Proof. We show that $\overline{\chi}_P(\lambda)$ and the right hand side satisfy the same recurrence:

$$q_P(\lambda) - \lambda q_{\operatorname{tr}_{\hat{1}} P}(\lambda) = -\mu_L(\hat{0}, \hat{1})$$

where L is the poset of flats of P.

Let $\operatorname{tr}_{\hat{1}} L$ be the poset of flats of $\operatorname{tr}_{\hat{1}} P$ and notice that $\mu_L(\hat{0}, g) = \mu_{\operatorname{tr}_{\hat{1}} L}(\hat{0}, g)$ for all g such that $\dim(g) > 1$. Therefore $\chi_P(\lambda) - \lambda\chi_{\operatorname{tr} P}(\lambda)$ is a polynomial of degree one divisible by $\lambda - 1$. Hence $\overline{\chi}_P(\lambda) - \lambda\overline{\chi}_{\operatorname{tr} P}(\lambda)$ is constant and equal to $\overline{\chi}_P(0) = -\mu_L(\hat{0}, \hat{1})$. This proves that $\overline{\chi}_P(\lambda)$ satisfies the recurrence.

Now observe that for $i > 0$ $\deg_P(\alpha_P^i \beta_P^{r-i}) = \deg_{\operatorname{tr} P}(\alpha_{\operatorname{tr} P}^{i-1} \beta_{\operatorname{tr} P}^{r-i})$ by Lemma 3.34. This proves that

$$\sum_{i=0}^{r} (-1)^i \deg_P(\alpha_P^i \beta_P^{r-i})\lambda^i - \lambda \sum_{i=0}^{r-1} (-1)^i \deg_{\operatorname{tr} P}(\alpha_{\operatorname{tr} P}^{i-1} \beta_{\operatorname{tr} P}^{r-i})\lambda^i =$$
$$= (-1)^r \deg_P(\beta^r),$$

and so Lemma 3.51 proves the recurrence.

The base case $\operatorname{cd}(E) = 1$ is trivial, so the proof follows. □

Corollary 3.54. *The coefficient of λ^i of the reduced characteristic polynomial $\overline{\chi}_P(\lambda)$ is (up to the sign) the reduced Euler characteristic of the order complex of the poset $(\operatorname{tr}_{\hat{1}}^i L) \setminus \{\hat{0}, \hat{1}\}$:*

$$[\lambda^i]\overline{\chi}_P(\lambda) = (-1)^{\operatorname{cd}(E)} \tilde{\chi}(\Delta((\operatorname{tr}_{\hat{1}}^i L) \setminus \{\hat{0}, \hat{1}\})).$$

Proof. It follows from Theorem 3.53 and Lemma 3.51. □

Remark 16. The coefficients of the characteristic polynomial χ_P and of the reduced characteristic polynomial $\overline{\chi}_P$ do not form a log-concave sequence. Indeed if P_1 is the polymatroid associated to 4 subspaces of codimension $2, 3, 4, 4$ in $\mathbb{C}^5$ in general position, then

$$\chi_{P_1}(\lambda) = \lambda^5 - \lambda^3 - \lambda^2 - 2\lambda + 3,$$

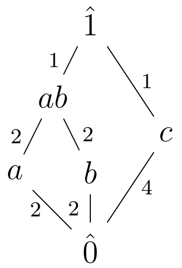


Figure 3.1: The Hasse diagram of the poset of flats L of Section 3.6

which is not log-concave. Let P_2 be the polymatroid on $E = \{a, b, c, d, e\}$ with rank defined by $\mathrm{cd}(a) = 2$, $\mathrm{cd}(b) = 3$, $\mathrm{cd}(c) = 4$, $\mathrm{cd}(d) = 4$, $\mathrm{cd}(e) = 1$, by $\mathrm{cd}(A) = 6$ if $|A| \geq 3$ and $\mathrm{cd}(\{x, y\}) = \min\{5, \mathrm{cd}(x) + \mathrm{cd}(y)\}$. The reduced characteristic polynomial is not log-concave because

$$\overline{\chi}_{P_2}(\lambda) = \lambda^5 - \lambda^3 - \lambda^2 - 2\lambda + 6.$$

