Alma Mater Studiorum · Università di Bologna

DOTTORATO DI RICERCA IN MATEMATICA

CICLO XXXIII

Settore Concorsuale: 01/A2 - GEOMETRIA E ALGEBRA

Settore Scientifico Disciplinare MAT/03 - GEOMETRIA

THE HITCHIN MAP FOR ONE-NODAL BASE CURVES OF COMPACT TYPE

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Esame finale anno 2022

Abstract

Studying moduli spaces of semistable Higgs bundles (E, ϕ) of rank n on a smooth curve C, a key role is played by the *spectral curve* X (Hitchin [30]), because an important result by Beauville-Narasimhan-Ramanan [11] allows us to study isomorphism classes of such Higgs bundles in terms of isomorphism classes of rank-1 torsion-free sheaves on X. This way, the generic fibre of the *Hitchin map*, which associates to any semistable Higgs bundle the coefficients of the characteristic polynomial of ϕ, is isomorphic to the *Jacobian* of X. Focusing on rank-2 Higgs data, this construction was extended by Barik [7] to the case in which the curve C is reducible, one-nodal, having two smooth components. Such curve is called *of compact type* because its Picard group is compact.

In this work, we describe and clarify the main points of the construction by Barik and we give examples, especially concerning generic fibres of the Hitchin map.

Referring to Hausel-Pauly [29], we consider the case of $\mathrm{SL}(2, \mathbb{C})$-Higgs bundles on a smooth base curve, which are such that the generic fibre of the Hitchin map is a subvariety of the Jacobian of X, the *Prym variety.* We recall the description of special loci, called *endoscopic loci*, such that the associated Prym variety is not connected.

Then, letting G be an affine reductive group having underlying Lie algebra $\mathfrak{so}(4, \mathbb{C})$, we consider G-Higgs bundles on a smooth base curve. Starting from the construction by Bradlow-Schaposnik [16], we discuss the associated endoscopic loci.

By adapting these studies to a one-nodal base curve of compact type, we describe the fibre of the $\mathrm{SL}(2, \mathbb{C})$-Hitchin map and of the G-Hitchin map,

together with endoscopic loci.

In the Appendix, we give an interpretation of generic spectral curves in terms of families of double covers.

Introduction

Let C be a smooth complex projective curve whose genus is at least 2 and let K_C be its canonical bundle. Let L be a line bundle on C whose degree is greater or equal to the degree of K_C and let $\mathcal{M}(n,d)$ be the moduli space of semistable $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles (E,ϕ) on C, where E is a holomorphic vector bundle on C having rank n and degree d and

$$\phi : E \to E \otimes L$$

is a L-twisted endomorphism of E, which we call *Higgs field*. There is a proper morphism

$$h : \mathcal{M}(n,d) \to \mathbb{A} = \bigoplus_{i=1}^{n} H^0(C, L^i),$$

obtained by assigning to (E,ϕ) the coefficients of the characteristic polynomial of ϕ. We call this morphism *Hitchin map*, referring to the works by Hitchin [30] and Simpson [49]. We refer to elements a of $\mathbb{A}$ as *characteristics*. Letting Tot (L) be the total space of L and letting π : Tot $(L) \to C$ be the natural projection, the *spectral curve* X_a (also called *spectral cover*) is defined as the zero divisor in Tot (L) of a nonzero section in $H^0(\text{Tot }(L), \pi^* L^n)$. This is a ramified cover of C whose degree equals the rank n of E. It is called spectral curve because its fibre over each point p of C represents the eigenvalues of ϕ over p, which are not necessarily distinct. For a generic choice of the characteristic a, the curve X_a is smooth, however, the curve X_a can be singular even if the base curve is smooth.

The fibre $h^{-1}(a)$ of the Hitchin map is given by isomorphism classes of semistable Higgs bundles (E,ϕ) such that the characteristic polynomial of ϕ yields the characteristic $a \in \mathbb{A}$, as described above. The correspondence described by Beauville, Narasimhan and Ramanan [11], called the *BNR correspondence*, in the case in which the spectral curve is integral, says that there is a natural correspondence between the fibre of the Hitchin map, that is isomorphism classes of semistable Higgs bundles (E, ϕ) such that $h(E, \phi)$ equals the characteristic a, and isomorphism classes of rank-1 torsion-free sheaves on X_a having suitable degree. This means that the fibre $h^{-1}(a)$ is isomorphic to the compactified Jacobian of X_a (Proposition 1.3.22, Section 3 of [36]).

An aim of this work is to clarify and give examples of Barik's construction [7], which adapts the generic case of the BNR correspondence for $\mathrm{GL}(2, \mathbb{C})$-Higgs data to the case of a reducible one-nodal base curve C which is given by the union of two smooth components C_1 and C_2, both having genus at least 2. The Picard group $\mathrm{Pic}\ C$ of such curve is compact, hence C is called *one-nodal curve of compact type*. Barik considers rank-2 vector bundles E_1 on C_1 and E_2 on C_2, together with a homomorphism $\overrightarrow{A}^{(q)}$ between the fibres of E_1 and E_2 at the node q of C. The compactness of $\mathrm{Pic}\ C$ allows us to associate to any line bundle L on C line bundles L_1 on C_1 and L_2 on C_2, which yields moduli spaces of $\mathrm{GL}(2, \mathbb{C})$-Higgs bundles (E_i, ϕ_i) on C_i and spectral curves $X_{a_i} \to C_i$, $i = 1, 2$. In this case, the Higgs fields ϕ_i need to commute with the map $\overrightarrow{A}^{(q)}$, which yields *Hitchin triples* that we see in Section 3.1. Moreover, we call *adapted spectral curve* the union of the spectral curves X_{a_i}, which, for a generic choice of a_i, are smooth and intersect transversally over the node.

Another aim of this work is to consider G-Higgs bundles, for some affine reductive groups G which are different from $\mathrm{GL}(n, \mathbb{C})$. We first assume that C is a smooth base curve. Considering G-Higgs bundles implies asking extra-conditions on Higgs bundles, mirroring the nature of the group G. By the BNR correspondence, this also gives further restrictions on rank-1 torsion-free sheaves in the compactified Jacobian of X_a. For example, considering the subgroup $\mathrm{SL}(2, \mathbb{C})$ of $\mathrm{GL}(2, \mathbb{C})$ implies that the generic fibre of the Hitchin map is a subvariety of the Jacobian, precisely the *Prym variety* (Definition 1.5.4).

The choice of $\mathrm{SL}(2, \mathbb{C})$-Higgs data allows the existence of special loci, called *endoscopic loci*, which are such that the normalization of the associated spectral curve factors through an étale double cover of C and the fibre of the Hitchin map is no longer connected. We refer to the description of endoscopic loci for the $\mathrm{SL}(2,\mathbb{C})$-Hitchin map by Hausel-Pauly [29]. Following the work by Bradlow-Schaposnik [16], which describes the fibre of the Hitchin map for groups G which are isogenous to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ by considering the fibred product of two smooth $\mathrm{SL}(2,\mathbb{C})$-spectral covers of C, we describe generic endoscopic loci for G-Higgs bundles, where $G = \mathrm{Spin}(4,\mathbb{C}), \mathrm{SO}(4,\mathbb{C}), \mathrm{PSO}(4,\mathbb{C})$. We give an analogue description for a one-nodal base curve of compact type, finding differences in the description of endoscopic loci.

Our description of endoscopic loci for $\mathrm{Spin}(4,\mathbb{C})$-Higgs bundles and $\mathrm{SO}(4,\mathbb{C})$-Higgs bundles on a smooth base curve and the study of $\mathrm{SL}(2,\mathbb{C})$-Higgs data and of G-Higgs data on a one-nodal base curve of compact type that we give in Chapter 5 was not considered before and we think that it can be useful.

This work is organized as follows.

In Chapter 1, we recall moduli spaces of vector bundles and of Higgs bundles on a smooth base curve. We describe the associated Hitchin map, the spectral curve and the BNR correspondence. We briefly recall Simpson's compactification of the Jacobian of the spectral curve, which enables us to describe moduli spaces of rank-1 torsion-free sheaves on any projective curve. We also give some preliminaries on the norm map and on Prym varieties, which will be useful in Chapter 5.

In Chapter 2, we recall the correspondence between a line bundle L on a one-nodal curve C of compact type and the restrictions of L to its smooth components C_1, C_2. We then see how the preliminaries on moduli spaces of vector bundles adapt when C is a one-nodal curve of compact type and recall the description of moduli spaces of torsion-free sheaves on C following Nagaraj-Seshadri [40]. In particular, we recall the correspondence between torsion-free sheaves and triples $(E_1, E_2, \overrightarrow{A}^{(q)})$, where E_i is a vector bundle on C_i, for $i = 1, 2$, and $\overrightarrow{A}^{(q)}$ is a linear map between the fibres of E_1, E_2 at the node q of C. We then focus on vector bundles E_1 and E_2 having rank 2. A suitable stability condition on triples, taking into account the polarization on C (Definition 2.3.9) enables us to define the moduli space of semistable triples, then we focus on triples having odd Euler characteristic, which implies that semistable triples are stable. The moduli space of semistable triples is given by the union of two smooth components, which intersect transversally at a smooth divisor $\mathcal{N}^V$ (Theorem 2.4.2). We describe the relation between stability of triples and stability of vector bundles appearing in the triples and we discuss a class of unstable triples (Remark 2.4.4, Lemma 2.4.6). We recall that the divisor $\mathcal{N}^V$ is isomorphic to a product of moduli spaces of vector bundles with parabolic structure only at the node q (Theorem 2.5.5). We show that there are no couples of vector bundles with parabolic structure at q corresponding to the class of unstable triples above (Remark 2.5.6). The description above allows to compute the dimension of the moduli space of triples in two ways (see Proposition 2.4.8, Remark 2.4.9 and Remark 2.5.7).

In Chapter 3, we add the Higgs datum. Following Barik, we adapt the description in Chapter 2 to moduli spaces of Hitchin triples, in which E_1 and E_2 have rank 2. We focus on triples of the form $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ where $\hat{E}_i$ denotes the Higgs bundle (E_i, ϕ_i), which is such that ϕ_i satisfy a commutativity condition involving $\overrightarrow{A}^{(q)}$ (Definition 3.1.1). The semistability condition (Definition 3.2.7) implies that the rank of the map $\overrightarrow{A}^{(q)}$ is at least one (Theorem 3.2.10). Considering Hitchin triples whose Euler characteristic is odd, the moduli space of such semistable Hitchin triples is still given by the union of two smooth components, intersecting transversally at a smooth divisor $\mathcal{N}$ (Theorem 3.3.2). We describe the relation between stability of Hitchin triples and stability of Higgs bundles appearing in the triples and we discuss a class of unstable Hitchin triples (Remark 3.3.5, Lemma 3.3.7). As in the case of triples, every Hitchin triple appearing in $\mathcal{N}$ corresponds to a couple of Higgs bundles with parabolic structure at q, but the converse does not hold (Remark 3.4.6).

In Chapter 4, we recall how Barik [7] and Bhosle [13] adapt the notions of Hitchin map and of spectral covers to one-nodal base curves of compact type and we recall the description of the generic fibre of the adapted Hitchin map (Proposition 4.4.4), giving an analogue of the BNR correspondence for generic spectral covers of one-nodal base curves of compact type, which are not ramified over the node. Considering the generic fibres of the adapted Hitchin map, we prove the following.

Proposition 4.4.6 *Let C be a one-nodal base curve of compact type having arithmetic genus g and assume that $L_1 \cong K_{C_1}$ and $L_2 \cong K_{C_2}$. Then the dimension of the moduli space of semistable Hitchin triples is $8g-12$.*

We also give an analogue of the BNR correspondence for a non-generic case in which the spectral cover is ramified over the node q (Proposition 4.4.7).

In Chapter 5, we first consider $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles on a smooth base curve C. Following Schaposnik [45], Section 2.2.2, we recall the description of the fibre of the $\mathrm{SL}(2,\mathbb{C})$-Hitchin map, which, for $L \cong K_C$ is a connected Prym variety (we further discuss definitions of Prym varieties in Remark 1.5.5). On the other hand, the number of connected components of the Prym variety of a $\mathrm{SL}(2,\mathbb{C})$-spectral cover depends on the choice of L: we discuss this aspect by referring to the work by Hausel and Pauly [29]. If a generic $\mathrm{SL}(2,\mathbb{C})$-spectral cover is endoscopic, the associated Prym variety is not connected.

Then we let C be a one-nodal base curve of compact type, we define $\mathrm{SL}(2,\mathbb{C})$-Higgs data for this case and we apply the discussion of Chapters 3 and 4 to describe the BNR correspondence for $\mathrm{SL}(2,\mathbb{C})$-Hitchin triples and to compute the dimension of the moduli space of semistable $\mathrm{SL}(2,\mathbb{C})$-Hitchin triples (Proposition 5.2.5). Moreover, we can describe endoscopic loci for base curves of compact type as follows.

Proposition 5.2.7 *Let C be a one-nodal base curve of compact type such that the genus of the components C_1 and C_2 is at least 2. If at least a $SL(2,\mathbb{C})$-spectral cover of C_i is endoscopic, then the $SL(2,\mathbb{C})$-adapted spectral curve is endoscopic.*

Following the work by Bradlow-Schaposnik [16], we recall how the isomorphism $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(4,\mathbb{C})$ of Lie algebras lets us describe G-Higgs bundles, for $G = \mathrm{Spin}(4,\mathbb{C}), \mathrm{SO}(4,\mathbb{C}), \mathrm{PSO}(4,\mathbb{C})$ via moduli spaces of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. After studying the fibres of the G-Hitchin map, we describe generic endoscopic loci both for smooth base curves and for one-nodal base curves of compact type in Sections 5.4, 5.6, 5.8. We describe endoscopic loci for G-Higgs bundles on a smooth base curve in the following proposition.

Proposition *Let C be a smooth base curve of genus $g \geq 2$ and consider G-spectral covers of C such that the underlying Lie algebra of G is $\mathfrak{so}(4, \mathbb{C})$. Then:*

- *a $Spin(4, \mathbb{C})$-spectral cover is endoscopic if at least an associated $SL(2, \mathbb{C})$-spectral cover is endoscopic,*
- *a $SO(4, \mathbb{C})$-spectral cover is endoscopic if and only if both its associated $SL(2, \mathbb{C})$-spectral covers are endoscopic.*
- *a $PSO(4, \mathbb{C})$-spectral cover is not endoscopic.*

Endoscopic loci for a one-nodal base curve of compact type are described in Propositions 5.5.5, 5.7.2, 5.8.7.

In the Appendix, we give an interpretation of generic spectral curves and generic adapted spectral curves in terms of families of double covers and their admissible compactification.

Contents

Chapter 1

Preliminaries

1.1 Vector bundles on a smooth base curve

Let C be a complex nonsingular projective curve having genus $g \geq 2$.

Definition 1.1.1. Let E be a holomorphic vector bundle of rank n on C. We define the *degree* of E as the degree of its determinant bundle $\det E = \bigwedge^n E$, i.e. the degree of the divisor associated to the line bundle $\det E$.

Remark 1.1.2. Alternatively, defining, for any vector bundle E on C, its *Euler characteristic* as

$$\chi_C(E) = \dim H^0(C, E) - \dim H^1(C, E),$$

the degree of E is defined as

$$\deg E = \chi_C(E) - \mathrm{rank}\,(E)\chi_C(\mathcal{O}_C). \tag{1.1}$$

Definition 1.1.3. We define the *slope* of E as

$$\mu(E) = \frac{\deg\,(E)}{\mathrm{rank}\,(E)} = \frac{d}{n}. \tag{1.2}$$

Definition 1.1.4. We say that E is a *stable vector bundle* if, for any proper, nonzero subbundle $F \subset E$, we have

$$\mu(F) < \mu(E).$$

We say that E is a *semistable vector bundle* if, for any proper, nonzero subbundle $F \subset E$, we have

$$\mu(F) \leq \mu(E).$$

We say that E is a *polystable vector bundle* if we have

$$E = E_1 \oplus \cdots \oplus E_l,$$

where $\mu(E_i) = \mu(E)$ for $1 \leq i \leq l$.

Remark 1.1.5. In order to define the moduli space of semistable vector bundles, we define an appropriate equivalence relation. Any semistable vector bundle admits a (non-unique) *Jordan-Hölder filtration*, i.e. a flag

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = E$$

such that F_i/F_{i-1} is stable and $\mu(F_i/F_{i-1}) = \mu(E)$. The graded object

$$\mathrm{Gr}(E) = \bigoplus_{i=1}^{l} F_i/F_{i-1}$$

is unique up to isomorphism, e.g. by Huybrechts-Lehn [33], Proposition 1.5.2.

Definition 1.1.6. Two semistable vector bundles E and E' are said to be *S-equivalent* if $\mathrm{Gr}(E) \cong \mathrm{Gr}(E')$.

Thus we have the following.

Theorem 1.1.7 ([41], Theorem 5.8, Remark 5.9, [32], Theorem 8.64, Proposition 8.65)**.** *There exists a coarse moduli space $\mathcal{M}^{V,s}(n,d)$ parametrizing isomorphism classes of stable vector bundles of rank n and degree d. The moduli space $\mathcal{M}^{V,s}(n,d)$ has a natural compactification to a projective variety $\mathcal{M}^{V}(n,d)$ parametrizing S-equivalence classes of semistable vector bundles of rank n and degree d.*

When n and d are coprime, the moduli space $\mathcal{M}^{V}(n,d) = \mathcal{M}^{V,s}(n,d)$ is smooth and we have

$$\dim \mathcal{M}^{V}(n,d) = n^2(g-1) + 1. \tag{1.3}$$

We also recall the definition of Jacobian of a curve.

Definition 1.1.8. Let C be a smooth curve. We define the *Jacobian* $J(C) := \mathrm{Pic}^0 C$ of C as the subgroup of line bundles of degree zero of

$$\mathrm{Pic}(C) \cong H^1(C, \mathcal{O}_C^*).$$

Remark 1.1.9 ([15], Chapter 11.1). If M is a line bundle on C of degree d, then we can define a noncanonical isomorphism

$$\mathrm{Pic}^d(C) \xrightarrow{\sim} J(C) \qquad L \mapsto L \otimes M^{-1}. \tag{1.4}$$

Notation 1.1.10. Considering the isomorphism (1.4), from now on we use the term "Jacobian", instead of Picard, even if we consider line bundles having any degree. We write $J^d(C)$ when we need to emphasize that we are referring to line bundles on C having degree d.

Remark 1.1.11. All line bundles are stable, thus $\mathcal{M}^V(1, d)$ from Theorem 1.1.7 contains all line bundles of degree d and is isomorphic, via (1.4), to the Jacobian $J(C)$ of C, which is an abelian variety whose dimension equals the genus g of C.

More generally, we define the generalized Jacobian of a nodal curve in Definition 1.3.18.

1.2 Higgs bundles on a smooth base curve

Let K_C be the canonical bundle of C and let L be a line bundle on C whose degree is greater or equal to the degree of K_C. Following classical references, (Hitchin [30], see e.g. Schaposnik [45], Section 2.1.3), we simply call *Higgs bundles* the $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles.

Definition 1.2.1. A *Higgs bundle* is a pair (E, ϕ), where E is a holomorphic vector bundle on C and ϕ is a holomorphic section in $H^0(C, \mathrm{End}E \otimes L)$, which is seen as a L-twisted endomorphism $\phi : E \to E \otimes L$ that we call *Higgs field.*

Definition 1.2.2. We say that a vector subbundle $F \subset E$ is a *ϕ-invariant subbundle* of E if it is such that $\phi(F) \subset F \otimes L$.

Stability for Higgs bundles is defined in terms of ϕ-invariant subbundles.

Definition 1.2.3. Let (E, ϕ) be a Higgs bundle on C. We say that (E, ϕ) is a *stable Higgs bundle* if for any proper, nonzero, ϕ-invariant subbundle $F \subset E$, we have

$$\mu(F) < \mu(E).$$

We say that (E, ϕ) is a *semistable Higgs bundle* if for any proper, nonzero, ϕ-invariant subbundle $F \subset E$, we have

$$\mu(F) \leq \mu(E).$$

We say that (E, ϕ) is a *polystable Higgs bundle* if we have

$$(E, \phi) = (E_1, \phi_1) \oplus (E_2, \phi_2) \oplus \cdots \oplus (E_l, \phi_l),$$

where, for each $1 \leq i \leq l$, we have that (E_i, ϕ_i) is a stable Higgs bundle and $\mu(E_i) = \mu(E)$.

Remark 1.2.4. If a Higgs bundle (E, ϕ) has underlying stable vector bundle E, then it is also stable as a Higgs bundle. In fact, since E has no destabilizing subbundles, a fortiori it has no ϕ-invariant destabilizing subbundles.

On the other hand, an unstable vector bundle having no ϕ-invariant subbundles is stable as a Higgs bundle, as we see in Example 1.2.8.

Remark 1.2.5. In order to define the moduli space of Higgs bundles, we define an appropriate equivalence relation. Any semistable Higgs bundle admits a (non-unique) *Jordan-Hölder filtration*, i.e. a flag

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = E$$

such that $\mu(F_i/F_{i-1}) = \mu(E)$ and, letting $\phi_i : F_i/F_{i-1} \to F_i/F_{i-1} \otimes L$ be the induced Higgs fields, we have that $(F_i/F_{i-1}, \phi_i)$ are stable, for $1 \leq i \leq l$.