3.6 An example

Let $E = \{a, b, c\}$ and $\mathrm{cd}\colon 2^E \to \mathbb{N}$ the function defined by

$$\begin{aligned} &\mathrm{cd}(a) = \mathrm{cd}(b) = 2, \qquad \mathrm{cd}(ab) = \mathrm{cd}(c) = 4, \\ &\mathrm{cd}(ac) = \mathrm{cd}(bc) = \mathrm{cd}(abc) = 5. \end{aligned}$$

This function defines a polymatroid P with poset of flats L shown in Figure 3.1. Near every cover relation, the relative codimension of the two flats is shown. This polymatroid is realizable: a realization is the collection in $\mathbb{C}^5$ of two subspace of dimension 3 and a line in general position.

Consider the (minimal) building set $\mathcal{G} = \{a, b, c, \hat{1}\}$; the nested set complex $n(P, \mathcal{G})$ is shown in Figure 3.2.

The algebra $B(P, \mathcal{G})$ is generated by $x_a, x_b, x_c, x_{\hat{1}}, e_a, e_b, e_c, e_{\hat{1}}$ with rela-

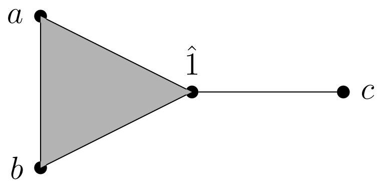


Figure 3.2: The nested set complex $n(P, \mathcal{G})$.

2	1	2	1		
1	3	7	7	3	
0	1	4	5	4	1
	0	**1**	**2**	**3**	**4**

Table 3.1: The dimensions of $B^{2p,q}(P, \mathcal{G})/(e_1)$ in position (p, q).

tions:

$$\begin{aligned}
&e_a e_c = e_b e_c = 0 && x_a x_c = x_b x_c = 0\\
&x_a e_c = x_b e_c = 0 && e_a x_c = e_b x_c = 0\\
&(x_a + x_{\hat{1}})^2 = (x_b + x_{\hat{1}})^2 = 0 && (x_c + x_{\hat{1}})^4 = 0\\
&x_{\hat{1}}^5 = 0 && x_c x_{\hat{1}} = e_c x_{\hat{1}} = 0\\
&x_a x_{\hat{1}}^3 = e_a x_{\hat{1}}^3 = 0 && x_b x_{\hat{1}}^3 = e_b x_{\hat{1}}^3 = 0\\
&x_a x_b x_{\hat{1}} = e_a x_b x_{\hat{1}} = 0 && e_a e_b x_{\hat{1}} = x_a e_b x_{\hat{1}} = 0
\end{aligned}$$

The homogeneous component $B^{4,1}(P, \mathcal{G})$ has dimension 12 and the additive basis provided by Corollary 3.6 is:

$$\begin{gathered}
e_{\hat{1}} x_a x_b,\ e_{\hat{1}} x_a x_{\hat{1}},\ e_{\hat{1}} x_b x_{\hat{1}},\ e_{\hat{1}} x_c^2,\ e_{\hat{1}} x_{\hat{1}}^2,\\
e_a x_a x_b,\ e_a x_a x_{\hat{1}},\ e_a x_{\hat{1}}^2,\ e_b x_a x_b,\ e_b x_a x_{\hat{1}},\ e_b x_{\hat{1}}^2,\ e_c x_c^2.
\end{gathered}$$

Notice that $B(P, \mathcal{G}) = B(P, \mathcal{G})/(e_{\hat{1}}) \otimes \langle 1, e_{\hat{1}} \rangle$ and their dimensions are reported in Tables 3.1 and 3.2.

The other presentation of $B(P, \mathcal{G})$ is given by generators $\sigma_a, \sigma_b, \sigma_c, \sigma_{\hat{1}}$,

3	1	2	1		
2	4	9	8	3	
1	4	11	12	7	1
0	1	4	5	4	1
	0	1	2	3	4

Table 3.2: The dimensions of $B^{2p,q}(P,\mathcal{G})$ in position (p,q).