Also in the case of Higgs bundles, the graded object

$$\mathrm{Gr}(E, \phi) = \bigoplus_{i=1}^{l} (F_i/F_{i-1}, \phi_i)$$

is unique up to isomorphism, e.g. by [33], Proposition 1.5.2.

Definition 1.2.6. Two semistable Higgs bundles (E, ϕ) and (E', ϕ') are said to be *S-equivalent* if $\mathrm{Gr}(E, \phi) \cong \mathrm{Gr}(E', \phi')$.

Remark 1.2.7. If a pair (E, ϕ) is stable, then the associated Jordan-Hölder filtration is trivial, so the isomorphism class of $\mathrm{Gr}(E, \phi)$ is the isomorphism class of (E, ϕ).

We consider the following example to give a more explicit view on stability and Higgs fields.

Example 1.2.8 ([45], Example 2.7)**.** Let C be a smooth curve of genus at least 2, choose a square root of K_C and let

$$E = K_C^{1/2} \oplus K_C^{-1/2} \tag{1.5}$$

be a vector bundle of degree zero. Since $\mathrm{End}E \cong E^\vee \otimes E$, (1.5) yields

$$H^0(C, \mathrm{End}E \otimes K_C) \cong H^0(C, K_C)^2 \oplus H^0(C, K_C^2) \oplus H^0(C, \mathcal{O}_C),$$

thus we have

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, d \in H^0(C, K_C), b \in H^0(C, K_C^2), c \in H^0(C, \mathcal{O}_C)$.

We consider the family of Higgs fields given by:

$$\phi_\omega = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} : (K_C^{1/2} \oplus K_C^{-1/2}) \to (K_C^{1/2} \oplus K_C^{-1/2}) \otimes K_C \cong K_C^{3/2} \oplus K_C^{1/2}, \tag{1.6}$$

where $\omega \in H^0(C, K_C^2)$ is a quadratic differential.

We now discuss stability: clearly E is not stable as a vector bundle, since we have assumed that $g \geq 2$, which implies that $K_C^{1/2}$ is a destabilizing subbundle since

$$g - 1 = \frac{\deg(K_C^{1/2})}{1} > \frac{\deg(E)}{2} = 0.$$

However, we now show that the Higgs bundles (E, ϕ_ω) are stable for any ω.

If $\omega = 0$, then $K_C^{-1/2}$ is preserved by ϕ_ω (since it is sent to zero), and it satisfies the stability condition since $g \geq 2$:

$$1 - g = \frac{\deg(K_C^{-1/2})}{1} < \frac{\deg(E)}{2} = 0.$$

If ω is different from zero, (1.6) shows that no subbundle of E is ϕ_ω-invariant, so (E, ϕ_ω) is a family of stable Higgs bundles.

Theorem 1.2.9 ([42], Theorem 5.10, Proposition 7.1)**.** *With the above notation, there exists a scheme $\mathcal{M}(n,d)$ which is a coarse moduli space for S-equivalence classes of semistable Higgs bundles of degree d and rank n. It is quasi-projective and it has an open subscheme $\mathcal{M}^s(n,d)$ which is the moduli scheme of stable Higgs bundles.*

When n and d are coprime, we have that $\mathcal{M}(n,d) = \mathcal{M}^s(n,d)$ is smooth.

The moduli space $\mathcal{M}(n,d)$ is a non-compact variety whose dimension depends on the choice of L. If $L \cong K_C$, then

$$\dim \mathcal{M}(n,d) = 2n^2(g-1) + 2. \tag{1.7}$$

1.3 Spectral data for Higgs bundles on a smooth base curve

A natural way to study $\mathcal{M}(n,d)$ is through the Hitchin map, which we now present.

The Hitchin map

Since a Higgs field is a twisted endomorphism of the vector bundle E, it has a characteristic polynomial with coefficients

$$a_i := (-1)^i \mathrm{tr}\left(\bigwedge^i \phi\right) \in H^0(C, L^i), \qquad i = 1, \cdots, n. \tag{1.8}$$

In particular, we have $a_1 = -\mathrm{tr}(\phi)$ and $a_n = (-1)^n \mathrm{tr}(\bigwedge^n \phi) = (-1)^n \det(\phi)$.

Definition 1.3.1. Let $(E, \phi) \in \mathcal{M}(n,d)$ be a semistable Higgs bundle and consider sections a_i as in (1.8). The *Hitchin map* is defined as follows.

$$h : \mathcal{M}(n,d) \to \mathbb{A} = \bigoplus_{i=1}^{n} H^0(C, L^i), \qquad (E, \phi) \mapsto (a_1, \cdots, a_n). \tag{1.9}$$

Definition 1.3.2. We call *characteristic* the n-tuple $a = (a_1, \cdots, a_n) \in \mathbb{A}$.

Remark 1.3.3. As it is already observed in [31], Theorem 8.1, the Hitchin map is surjective and proper (see also [42], Theorem 6.1 and Remark 6.2).

Moreover, if $L \cong K_C$, referring to the definition of $\mathbb{A}$ in (1.9), we have:

$$\dim \mathbb{A} = n^2(g-1)+1 = \frac{1}{2}\dim \mathcal{M}(n,d).$$

Spectral curves

Let

$$\text{Tot }(L) = \text{Spec}(\text{Sym}(L^{-1})) \tag{1.10}$$

be the total space of the line bundle L and let $\pi : L \to C$ be the projection onto C. There is a tautological coordinate $x \in H^0(\text{Tot }(L), \pi^* L)$ on Tot (L). Consider the sections of $\pi^* L^n$ having the form

$$s_a = x^n + \pi^*(a_1)x^{n-1} + \cdots + \pi^*(a_n). \tag{1.11}$$

Definition 1.3.4. We define the *spectral curve* X_a as the zero divisor in Tot (L) of a nonzero section as in (1.11). We also refer to X_a as the *spectral cover.*

We now characterize the spectral curve X_a as a projective scheme defined in Tot (L). Let

$$Z = \mathbb{P}(\mathcal{O}_C \oplus L^{-1}) \tag{1.12}$$

and let $p : Z \to C$ be the projection map extending $\pi : L \to C$. Let $\mathcal{O}_Z(1)$ be the relatively ample line bundle on Z. We denote by y the section of $\mathcal{O}_Z(1)$, whose pushforward via p corresponds to the constant section $(1,0)$ of the vector bundle

$$p_*\mathcal{O}_Z(1) \cong \mathcal{O}_C \oplus L^{-1}.$$

We denote by x the section of $\mathcal{O}_Z(1) \otimes p^* L$ whose pushforward via p corresponds to the constant section $(0,1)$ of the vector bundle

$$p_*(\mathcal{O}_Z(1) \otimes p^* L) \cong (\mathcal{O}_C \oplus L^{-1}) \otimes L \cong L \oplus \mathcal{O}_C.$$

In other words, $\{y = 0\}$ is the section of p, called the *infinity section*, corresponding to the surjection

$$\mathcal{O}_C \oplus L^{-1} \to L^{-1}$$

and $\{x = 0\}$ is the section of p, called the *zero section*, corresponding to the surjection

$$\mathcal{O}_C \oplus L^{-1} \to \mathcal{O}_C.$$

Remark 1.3.5. Given $a = (a_1, \cdots, a_n) \in \mathbb{A}$ as in (1.9), the spectral curve is the projective curve inside Z given by the zero locus of the section

$$x^n + p^*(a_1)x^{n-1}y + \cdots + p^*(a_n)y^n \in H^0(Z, p^*L^n \otimes \mathcal{O}_Z(n)),$$

thus by the equation

$$x^n + p^*(a_1)x^{n-1}y + \cdots + p^*(a_n)y^n = 0. \tag{1.13}$$

We denote by $\pi_a : X_a \to C$ the restriction of π to the spectral curve X_a.

Remark 1.3.6. By the description above, the spectral curve has pure dimension one and it has at worst locally planar singularities, since it is embedded in Z.

Remark 1.3.7 ([30], Section 5.1). As the sections a_i vary, (1.13) forms a linear system of divisors on Tot (L) and on Z. The linear system

$$\mathbb{P}(H^0(Z, p^*L^n \otimes \mathcal{O}_Z(n))) \tag{1.14}$$

has no base points: the main points of the argument by Hitchin are that x^n lies in the system (1.14) and that the linear system $\mathbb{P}(H^0(C, L^n))$ has no base points for $n \geq 2$. This follows from our assumption $\deg L \geq \deg K_C = 2g-2$. Since we also assume that $g \geq 2$ and that the vector bundle E has rank $n \geq 2$, we have $\deg L^n \geq 4g - 4 \geq 2g$, yielding the claim (e.g. by [28], Chapter IV, Corollary 3.2).

Remark 1.3.8. By Bertini's theorem (e.g. [28], Chapter III, Corollary 10.9), the generic spectral curve is a nonsingular projective curve, since the linear system (1.14) has no base points by Remark 1.3.7.

On the other hand, if E has a ϕ-invariant subbundle F, then the characteristic polynomial of ϕ decomposes and one factor corresponds to the characteristic polynomial of the restriction of ϕ to F. In this case, the spectral curve is singular.

We will also use the term "generic" for singular curves, in the sense of Notation 5.1.19.

Remark 1.3.9. Referring to (1.13), note that y restricted to X_a is everywhere nonzero, so it trivializes $\mathcal{O}_Z(1)_{|X_a}$. Thus $x_{|X_a}$ is a section of $(\mathcal{O}_Z(1) \otimes p^*L)_{|X_a} = \pi_a^*L$ associated to the spectral curve

$$x^n + \pi_a^*(a_1)x^{n-1} + \cdots + \pi_a^*(a_n) = 0.$$

Notation 1.3.10. From now on, with a slight abuse of notation, we assume that the equation of the spectral curve X_a is of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0. \tag{1.15}$$

Spectral curves as finite morphisms

It is also useful to consider the following description of π_a as a finite cover of degree n.

Remark 1.3.11. Let $a_0 = 1 \in H^0(C, \mathcal{O}_C)$. We can also see the sections $a_i \in H^0(C, L^i)$, $i = 0, \cdots, n$, as embeddings

$$a_i : L^{-n} \to L^{-(n-i)}.$$

Consider the ideal sheaf

$$\mathcal{I}_a = \left(\bigoplus_{i=0}^{n} a_i(L^{-n}) \right) \subset \mathrm{Sym}(L^{-1})$$

generated by the images of a_i. Then we have

$$X_a = \mathrm{Spec}(\mathrm{Sym}(L^{-1})/\mathcal{I}_a) \subset \mathrm{Tot}\ (L) = \mathrm{Spec}(\mathrm{Sym}(L^{-1})).$$

In particular, the spectral curve $\pi_a : X_a \to C$ is a finite morphism of degree n and we have

$$\pi_{a,*}\mathcal{O}_{X_a} \cong \mathrm{Sym}(L^{-1})/\mathcal{I}_a \tag{1.16}$$

as an $\mathcal{O}_C$-algebra and

$$\pi_{a,*}\mathcal{O}_{X_a} \cong \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)} \tag{1.17}$$

as a vector bundle (e.g. by [9], Chapter I, Lemma 17.2).

Moreover, since the map π_a is finite, we can compute the arithmetic genus of X_a via the invariance of the Euler characteristic (1.1) by pushforward by finite morphisms, obtaining

$$\chi_{X_a}(\mathcal{O}_{X_a}) = \chi_C(\pi_{a,*}\mathcal{O}_{X_a}).$$

By (1.17), this yields

$$\chi_{X_a}(\mathcal{O}_{X_a}) = \sum_{i=0}^{n-1} \chi_C(L^i) = -(\deg L) \cdot n(n-1)/2 + n\chi_C(\mathcal{O}_C). \tag{1.18}$$

Thus we have

$$g_{X_a} = (\deg L) \cdot n(n-1)/2 - n\chi_C(\mathcal{O}_C) + 1,$$

which, for $L \cong K_C$, yields

$$g_{X_a} = n^2(g-1) + 1. \tag{1.19}$$

We also consider the spectral curve locally, for later use.

Definition 1.3.12. Let $E \to C$ be a vector bundle. We define the *fibre* of E at a point $q \in C$ as $E^{(q)} := E_q \otimes_{\mathcal{O}_{C,q}} \mathbb{C}^{(q)}$, where E_q is the stalk of E at q and $\mathbb{C}^{(q)}$ is the residue field.

Notation 1.3.13. We denote by $^{(q)}$ all data referring to fibres of E at a point q, e.g. maps, Higgs fields, etc.

Remark 1.3.14 ([7], Remark 5.1.2, [30], Section 5.1). We now give a more explicit description of the relation between spectral curves and eigenvalues of the Higgs field ϕ. Consider the characteristic polynomial of ϕ having the form

$$\det(\lambda \cdot I_n - \phi) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n.$$

Pulling back the vector bundle E to X_a, we have $\phi \in H^0(X_a, \mathrm{End}E \otimes L)$ satisfying

$$\det(x \cdot I_n - \phi) = 0,$$

where x is the tautological coordinate on Tot (L).
For example, if E has rank 2, considering a point $p \in C$ such that X_a is not ramified over it, we have $\pi_a^{-1}(p) = \{p_1, p_2\}$ and we have

$$\phi^{(p)} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \tag{1.20}$$

where $c = x(p_1), d = x(p_2)$ are the distinct eigenvalues of ϕ over p.

Fibres of the Hitchin map

Note that, while the generic spectral curve is smooth by Remark 1.3.8, it is also useful to consider some non-generic cases for later use.

Definition 1.3.15. We say that a curve is *integral* if it is reduced and irreducible.

Notation 1.3.16. We assume that spectral covers of smooth base curves are integral.

In particular, we focus on nodal curves.

Definition 1.3.17. By a *nodal curve* we mean a connected curve which is allowed to be reducible and has only nodes (ordinary double points) as singularities.

We need to consider a more general definition of Jacobian of a curve than Definition 1.1.8.

Definition 1.3.18. Let Y a nodal curve. We define the *generalized Jacobian* $J(Y)$ of Y as the moduli space of isomorphism classes of line bundles having degree zero on every irreducible component of Y. Its dimension equals the arithmetic genus of Y.

The level of generality of Notation 1.3.16 is enough for the aims of this work. The case in which the spectral curve is reducible is discussed in the Appendix of [35] and the case in which the spectral curve is not reduced is discussed in [29], Proposition 6.1 and [17], Part 1.

Many ways of compactifying $J(X_a)$ were described, e.g. by Altman-Kleiman [2] for integral curves and by Seshadri [46] for reducible curves. Simpson's description, which we recall in Section 1.4, includes both of them.

As we saw in Remark 1.3.11, the morphism π_a is finite, so it is affine. This yields the following local description of the fibres of the Hitchin map.

Remark 1.3.19 ([36], Section 3). Let $U = \mathrm{Spec}A$ be an affine open set of the smooth base curve C. Then $\pi_a^{-1}(U) = \mathrm{Spec}B$, where B is the A-algebra

$$B = \frac{A[x]}{x^n + a_1 x^{n-1} + \cdots + a_n}. \tag{1.21}$$

A torsion-free sheaf on X_a corresponds, when it is restricted to $\pi_a^{-1}(U)$, to a torsion-free B-module M. By the presentation (1.21), the datum of a

B-module is equivalent to the datum of an A-module N together with an endomorphism ψ satisfying

$$\psi^n + a_1\psi^{n-1} + \cdots + a_n \mathrm{id} = 0.$$

The module M is torsion-free if and only if N is torsion-free, which, since A is regular of dimension one, is equivalent to N being locally free.

We see in Proposition 1.3.22 how this local construction can be globalized over C.

In order to characterize rank-1 torsion-free sheaves on X_a, we recall the definition of Euler characteristic of a sheaf.

Definition 1.3.20. Let $\mathcal{E}$ be a coherent sheaf on an integral curve Y. We define the *Euler characteristic* of $\mathcal{E}$ as $\chi_Y(\mathcal{E}) = \deg \mathcal{E} + \operatorname{rank} \mathcal{E} \cdot \chi_Y(\mathcal{O}_Y)$.

Let

$$\Lambda := \mathrm{Sym}(L^{-1}). \tag{1.22}$$

Lemma 1.3.21 ([6], Section 2.3)**.** *Let Y be a curve having at worst a nodal singularity. The category of Higgs bundles on Y is equivalent to the category of Λ-modules on Y.*

We discuss Simpson's compactified Jacobian in Remark 1.4.15.

Proposition 1.3.22 ([11], Proposition 3.6, [17], Section 1.4.1)**.** *Let C be a smooth projective curve, let L be a line bundle on it having degree which is greater or equal to the degree of K_C. Let $a = (a_1, \cdots, a_n)$ be a set of sections of L^i, $i = 1, \cdots, n$, let $\pi_a : X_a \to C$ be the spectral curve and assume that X_a is integral. Then there is a bijective correspondence between isomorphism classes of rank-1 torsion-free sheaves η on X_a and isomorphism classes of semistable Higgs bundles (E, ϕ) on C, where ϕ has characteristic coefficients a_i. The correspondence is given by associating to any rank-1 torsion-free sheaf η on X_a the semistable Higgs bundle (E, ϕ), where E is the vector bundle $\pi_{a,*}\eta$ on C, endowed with the natural isomorphism*

$$\pi_{a,*}\eta \to \pi_{a,*}\eta \otimes L \cong \pi_{a,*}(\eta \otimes \pi_a^* L)$$

given by multiplication by the section x of $\pi_a^* L$. This yields an isomorphism

$$h^{-1}(a) \cong \overline{J}^{\gamma}(X_a), \tag{1.23}$$

where $\overline{J}^{\gamma}(X_a)$ is Simpson's compactified Jacobian parametrizing rank-1 torsion-free sheaves on X_a having suitable fixed degree γ.

Proof. Recall that π_a is a finite morphism of degree n. Thus, if η is a torsion-free sheaf of rank 1 on X_a, then we have that $E = \pi_{a,*}\eta$ is a torsion-free sheaf of rank n on C. Since C is smooth, E is a vector bundle and we have

$$\chi_{X_a}(\eta) = \chi_C(\pi_{a,*}\eta),$$

yielding

$$\deg \eta + \chi_{X_a}(\mathcal{O}_{X_a}) = \deg E + n\chi_C(\mathcal{O}_C).$$

Evaluating $\chi_{X_a}(\mathcal{O}_{X_a}) - n\chi_C(\mathcal{O}_C)$ as in (1.18) yields

$$\deg E = \deg \eta - \frac{n(n-1)}{2}\deg L. \tag{1.24}$$

Letting

$$\gamma := \deg \eta = \deg E + \frac{n(n-1)}{2}\deg L,$$

we have $\eta \in \overline{J}^{\gamma}(X_a)$. In particular, letting $L \cong K_C$ yields

$$\gamma = \deg E + n(n-1)(g-1). \tag{1.25}$$

Pushing forward η via π_a also yields the Higgs field, as we now see. By (1.16), a $\pi_{a,*}\mathcal{O}_{X_a}$-module structure on E corresponds to an algebra homomorphism

$$\Lambda/\mathcal{I}_a \to \mathrm{End}E, \tag{1.26}$$

which is equivalent to an $\mathcal{O}_C$-module $L^{-1} \to \mathrm{End}E$, which is equivalent, by Lemma 1.3.21, to the datum of a Higgs field $\phi : E \to E \otimes L$ satisfying $P_a(\phi) = 0$, where P_a is the polynomial determined by a. Since we assume that X_a is integral, P_a is irreducible over the function field of X_a, so it is the characteristic polynomial of ϕ.

Conversely, if (E, ϕ) is a semistable Higgs bundle, where E is a vector bundle of rank n on C and $\phi : E \to E \otimes L$ is a linear homomorphism with characteristic coefficients a_i, then $P_a(\phi) = 0$ by the Cayley-Hamilton theorem.

This yields the sought-for isomorphism (1.23). $\square$

Remark 1.3.23. Since the Hitchin map is surjective, by Proposition 1.3.22 we have a further way to compute the dimension of $\mathcal{M}(n,d)$, which will be useful in Section 4.4. Assuming $L \cong K_C$, we have:

$$\dim \mathcal{M}(n,d) = \dim h^{-1}(a) + \dim \mathbb{A} = g_{X_a} + \dim \mathbb{A} = 2n^2(g-1) + 2. \quad (1.27)$$

Definition 1.3.24. We call *spectral data* the datum of the spectral curve X_a together with the datum of the rank-1 torsion-free sheaf η as in Proposition 1.3.22.

In particular, the rank-1 torsion-free sheaf η corresponding to (E,ϕ) can always be obtained, up to twisting, as the eigenspace of $\pi_a^*\phi$ with eigenvalue x.

Proposition 1.3.25 ([20], Proposition 1.4.5, [11], Remark 3.7)**.** *In the notations of Theorem 1.3.22 we have the following exact sequence*

$$0 \to \eta \otimes \pi_a^* L^{1-n} \to \pi_a^* E \xrightarrow{\pi_a^*\phi - x} \pi_a^* E \otimes \pi_a^* L \xrightarrow{ev} \eta \otimes \pi_a^* L \to 0, \quad (1.28)$$

where ev is induced by the evaluation map $\pi_a^* E \cong \pi_a^* \pi_{a,*}\eta \to \eta$.

1.4 Simpson's moduli space

We first recall some definitions about coherent sheaves on a curve Y having at worst nodal singularities, which will also be useful later.