$\tau_a, \tau_b, \tau_c, \tau_{\hat{1}}$ and relations:

$$
\begin{array}{lll}
(\tau_a-\tau_{\hat{1}})(\tau_c-\tau_{\hat{1}})=0 & (\tau_b-\tau_{\hat{1}})(\tau_c-\tau_{\hat{1}})=0 & \sigma_c^4=0 \\
(\sigma_a-\sigma_{\hat{1}})(\sigma_c-\sigma_{\hat{1}})=0 & (\sigma_b-\sigma_{\hat{1}})(\sigma_c-\sigma_{\hat{1}})=0 & \sigma_a^2=0 \\
(\sigma_a-\sigma_{\hat{1}})(\tau_c-\tau_{\hat{1}})=0 & (\sigma_b-\sigma_{\hat{1}})(\tau_c-\tau_{\hat{1}})=0 & \sigma_b^2=0 \\
(\tau_a-\tau_{\hat{1}})(\sigma_c-\sigma_{\hat{1}})=0 & (\tau_b-\tau_{\hat{1}})(\sigma_c-\sigma_{\hat{1}})=0 & \sigma_{\hat{1}}^5=0 \\
(\sigma_c-\sigma_{\hat{1}})\sigma_{\hat{1}}=0 & (\tau_c-\tau_{\hat{1}})\sigma_{\hat{1}}=0 & (\sigma_a-\sigma_{\hat{1}})\sigma_{\hat{1}}^3=0 \\
(\tau_a-\tau_{\hat{1}})\sigma_{\hat{1}}^3=0 & (\sigma_b-\sigma_{\hat{1}})\sigma_{\hat{1}}^3=0 & (\tau_b-\tau_{\hat{1}})\sigma_{\hat{1}}^3=0 \\
(\sigma_a-\sigma_{\hat{1}})(\sigma_b-\sigma_{\hat{1}})\sigma_{\hat{1}}=0 & (\tau_a-\tau_{\hat{1}})(\sigma_b-\sigma_{\hat{1}})\sigma_{\hat{1}}=0 & \\
(\tau_a-\tau_{\hat{1}})(\tau_b-\tau_{\hat{1}})\sigma_{\hat{1}}=0 & (\sigma_a-\sigma_{\hat{1}})(\tau_b-\tau_{\hat{1}})\sigma_{\hat{1}}=0 &
\end{array}
$$

The homogeneous component $B^{4,1}(P,\mathcal{G})$ has dimension 12 and the additive basis provided by Corollary 3.10 is:

$$
\begin{gathered}
\tau_{\hat{1}}\sigma_a\sigma_b,\ \tau_{\hat{1}}\sigma_a\sigma_{\hat{1}},\ \tau_{\hat{1}}\sigma_b\sigma_{\hat{1}},\ \tau_{\hat{1}}\sigma_c^2,\ \tau_{\hat{1}}\sigma_{\hat{1}}^2, \\
\tau_a\sigma_a\sigma_b,\ \tau_a\sigma_a\sigma_{\hat{1}},\ \tau_a\sigma_{\hat{1}}^2,\ \tau_b\sigma_a\sigma_b,\ \tau_b\sigma_a\sigma_{\hat{1}},\ \tau_b\sigma_{\hat{1}}^2,\ \tau_c\sigma_c^2.
\end{gathered}
$$

The set of critical monomials is:

$$
1, \tau_a\sigma_a,\ \tau_b\sigma_b,\ \tau_c\sigma_c^3,\ \tau_{\hat{1}}\sigma_{\hat{1}}^4,\ \tau_a\tau_b\sigma_a\sigma_b,\ \tau_a\tau_{\hat{1}}\sigma_a\sigma_{\hat{1}}^2,\ \tau_b\tau_{\hat{1}}\sigma_b\sigma_{\hat{1}}^2, \tau_c\tau_{\hat{1}}\sigma_c^3,\ \tau_a\tau_b\tau_{\hat{1}}\sigma_a\sigma_b,
$$

and the dimensions of $\mathrm{CM}^{2p,q}(P,\mathcal{G})$ are given in Table 3.3. The rank of the cohomology group of $(B(P,\mathcal{G}),\mathrm{d})$ are given in Table 3.4