Definition 1.4.1. Let E be a coherent sheaf on Y, let y be a point of Y and let E_y be the stalk of E at y. We define the *support* of E as the closed set

$$\mathrm{Supp}\ E = \{y \in Y : E_y \neq 0\} \subset Y.$$

The dimension $\dim E$ of E is the dimension of its support.

Definition 1.4.2. We say that E is *pure of dimension* d if, for all nontrivial subsheaves $F \subset E$, we have $\dim F = d$.

We say that E is *torsion-free* if it is pure of dimension 1.

Let $\mathcal{O}_Y(1)$ be an ample invertible sheaf on Y.

Definition 1.4.3. We define the *Hilbert polynomial of* E as the polynomial with rational coefficients, whose degree equals $\dim E$, given by

$$p(E,s) := \chi_Y(E \otimes \mathcal{O}_Y(1)^s), \qquad s >> 0.$$

Definition 1.4.4. Let E be a torsion-free sheaf on Y and let

$$p(E,s) = c_1 \cdot s + c_0$$

be its Hilbert polynomial. We define the *polarized rank* of E as the rational number $\operatorname{rank}_p(E) = c_1/\deg(\mathcal{O}_Y(1))$ and the *polarized degree* of E as the rational number $\deg_p(E) = c_0 - \operatorname{rank}_p(E)\chi_Y(\mathcal{O}_Y)$. The Hilbert polynomial thus has the form:

$$p(E,s) = \deg(\mathcal{O}_Y(1))\operatorname{rank}_p(E) \cdot s + \deg_p(E) + \operatorname{rank}_p(E)\chi_Y(\mathcal{O}_Y). \quad (1.29)$$

Remark 1.4.5 ([29], Remark 3.8, [33], Section 1.2)*.* We now discuss how polarized rank and polarized degree relate to the (classical) notions of rank and degree given in Section 1.1.

First, assume that Y is an integral curve. If E is a torsion-free sheaf on Y, then there exists an open dense subset $U \subset Y$ such that $E_{|U}$ is locally free and the rank of E equals the rank of $E_{|U}$, so the notions of rank and degree coincide with the ones in Section 1.1.

If Y is not integral, then the polarized rank and polarized degree of E need not be integers. Example 2.2.4 provides an explicit computation of $\operatorname{rank}_p(E), \deg_p(E)$ for this case.

Definition 1.4.6. We define the *p-slope* of E as

$$\mu_p(E) := \frac{p(E,s)}{\operatorname{rank}_p(E)} = \frac{\chi_Y(E)}{\operatorname{rank}_p(E)} + \deg(\mathcal{O}_Y(1)) \cdot s. \quad (1.30)$$

We say that a coherent sheaf E is *p-semistable* (respectively, *p-stable*) if it is torsion-free and, for any proper subsheaf $F \subset E$, we have $\mu_p(F) \leq \mu_p(E)$ (respectively, $\mu_p(F) < \mu_p(E)$).

Remark 1.4.7 ([7], Section 4.1, [48], Section 1.3)*.* Equation (1.30) shows that, if Y is a smooth curve, Definition 1.4.6 is equivalent to Definition 1.1.3, since the difference between slope and p-slope does not depend on E. If Y

is a one-nodal reducible curve having two smooth components, we will give an alternative notion to p-slope in Definition 2.3.1 and we will show that it is equivalent to p-slope in Remark 2.3.2.

Notation 1.4.8. Until the end of the section, Y denotes a smooth curve or Y is a reducible, one-nodal, curve having two smooth components.

We recall the interpretation of Simpson's works [48], [49] by Balaji-Barik-Nagaraj [6], which generalizes Sections 1.1 and 1.2 and will also be useful later.

Definition 1.4.9. Let L be a line bundle on Y. We define a *torsion-free Hitchin pair* (E, ϕ) on Y as a torsion-free sheaf E together with a homomorphism $\phi : E \to E \otimes L$.

Remark 1.4.10. Simpson constructs in [48], Theorem 3.8, a quasi-projective scheme whose points correspond to S-equivalence classes of p-semistable Λ-modules with fixed Hilbert polynomial p. The equivalence Lemma 1.3.21 thus yields an alternative construction of the moduli space $\mathcal{M}(n, d)$, which will be useful in Chapter 3.

Remark 1.4.11. Let Y be a smooth curve. Then [48], Theorem 4.10, characterizes singularities of $\mathcal{M}(2, 0)$ in terms of nontrivial polystable Higgs bundles. In particular, assuming that Φ has trace zero, the singularities of $\mathcal{M}(2, 0)$ are given by

$$\Sigma = \{(E, \Phi) : (E, \Phi) = (N, \phi) \oplus (N^{-1}, -\phi)\}, \tag{1.31}$$

for N a Φ-invariant line subbundle of E of degree 0.

Let Z be as in (1.12). Let $D_\infty := Z \setminus L$ be the divisor at infinity. Then we have the following.

Lemma 1.4.12 ([49], Lemma 6.8, [6], Lemma 2.4, [7], Lemma 4.1.3)**.** *There is a functorial correspondence between the category of Hitchin pairs (E, ϕ) on Y and the category of coherent $\mathcal{O}_Z$-modules $\mathcal{E}$ whose support does not intersect the divisor at infinity. The sheaf E is flat if and only if $\mathcal{E}$ is flat. The sheaf E is torsion-free if and only if $\mathcal{E}$ is pure of dimension one.*

Moreover, as in [6], Section 2.4, we have that Remark 1.4.7 enables us to relate semistability on Z to semistability on Y, as we now see.

Remark 1.4.13. Choose k such that $\mathcal{O}_Z(1) := \pi^*\mathcal{O}_Y(k) \otimes \mathcal{O}_Z(D_\infty)$ is ample on Z. In particular, since Z is the projective completion of the total space of L, we have:

$$\mathcal{O}_Z(1)_{|L} = \pi^*\mathcal{O}_Y(k). \tag{1.32}$$

This way, for any coherent sheaf $\mathcal{E}$ on Z whose support does not meet the divisor D_∞, the Hilbert polynomials of the $\mathcal{O}_Z$-module $\mathcal{E}$ and that of $\pi_*\mathcal{E}$ are such that

$$p(\mathcal{E}, s) = p(\pi_*\mathcal{E}, ks),$$

thus the notion of p-semistability on Z is equivalent to that on Y. By Remark 1.4.7, p-semistability of $\mathcal{E}$ on Z is equivalent to slope-semistability of (E, ϕ) on Y (both in the case in which Y is smooth and in the case in which Y is one-nodal, reducible).

Remark 1.4.14. Fix a polynomial p of degree one and let $p_k(s) := p(ks)$.

Then [48], Theorem 1.21 yields a coarse moduli space $\mathcal{M}(\mathcal{O}_Z(1), p_k)$ of p-semistable coherent $\mathcal{O}_Z$-modules of pure dimension 1 on the projective variety Z with respect to the ample line bundle $\mathcal{O}_Z(1)$, having fixed Hilbert polynomial p_k. The moduli space $\mathcal{M}(\mathcal{O}_Z(1), p_k)$ is a projective variety.

Since, by Lemma 1.4.12, Higgs bundles on Y correspond to p-semistable pure sheaves whose support does not intersect D_∞, there is an open inclusion

$$\mathcal{M}(n, d) \subset \mathcal{M}(\mathcal{O}_Z(1), p_k).$$

Moreover, considering stable vector bundles as stable Higgs bundles having underlying stable vector bundle and zero Higgs field, Simpson's construction also yields moduli spaces of stable vector bundles.

Remark 1.4.15. Consider the spectral curve $X_a \to Y$. We denote by $\overline{J^\delta}(X_a)$ the Simpson moduli space of p-semistable rank-1 torsion-free sheaves on X_a of degree δ. We consider two cases:

(a) if Y is a smooth curve and X_a is an integral curve, then every rank-1 torsion-free sheaf on X_a is stable and $\overline{J^\delta}(X_a)$ is isomorphic to the compactification by Altman-Kleiman [2],

(b) if Y is one-nodal, reducible and X_a is an adapted spectral curve (Definition 4.3.1), [1] shows that Simpson's compactification is isomorphic to the one by Oda-Seshadri [43], which parametrizes rank-1 torsion-free sheaves on X_a with a semistability condition depending on the polarization on Y, as we see in Proposition 4.4.4.

1.5 The norm map and Prym varieties

We now give preliminaries to characterize the fibre of the $\mathrm{SL}(2,\mathbb{C})$-Hitchin map, which will be useful in Chapter 5.

Notation 1.5.1. In this section, we denote by $\pi : X \to C$ a cover of degree $n \geq 2$ of a smooth base curve C.

Definition 1.5.2. Let X be a smooth curve, let $p_i \in X$ and let $n_i \in \mathbb{Z}$. We define the *norm map associated to* π as the map

$$\mathrm{Nm}_\pi : J(X) \to J(C) \qquad \mathcal{O}_X\left(\sum n_i p_i\right) \mapsto \mathcal{O}_C\left(\sum n_i \pi(p_i)\right).$$

Remark 1.5.3. More generally, the norm map is defined for any finite cover $\pi : X \to C$ of degree $n \geq 2$ in [29], Section 3.1 and the following properties hold:

- for any invertible sheaves $\mathcal{N}, \mathcal{N}'$ on X, we have

$$\mathrm{Nm}_\pi(\mathcal{N} \otimes \mathcal{N}') \cong \mathrm{Nm}_\pi(\mathcal{N}) \otimes \mathrm{Nm}_\pi(\mathcal{N}').$$

- for any invertible sheaf $\mathcal{N}$ on C, we have

$$\mathrm{Nm}_\pi(\pi^*\mathcal{N}) \cong \mathcal{N}^n. \tag{1.33}$$

Definition 1.5.4. We define the *Prym variety associated to* π as the locus $\mathrm{Pr}_\pi \subseteq J(X)$ of line bundles whose norm with respect to π is trivial, i.e.

$$\mathrm{Pr}_\pi = \{L \in J(X) : \mathrm{Nm}_\pi(L) \cong \mathcal{O}_C\}.$$

We define the *compactified Prym variety associated to* π as the closure of Pr_π inside the compactified Jacobian of X:

$$\overline{\mathrm{Pr}}_\pi = \{\eta \in \overline{J}(X), \mathrm{Nm}_\pi(\eta) \cong \mathcal{O}_C\}.$$

In particular, $\mathrm{Pr}_\pi = \overline{\mathrm{Pr}}_\pi \cap J(X)$.

Remark 1.5.5. Note that, as in the work by Gothen-Oliveira [26], Remark 3.5, we define the Prym variety as full kernel of the norm map, which agrees e.g. with the work by Hausel-Pauly [29], Theorem 1.1, de Cataldo-Hausel-Migliorini [23], Lemma 4.4.3, de Cataldo [22], Definition 2.4.8. This enables us to consider cases in which the Prym variety is not connected.

Definition 1.5.4 is different from the classical one by Mumford [37], Section I.3, in which the Prym variety is always connected.

Proposition 1.5.6 ([20], Definition-Lemma 2.5.1)**.** *For any rank-1 torsion-free sheaf η in $\overline{J}(X)$ we have:*

$$Nm_\pi(\eta) \cong \det(\pi_* \eta) \otimes \det(\pi_* \mathcal{O}_X)^{-1}. \tag{1.34}$$

From [15], Proposition 11.4.3, we obtain the following characterization.

Proposition 1.5.7. *Let X and C be smooth curves and let $\pi : X \to C$ be a cover of degree $n \geq 2$. Then Ker Nm_π is nontrivial if and only if π factors through a cyclic étale cover of degree at least 2.*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\pi_2"'] & & C \\ & X_1 \arrow[ur, "\pi_1"'] & \end{tikzcd} \tag{1.35}$$

Prym varieties appear in the study of G-Higgs bundles, which we briefly introduce here and discuss in Chapter 5. We emphasize the aspects of the description of G-Higgs bundles from Arroyo [5] and Branco [17] which are useful to our aims and we refer to these works for a more comprehensive treatment.

Remark 1.5.8. Let C be a smooth base curve and let G be an affine reductive group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Similarly to Theorem 1.2.9, there exists a coarse moduli space $\mathcal{M}_G(n, d)$ for S-equivalence classes of semistable G-Higgs bundles of degree d and rank n.
The datum of the G-Higgs bundles that we will study is obtained from the datum of $\mathrm{GL}(2, \mathbb{C})$-Higgs bundles satisfying suitable extra-conditions, e.g. that the underlying vector bundles have fixed determinant. Considering

Proposition 1.3.22, these conditions are mirrored both by extra-conditions on the coefficients of the spectral curve, some of which are necessarily zero, and by considering (in the generic case) appropriately constrained line bundles, e.g. having fixed norm, on the spectral curve. This way, we will see that the fibre of the Hitchin map associated to G-Higgs bundles is given by the Prym variety introduced in Definition 1.5.4.

Chapter 2

Torsion-free sheaves on a one-nodal curve of compact type

We now discuss how the tools introduced in Chapter 1 adapt when we consider a one-nodal curve C of compact type as our base curve.

2.1 Line bundles on a one-nodal curve of compact type

We first recall some definitions about nodal curves, then we focus on one-nodal curves of compact type. We consider a nodal curve C (Definition 1.3.17) having γ irreducible components $C_1, \cdots, C_\gamma$ and δ nodes $q_1, \cdots, q_\delta$.

Definition 2.1.1. We define the *dual graph* Γ_C of C as the graph whose vertices are identified with the irreducible components of C and whose edges are identified with the nodes of C. An edge joins two (possibly equal) vertices if the corresponding node is in the intersection of the corresponding irreducible components.

Definition 2.1.2. Let C be a nodal curve. We say that C is *of compact type* if every irreducible component of C is smooth and its dual graph Γ_C is a tree, i.e. it is non oriented, connected and without cycles. In particular,

Γ_C being a tree implies that its first Betti number is zero, i.e.

$$\gamma - \delta + 1 = 0. \tag{2.1}$$

We now show that the generalized Jacobian of C (Definition 1.3.18) is projective if and only if C is a curve of compact type.

Remark 2.1.3 ([19], Section 2.1). Let C be a nodal curve. Assume that C has δ nodes and γ irreducible components. Let

$$\nu : \widetilde{C} = \sqcup_{i=1}^{\gamma} C_i \to C$$

be the (partial) normalization. The associated map of structure sheaves

$$\mathcal{O}_C \hookrightarrow \mathcal{O}_{\widetilde{C}}$$

is associated to the following exact sequence

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\widetilde{C}} \to \oplus_{i=1}^{\delta} \mathbb{C}^{(q_i)} \to 0,$$

which yields the following exact sequence in cohomology:

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(\widetilde{C}, \nu_* \mathcal{O}_{\widetilde{C}}) \to \mathbb{C}^{\delta} \to H^1(C, \mathcal{O}_C) \to H^1(\widetilde{C}, \nu_* \mathcal{O}_{\widetilde{C}}) \to 0. \tag{2.2}$$

From this sequence we have a formula for the arithmetic genus g of C:

$$g = h^1(\widetilde{C}, \nu_* \mathcal{O}_{\widetilde{C}}) + \delta - \gamma + 1 = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1, \tag{2.3}$$

where g_i is the geometric genus of C_i.

We now consider $\mathcal{O}_C^* \hookrightarrow \mathcal{O}_C$ in (2.2), and identify $J(C) \cong H^1(C, \mathcal{O}_C^*)$, $J(\widetilde{C}) \cong H^1(\widetilde{C}, \nu_* \mathcal{O}_{\widetilde{C}}^*)$. Then, arguing as above, we obtain the exact sequence

$$0 \to (\mathbb{C}^*)^{\delta-\gamma+1} \to J(C) \xrightarrow[\nu^*]{\sim} J(\widetilde{C}) \to 0.$$

In particular, if C is of compact type, then (2.1) yields the isomorphism

$$J(C) \xrightarrow{\sim} J(\widetilde{C}), \qquad L \mapsto (L_{|C_1}, \cdots, L_{|C_\gamma}). \tag{2.4}$$

Notation 2.1.4. From now on, unless otherwise stated, we assume that C is a one-nodal curve of compact type having two smooth components C_1, C_2 which have genus at least 2. Let q be the node of C.

Remark 2.1.5. By Remark 2.1.3, we have the following characterizations.

- The datum of a line bundle L on C is equivalent to the datum of its restrictions L_1 to C_1 and L_2 to C_2, together with a canonical identification between the fibres of L_1 and L_2 at q.
- Letting g be the arithmetic genus of C, we have $g = g_1 + g_2$ because of (2.3), where g_i is the genus of C_i.

2.2 Equivalence of categories: torsion-free sheaves and triples

As we saw in Remark 1.4.5, when C is reducible, the rank and degree of torsion-free sheaves depend on the choice of a polarization on C.

Definition 2.2.1. Let C be a one-nodal curve of compact type. A *polarization* $\alpha = (\alpha_1, \alpha_2)$ on C is the choice of positive rational numbers such that

$$\alpha_1 + \alpha_2 = 1. \tag{2.5}$$

We can reformulate Definition 2.2.1 as follows, which will also be useful for Remark 4.4.3.

Remark 2.2.2. Let $\mathcal{O}_C(1)$ be an ample line bundle on C. This yields ample line bundles $\mathcal{O}_C(1)_{|C_1}$, $\mathcal{O}_C(1)_{|C_2}$ on C_1, C_2 respectively. Equivalently to Definition 2.2.1, setting $\delta_i := \deg \mathcal{O}_C(1)_{|C_i}$ for $i = 1, 2$, we say that $\mathcal{O}_C(1)$ gives a polarization on C if we have

$$\frac{\delta_1}{\delta_2} = \frac{\alpha_1}{\alpha_2}.$$

In this case, we set $\alpha_i = \dfrac{\delta_i}{\delta_1 + \delta_2}$.

Since C is of compact type, we can reformulate Definition 1.4.4 as follows.

Definition 2.2.3. Let E be a torsion-free sheaf on C. Let r_1, r_2 be the ranks of the restrictions of E to the components C_1, C_2 respectively. We say that E has rank (r_1, r_2) and we define the *α-rank of E* as

$$r_\alpha(E) := \operatorname{rank}_\alpha(E) = \alpha_1 r_1 + \alpha_2 r_2. \tag{2.6}$$

In particular, if we have $r_1 = r_2 = r$, we also simply say that E has rank r.

We define the *α-degree of E* as

$$d_\alpha(E) := \deg_\alpha(E) = \chi_C(E) - r_\alpha(E)(1-g).$$

Example 2.2.4 ([17], Example 1.8)**.** We now consider the relation of Definition 2.2.1 and Remark 2.2.2 to Definition 1.4.4. Let C be a one-nodal curve of compact type having smooth components C_1, C_2 of genus g_1, g_2 respectively. Let q be the node of C. Let $\mathcal{O}_C(1)$ be an ample line bundle whose corresponding polarization is given by $\alpha_1 = \alpha_2 = 1/2$. Let H be a locally free sheaf of rank r_1 and degree d_1 on C_1. Let $j : C_1 \hookrightarrow C$ be the natural inclusion and consider the torsion-free sheaf $\mathcal{H} = j_* H$ on C.

By the assumption on the polarization, we have

$$\deg \mathcal{O}_C(1) = 2\deg\left(\mathcal{O}_C(1)_{|C_1}\right) = 2\delta_1.$$

The polarized rank and polarized degree of the sheaf $\mathcal{H}$ are given by (1.29):

$$p(\mathcal{H}, s) = \chi_C(\mathcal{H} \otimes \mathcal{O}_C(s)) = 2\delta_1 \operatorname{rank}_p(\mathcal{H}) s + \deg_p(\mathcal{H}) + \operatorname{rank}_p(\mathcal{H})(1-g_1-g_2).$$

On the other hand, we have that

$$p(\mathcal{H}, s) = \chi_{C_1}(H \otimes j^* \mathcal{O}_C(s)) = \delta_1 r_1 s + d_1 + r_1(1-g_1),$$

thus we have

$$\operatorname{rank}_p(\mathcal{H}) = \frac{r_1}{2},$$

which is not necessarily an integer. Moreover, we have

$$\deg_p(\mathcal{H}) = d_1 + r_1(1-g_1) - \frac{r_1}{2}(1-g_1-g_2) = d_1 + \frac{r_1}{2}(1-g_1+g_2).$$

Note that $\mathcal{H}$ restricted to C_2 is a torsion sheaf supported at the node q.

We now define triples on C.

Definition 2.2.5. Let E_1, E_2 be locally free sheaves on C_1, C_2 respectively. Let $E_i^{(q)}$ be the fibre of E_i at q, $i = 1, 2$. A *triple* on C is the datum of $(E_1, E_2, \overrightarrow{A}^{(q)})$, where $\overrightarrow{A}^{(q)} : E_1^{(q)} \to E_2^{(q)}$ is a homomorphism.

Definition 2.2.6. A *morphism of triples* $\beta : (F_1, F_2, \overrightarrow{D}^{(q)}) \to (E_1, E_2, \overrightarrow{A}^{(q)})$ consists of $\mathcal{O}_{C_i}$-module homomorphisms $\beta_i : F_i \to E_i$, $i = 1, 2$, such that there is a commutative diagram

$$\begin{tikzcd} F_1^{(q)} \arrow[r, "\beta_1\otimes\mathbb{C}^{(q)}"'] \arrow[d, "\overrightarrow{D}^{(q)}"'] & E_1^{(q)} \arrow[d, "\overrightarrow{A}^{(q)}"] \\ F_2^{(q)} \arrow[r, "\beta_2\otimes\mathbb{C}^{(q)}"'] & E_2^{(q)} \end{tikzcd} . \tag{2.7}$$

We now recall the equivalence of categories between torsion-free sheaves and triples.