As an example we have

$$
\begin{aligned}
\mathrm{d}(c\mu(ab\hat{1})) = \mathrm{d}(\tau_a\tau_b\tau_{\hat{1}}\sigma_a\sigma_b) &= \tau_a\tau_b\sigma_{\hat{1}}\sigma_a\sigma_b \\
&= \tau_b\tau_{\hat{1}}\sigma_b\sigma_{\hat{1}}^2 - \tau_a\tau_{\hat{1}}\sigma_a\sigma_{\hat{1}}^2 = c\mu(b\hat{1}) - c\mu(a\hat{1}),
\end{aligned}
$$

3	0	0	1		
2	0	0	1	3	
1	0	2	0	1	1
0	1	0	0	0	0
	0	1	2	3	4

Table 3.3: The dimensions of $\mathrm{CM}(P,\mathcal{G})$ in position (p,q).

3	0	0	0		
2	0	0	1	1	
1	0	2	0	1	0
0	1	0	0	0	0
	0	1	2	3	4

Table 3.4: The dimensions of $H^{2p,q}(B(P,\mathcal{G}),\mathrm{d})$ in position (p,q).

that coincides with $\mathrm{d}((a,b,\hat{1})) = (b,\hat{1}) - (a,\hat{1})$ in the differential algebra $\mathrm{CM}(P,\mathcal{G})$. Moreover, in $\mathrm{CM}(P,\mathcal{G})$ we have

$$(a)\cdot(b) = \lambda(a,b) - \lambda(b,a) = (a,b),$$

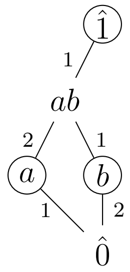
because $a \prec b$ and it corresponds to the equality

$$c\mu(a)c\mu(b) = \tau_a\sigma_a\tau_b\sigma_b = \tau_a\tau_b\sigma_a\sigma_b = c\mu(ab).$$

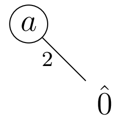
The posets related to P and a are shown in Figure 3.3. The polymatroids $P(a)$ and $P\setminus a$ are equal by coincidence; see below for the poset $P(a)$ relative to the maximal building set.

The σ-cone $\Sigma_{P,\mathcal{G}}$ is given by the linear combinations $-d_a\sigma_a - d_b\sigma_b - d_c\sigma_c - d_{\hat{1}}\sigma_{\hat{1}}$ with positive coefficients $d_g > 0$.

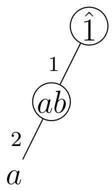
We have $\mathrm{Ann}(x_a) = (x_c, x_b\sigma_{\hat{1}}, \sigma_{\hat{1}}^3)$ and so in $\mathrm{DP}(P,\mathcal{G})/\mathrm{Ann}(x_a)$ we have $\sigma_c = \sigma_{\hat{1}}$, $(\sigma_b - \sigma_{\hat{1}})\sigma_{\hat{1}} = 0$, and $\sigma_{\hat{1}}^3 = 0$. The last two equations correspond to the defining relation for $\mathrm{DP}(P_a,\mathcal{G}_a)$. Similarly, $\mathrm{Ann}(\sigma_a) = (\sigma_c - \sigma_{\hat{1}}, \sigma_a, \sigma_{\hat{1}}^4, (\sigma_b - \sigma_{\hat{1}})\sigma_{\hat{1}}^2)$ and these are exactly the equations defining $\mathrm{DP}(\mathrm{tr}_a P, \mathrm{tr}_a \mathcal{G})$ that do not appear in $\mathrm{DP}(P,\mathcal{G})$.



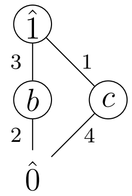
(a) The poset of flats of $\mathrm{tr}_a\, P$.



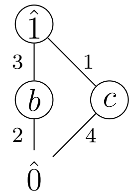
(b) The poset of flats of P^a.



(c) The poset of flats of P_a.



(d) The poset of flats of $P(a)$.



(e) The poset of flats of $P \setminus a$.

Figure 3.3: The Hasse diagram of some posets related to a. The circle nodes are in the corresponding building sets.