Lemma 2.2.7 ([40], Lemma 2.3, [13], Theorem 6.5(1))**.** *Fix an orientation of C such that C_1 is the first component and C_2 is the second component. Then there is an equivalence between the category of torsion-free sheaves E as in Definition 1.4.2 and the category of triples $(E_1, E_2, \overrightarrow{A}^{(q)})$ as in Definition 2.2.5.*

Example 2.2.8. Consider the torsion-free sheaf $\mathcal{H}$ from Example 2.2.4. Then, since the restriction of $\mathcal{H}$ to C_2 is a torsion sheaf supported at q, the construction in the proof of Lemma 2.2.7 associates $\mathcal{H}$ to the triple $(H, 0, \overrightarrow{0})$.

Example 2.2.9. Let $\mathcal{L}$ be a rank-1 torsion-free sheaf on C. Then, by Lemma 2.2.7, the datum of $\mathcal{L}$ is equivalent to that of its restrictions L_1 and L_2, together with a map $\overrightarrow{A}^{(q)}$ whose rank is at most 1. If $\overrightarrow{A}^{(q)}$ has rank 1, then $\mathcal{L}$ is a line bundle and $\overrightarrow{A}^{(q)}$ is the canonical identification between L_1 and L_2 at q as in Remark 2.1.5, otherwise $\mathcal{L}$ is not locally free at q.

We give a stability condition on triples, depending on the polarization on C, and we give further examples in Sections 2.3, 2.4.

Remark 2.2.10. By Lemma 2.2.7, every torsion-free sheaf E gives rise to a triple $(E_1, E_2, \overrightarrow{A}^{(q)})$. Note that, if we choose C_2 as the first component and C_1 as the second component, Lemma 2.2.7 yields a triple $(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$, where $\dot{E}_1$, $\dot{E}_2$ are locally free sheaves on C_1, C_2 respectively, which we consider together with a homomorphism $\overleftarrow{B}^{(q)} : \dot{E}_2^{(q)} \to \dot{E}_1^{(q)}$.

We recall the construction which relates triples of the form $(E_1, E_2, \overrightarrow{A}^{(q)})$ to triples of the form $(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$.

Remark 2.2.11 ([40], Remark 2.4). Let V be a vector bundle on a smooth curve Y, let p be a point of Y and let H be a subspace of the fibre $V^{(p)}$ of V at p. Then there are two canonical constructions, called *Hecke modifications of vector bundles*, which are defined as follows:

(i) $\gamma : W \to V$, $\mathrm{Im}(W^{(p)}) = H$, where W is a vector bundle and γ is a homomorphism of vector bundles, which is an isomorphism outside p.

We obtain γ and W by letting $T := V^{(p)}/H$ and letting $j : V \to T$ be the canonical $\mathcal{O}_Y$-module homomorphism. Then we naturally obtain

$$W = \mathrm{Ker} j \subseteq V, \qquad \gamma : W \to V. \tag{2.8}$$

In fact, W is a subsheaf of V which is torsion-free. Since Y is smooth, W is also a vector bundle. The map γ is just the natural inclusion of the kernel of a homomorphism into its domain.

(ii) $\delta : V \to W$, $\mathrm{Ker}(\delta^{(p)}) = H$, where W is a vector bundle and δ is a homomorphism of vector bundles, which is an isomorphism outside p.

We obtain δ and W as follows: let $V^{(p)} \times (V^{(p)})^\vee \to k$ be the canonical dual pairing and let $H^\perp$ be the orthogonal of H under the dual pairing. Let $V^\vee$ be the dual of V and define the vector bundle $W^\vee$ and the homomorphism $\delta^\vee$ as in (2.8) so that

$$\delta^\vee : W^\vee \to V^\vee, \qquad \mathrm{Im}((W^{(p)})^\vee) = H^\perp.$$

These can be obtained as in (i). Let δ be the dual of $\delta^\vee$ and let W be the dual of $W^\vee$. Then $\delta : V \to W$ is a homomorphism of vector bundles and $\delta^{(p)}$ is the dual of $(\delta^{(p)})^\vee$, thus it satisfies

$$\mathrm{Ker}(\delta^{(p)}) = (\mathrm{Im}(\delta^{(p)})^\vee)^\perp = (H^\perp)^\perp = H,$$

hence we have δ and W with the required properties.

Remark 2.2.12 ([40], Remark 2.5). Given a triple $(E_1, E_2, \overrightarrow{A}^{(q)})$, the triple $(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$ can be obtained as follows. Consider the diagram

$$\begin{tikzcd} E_1^{(q)} \arrow[r, "i^{(q)}"] \arrow[d, "\overrightarrow{A}^{(q)}"'] & \dot{E}_1^{(q)} \\ E_2^{(q)} & \dot{E}_2^{(q)} \arrow[l, "j^{(q)}"] \arrow[u, "\overleftarrow{B}^{(q)}"'] \end{tikzcd} \tag{2.9}$$

where $j : \dot{E}_2 \to E_2$ (respectively $i : E_1 \to \dot{E}_1$) is the canonical Hecke modification such that $\mathrm{Im}(j^{(q)}) = \mathrm{Im}(\overrightarrow{A}^{(q)})$ (respectively $\mathrm{Ker}(i^{(q)}) = \mathrm{Ker}(\overrightarrow{A}^{(q)})$). The homomorphism $\overleftarrow{B}^{(q)} : \dot{E}_2^{(q)} \to \dot{E}_1^{(q)}$ is obtained as follows: for $x \in \dot{E_2}^{(q)}$ let $f \in {E_1}^{(q)}$ be such that $\overrightarrow{A}^{(q)}(f) = j^{(q)}(x)$. If we set $\overrightarrow{B}^{(q)}(x) = i^{(q)}(f)$, this is well defined.

Definition 2.2.13. The *Euler characteristic of a triple* $(E_1, E_2, \overrightarrow{A}^{(q)})$ is defined by

$$\chi_C(E_1, E_2, \overrightarrow{A}^{(q)}) = \chi_{C_1}(E_1) + \chi_{C_2}(E_2) - \mathrm{rank}\,(E_2). \tag{2.10}$$

Remark 2.2.14. Note that, if E_1, E_2 are rank-2 locally free sheaves on C_1, C_2 respectively, Definition 2.2.13 is compatible with the definitions of Euler characteristics of E_i as vector bundles on C_i, $i = 1, 2$, since we have

$$\chi_{C_1}(E_1)+\chi_{C_2}(E_2)-2 = \deg\,(E_1)+2(1-g_1)+\deg\,(E_2)+2(1-g_2)-2 = \deg E+2(1-g) = \chi_C(E).$$

Remark 2.2.15 ([40], Remark 2.11). Recalling Remark 2.2.12, we define the Euler characteristic of triples of the form $(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$ as

$$\chi_C(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)}) := \chi_C(E_1) + \chi_C(E_2) - \mathrm{rank}\,(E_1)$$

and we have $\chi_C(E_1, E_2, \overrightarrow{A}^{(q)}) = \chi_C(E) = \chi_C(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$.

2.3 Moduli spaces of semistable triples

We now define semistability for torsion-free sheaves and for the associated triples on the curve C having polarization α as in Definition 2.2.1.

Semistability for torsion-free sheaves and for triples

Definition 2.3.1. We define the *α-slope* of the torsion-free sheaf E on C as

$$\mu_\alpha(E) = \frac{\chi_C(E)}{r_\alpha(E)}.$$

for r_α as in (2.6).

Remark 2.3.2. Considering (1.29) for a one-nodal curve C of compact type, we have

$$\mu_p(E) = \frac{p(E,s)}{\operatorname{rank}_p(E)} = \mu_\alpha(E) + \deg\left(\mathcal{O}_C(1)\right)\cdot s,$$

in particular the difference between the p-slope and the α-slope does not depend on E.

Definition 2.3.3. We say that E is *α-semistable* (respectively *α-stable*) if

$$\mu_\alpha(F) \le \mu_\alpha(E) \qquad \text{(respectively } \mu_\alpha(F) < \mu_\alpha(E)\text{)}$$

for all nontrivial proper subsheaves $F \subset E$.

Remark 2.3.4 ([7], Remark 1.1.4). If $\chi_C(E)$ is odd and E is a α-semistable torsion-free sheaf of rank 2, then E is automatically α-stable since there is no subsheaf F of E such that $2\chi_C(F) = \chi_C(E)$.

Notation 2.3.5. From now on, we focus on triples of type $(E_1, E_2, \overrightarrow{A}^{(q)})$ and refer to triples of the form $(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$ only when it is necessary.

Recall that Remark 2.2.12 provides a way to pass from triples of the form $(E_1, E_2, \overrightarrow{A}^{(q)})$ to triples of the form $(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$.

Definition 2.3.6. Let E be a torsion-free sheaf. A *subbundle* F of E is a subsheaf F such that the quotient E/F is a torsion-free sheaf.

Definition 2.2.6, considering β_i as the inclusion of F_i into E_i, $i = 1, 2$, gives the following.

Definition 2.3.7. A triple $(F_1, F_2, \overrightarrow{D}^{(q)})$ is said to be a *subtriple* of $(E_1, E_2, \overrightarrow{A}^{(q)})$ if $\iota_i : F_i \to E_i$ is an inclusion of $\mathcal{O}_{C_i}$-modules and the following diagram commutes.

$$\begin{tikzcd} F_1^{(q)} \arrow[r, hook, "\iota_1\otimes\mathbb{C}^{(q)}"'] \arrow[d, "\overrightarrow{D}^{(q)}"'] & E_1^{(q)} \arrow[d, "\overrightarrow{A}^{(q)}"] \\ F_2^{(q)} \arrow[r, hook, "\iota_2\otimes\mathbb{C}^{(q)}"'] & E_2^{(q)} \end{tikzcd} \tag{2.11}$$

We say that $(F_1, F_2, \overrightarrow{D}^{(q)})$ is a *proper subtriple* if at least one submodule F_i is strictly contained in E_i.

Definition 2.3.8. Let E be a torsion-free sheaf of rank (r_1, r_2) on C. We define the slope μ_α of the triple $(E_1, E_2, \overrightarrow{A}^{(q)})$ associated to E as

$$\mu_\alpha(E_1, E_2, \overrightarrow{A}^{(q)}) := \frac{\chi_C(E_1, E_2, \overrightarrow{A}^{(q)})}{r_\alpha(E)}.$$

Definition 2.3.9. A triple $(E_1, E_2, \overrightarrow{A}^{(q)})$ is said to be α-*semistable* (respectively α-*stable*) if

$$\mu_\alpha(F_1, F_2, \overrightarrow{D}^{(q)}) \leq \mu_\alpha(E_1, E_2, \overrightarrow{A}^{(q)}) \quad \text{(respectively } \mu_\alpha(F_1, F_2, \overrightarrow{D}^{(q)}) < \mu_\alpha(E_1, E_2, \overrightarrow{A}^{(q)})\text{)} \tag{2.12}$$

for all nontrivial proper subtriples $(F_1, F_2, \overrightarrow{D}^{(q)})$ of $(E_1, E_2, \overrightarrow{A}^{(q)})$.