The relative Lefschetz decomposition with respect to the atom a is

$$\mathrm{DP}(P, \mathcal{G}) = \mathrm{DP}_{(a)} \oplus x_a \,\mathrm{DP}_{(a)},$$

where

$$\mathrm{DP}_{(a)} = \langle 1, \sigma_b, \sigma_c, \sigma_{\hat{1}}, \sigma_b\sigma_{\hat{1}}, \sigma_c^2, \sigma_{\hat{1}}^2, \sigma_b\sigma_{\hat{1}}^2, \sigma_c^3, \sigma_{\hat{1}}^3, \sigma_{\hat{1}}^4, \rangle$$

and

$$\mathrm{DP}_a = x_a \,\mathrm{DP}_{(a)} = \langle x_a, x_a\sigma_b, x_a\sigma_{\hat{1}}, x_a\sigma_{\hat{1}}^2\rangle \simeq \mathrm{DP}(P_a, \mathcal{G}_a).$$

The relative Lefschetz decomposition with respect to the atom c is

$$\mathrm{DP}(P, \mathcal{G}) = \mathrm{DP}_{(c)} \oplus \mathrm{DP}_{\hat{1}} \oplus \mathrm{DP}_c,$$

where $\mathrm{DP}_{(c)} = \langle 1, \sigma_a, \sigma_b, \sigma_a\sigma_b\rangle$ and the other $\mathrm{DP}_{(c)}$-modules are $\mathrm{DP}_c = \langle x_c, x_c^2, x_c^3\rangle$ and $\mathrm{DP}_{\hat{1}} = x_{\hat{1}} \,\mathrm{DP}_{(c)} \oplus x_{\hat{1}}^2 \,\mathrm{DP}_{(c)}$. Moreover we have $x_{\hat{1}} \,\mathrm{DP}_{(c)} \simeq x_{\hat{1}}^2 \,\mathrm{DP}_{(c)} \simeq \mathrm{DP}((P \setminus c)^{ab}, (\mathcal{G} \setminus c)^{ab})$.

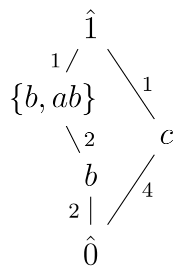


Figure 3.4: The Hasse diagram of the poset of flats of $P(a)$ with maximal building set on the groundset $\{b, c, ab\}$.

Maximal building set

Now consider the same polymatroid P with the maximal building set $\mathcal{G}_{\max} = \{a, b, c, ab, \hat{1}\}$. The polymatroid $P(a)$ relative to the maximal building set is shown in Figure 3.4 and the groundset $E(a)$ is $\{b, c, ab\}$. This polymatroid $P(a)$ associated with $\mathcal{G}_{\max}$ is different from the polymatroid $P(a)$ defined from the minimal building set $\mathcal{G}$ (shown in Figure 3.3d).

The characteristic polynomial is $\chi_P(\lambda) = \lambda^5 - 2\lambda^3 + 1$ and the reduced one is

$$\overline{\chi}_P(\lambda) = \lambda^4 + \lambda^3 - \lambda^2 - \lambda - 1.$$

We have $\alpha = -x_{\hat{1}}$, $\beta = x_a + x_b + x_c + x_{ab} + x_{\hat{1}}$ and $\deg(\alpha^4) = 1$, $\deg(\alpha^3\beta) = -1$, $\deg(\alpha^2\beta^2) = -1$, $\deg(\alpha\beta^3) = 1$, and $\deg(\beta^4) = -1$.

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Ringraziamenti

Vorrei ringraziare, in particolare, il mio relatore Roberto Pagaria per la passione e la competenza con la quale mi ha guidato in questi anni. Lo ringrazio inoltre per essere stato un punto di riferimento costante.

Vorrei ringraziare Luca Moci per avermi spronato ad effettuare questo percorso; gli sono grato per la comprensione e la fiducia che ha riposto nei miei confronti.

Ringrazio i miei colleghi Nicolò e Luis per il supporto che mi hanno dato durante la stesura di questa tesi.

Ringrazio infine la mia famiglia, i miei amici e Marta per essermi stati vicini in tutti questi anni.