Notation 2.3.10. As we did in Section 1.1, in order to have different notations for spaces of torsion-free sheaves (without Higgs datum) and for spaces of torsion-free Hitchin pairs, we use the superscript ^{V} (e.g. $S^V, \mathcal{M}^V$) when we refer to moduli spaces of torsion-free sheaves or triples (not involving Higgs data), while we omit the superscript ^{V} when we discuss Hitchin pairs and Hitchin triples.

Notation 2.3.11. Until the end of the chapter, we assume that we are given a torsion free sheaf E on C of rank $(2, 2)$.

Stability of torsion-free sheaves on C corresponds to stability of the associated triples.

Remark 2.3.12. Let $S^V(2, \chi, \alpha)$ denote the set of all isomorphism classes of α-semistable torsion-free sheaves on C of rank $(2, 2)$ and Euler characteristic χ. Let E be a torsion-free sheaf such that $[E] \in S^V(2, \chi, \alpha)$. Then the triple corresponding to it is α-semistable and conversely, because of our definition of Euler characteristic (2.10).

Theorem 2.3.13 ([40], Theorem 3.1(a))**.** *Let $\chi \neq 0$ and let $\alpha = (\alpha_1, \alpha_2)$ be a polarization on C such that $\alpha_1\chi$ is not an integer. Let (b_1, b_2) be the unique tuple satisfying*

$$\alpha_1\chi < b_1 < \alpha_1\chi + 1, \qquad \alpha_2\chi + 1 < b_2 < \alpha_2\chi + 2, \qquad b_1 + b_2 = \chi + 2. \tag{2.13}$$

Let $[E] \in S^V(2, \chi, \alpha)$ and let $(E_1, E_2, \overrightarrow{A}^{(q)})$ be the triple corresponding to E. Then we must have $\operatorname{rank} \overrightarrow{A}^{(q)} \geq 1$ *and there are two possibilities: either*

$$\chi_{C_1}(E_1) = b_1, \chi_{C_2}(E_2) = b_2 \tag{2.14}$$

or

$$\chi_{C_1}(E_1) = b_1 + 1, \chi_{C_2}(E_2) = b_2 - 1. \tag{2.15}$$

Moreover, if rank $\overrightarrow{A}^{(q)} = 1$ *only* (2.14) *is possible.*

Corollary 2.3.14 ([40], Section 1, Corollary 3.1)**.** *In the hypotheses of Theorem 2.3.13, if* $[E] \in S^V(2, \chi, \alpha)$*, then either* E *is locally free or it is a torsion-free* $\mathcal{O}_C$*-module such that* $E_q \cong \mathcal{O}_{C,q} \oplus \mathfrak{m}_{C,q}$*. The choice of the polarization* α *on* C *prevents the existence of torsion-free* $\mathcal{O}_C$*-modules whose stalk at the node* q *is of the form* $\mathfrak{m}_{C,q} \oplus \mathfrak{m}_{C,q}$*.*

Remark 2.3.15. Bounds on $\chi_{C_i}(E_i)$ which are similar to the ones given in Theorem 2.3.13 are given in Favale-Brivio [25], Lemma 3.3 and they involve the rank of $\overrightarrow{A}^{(q)}$ explicitly. Let E be a α-semistable torsion-free sheaf of rank $(2, 2)$, for some polarization $\alpha = (\alpha_1, \alpha_2)$. Let $\chi = \chi_C(E)$. Then we have:

$$\alpha_1\chi \leq \chi_{C_1}(E_1) \leq \alpha_1\chi + \operatorname{rank} \overrightarrow{A}^{(q)}, \alpha_2\chi + 2 - \operatorname{rank} \overrightarrow{A}^{(q)} \leq \chi_{C_2}(E_2) \leq \alpha_2\chi + 2. \tag{2.16}$$

Referring to the arguments in the proof of Theorem 3.2.10, which are similar to the ones in the proof of Theorem 2.3.13, we have that (2.16) coincides with the bounds (3.15), (3.16) for rank $\overrightarrow{A}^{(q)} = 2$ (respectively with (3.17) for rank $\overrightarrow{A}^{(q)} = 1$).

An explicit description of loci which allow the existence of a polarization α such that the conditions (2.16) hold is given in [25], Lemma 3.4, which we consider for our case of rank-2 torsion-free sheaves E having nonzero Euler characteristic.

Example 2.3.16. Let E be a torsion-free sheaf of rank 2 and let $1 \leq k \leq 2$ be an integer. Then there exists a non-empty subset $W_{2,k}$ of $\mathbb{Z}^2$ such that for any pair $(\chi_1, \chi_2) \in W_{2,k}$, we can find a polarization $\alpha = (\alpha_1, \alpha_2)$ such that

$$\alpha_1\chi \leq \chi_1 \leq \alpha_1\chi + k, \quad \alpha_2\chi + 2 - k \leq \chi_2 \leq \alpha_2\chi + 2, \quad \chi = \chi_1 + \chi_2 - 2. \tag{2.17}$$

If $\chi > 0$, then we need that the system

$$\begin{cases} \frac{\chi_1 - k}{\chi} \leq \alpha_1 \leq \frac{\chi_1}{\chi} \\ \frac{\chi_2 - 2}{\chi} \leq \alpha_2 \leq \frac{\chi_2 + k - 2}{\chi} \\ \alpha_1 + \alpha_2 = 1, \\ 0 < \alpha_i < 1, \alpha_i \in \mathbb{Q} \end{cases}$$

has solutions, which occurs if and only if $\chi_1 > 0, \chi_2 > 2 - k$. Similarly, if $\chi < 0$, then we need that the system

$$\begin{cases} \frac{\chi_1}{\chi} \leq \alpha_1 \leq \frac{\chi_1 - k}{\chi} \\ \frac{\chi_2 - 2 + k}{\chi} \leq \alpha_2 \leq \frac{\chi_2 - 2}{\chi} \\ \alpha_1 + \alpha_2 = 1, \\ 0 < \alpha_i < 1, \alpha_i \in \mathbb{Q} \end{cases}$$

has solutions, which occurs if and only if $\chi_1 < k, \chi_2 < 2$.

Remark 2.3.17. In Section 2.4.4, especially Lemma 2.4.6, we discuss the relation between α-stability of triples and stability of vector bundles appearing in these triples and we obtain a description which is similar to [25], Corollary 3.5.

We introduce the following notations for moduli spaces of α-semistable triples.

Notation 2.3.18. Let $\mathcal{M}_{12}^V(2, \chi)$ be the moduli space of α-semistable triples $(E_1, E_2, \overrightarrow{A}^{(q)})$ of rank 2 and Euler characteristic $\chi = \chi_C(E_1, E_2, \overrightarrow{A}^{(q)})$, which we simply denote by $\mathcal{M}_{12}^V$, satisfying

$$\alpha_1\chi < \chi_{C_1}(E_1) < \alpha_1\chi + 1, \qquad \alpha_2\chi + 1 < \chi_{C_2}(E_2) < \alpha_2\chi + 2. \tag{2.18}$$

Let $\mathcal{M}_{21}^V(2, \chi)$ be the moduli space of α-semistable triples $(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$ of rank 2 and Euler characteristic $\chi = \chi_C(\dot{E}_1, \dot{E}_2, \overleftarrow{B}^{(q)})$, which we simply denote by $\mathcal{M}_{21}^V$, satisfying

$$\alpha_1\chi + 1 < \chi_{C_1}(\dot{E}_1) < \alpha_1\chi + 2, \qquad \alpha_2\chi < \chi_{C_2}(\dot{E}_2) < \alpha_2\chi + 1. \tag{2.19}$$

Remark 2.3.19. Moduli spaces of α-semistable triples are constructed in [40], Theorem 5.3.

We also define moduli spaces of semistable vector bundles on C_i.

Notation 2.3.20. We denote by $\mathcal{M}_1^V$ the moduli space of semistable vector bundles of rank 2 on C_1. We denote by $\mathcal{M}_2^V$ the moduli space of semistable vector bundles of rank 2 on C_2.

2.4 Moduli spaces of triples having odd Euler characteristic

Notation 2.4.1. Until the end of the chapter, we assume that the Euler characteristic $\chi_C(E)$ of E is odd, in particular we assume that $\chi_{C_1}(E_1)$ is odd and $\chi_{C_2}(E_2)$ is even. As in Notation 2.3.18, we focus on the Euler characteristic of vector bundles E_1 and E_2, rather than on their degree. Recall that we obtain an equivalent notion of stability, as we saw in Remark 1.4.7.

Theorem 4.1 in [40] yields the following description of the moduli space of α-semistable rank-2 torsion-free sheaves on C.

Theorem 2.4.2. *Consider a polarization $\alpha = (\alpha_1, \alpha_2)$ on C such that $\alpha_1\chi$ is not an integer. Then the moduli space $\mathcal{M}^V(2, \chi, \alpha)$ of α-semistable rank-2 torsion-free sheaves of odd Euler characteristic χ on C is a reduced, connected, projective scheme with exactly two smooth irreducible components $\mathcal{M}_{12}^V$ and $\mathcal{M}_{21}^V$ intersecting transversally along a smooth divisor $\mathcal{N}^V$ parametrizing triples of the form $(E_1, E_2, \overrightarrow{A}^{(q)})$, where $\overrightarrow{A}^{(q)}$ has rank 1.*

In particular, assuming that $\alpha_1\chi$ is not an integer (together with the assumption that χ is odd) implies that all α-semistable torsion-free sheaves (equivalently, all α-semistable triples) are stable, so $\mathcal{M}_{12}^V$ and $\mathcal{M}_{21}^V$ are smooth.

Nagaraj-Seshadri characterize the smooth divisor $\mathcal{N}^V$ in terms of a product of smooth moduli spaces P_i^V of stable vector bundles on C_i having parabolic structure only at the node q as we recall in Section 2.5.

Stability of triples implies stability of vector bundles in the triples

Theorem 2.4.3 ([40], Theorem 5.1)**.** *Let* $\alpha = (\alpha_1, \alpha_2)$ *be a polarization on* C *such that* $\alpha_1\chi$ *is not an integer. Let* $(E_1, E_2, \overrightarrow{A}^{(q)})$ *be a* α*-semistable triple of rank (2,2) such that* $\chi_C(E_1, E_2, \overrightarrow{A}^{(q)}) = \chi$ *and*

$$\alpha_1\chi < \chi_{C_1}(E_1) < \alpha_1\chi + 1, \quad \alpha_2\chi + 1 < \chi_{C_2}(E_2) < \alpha_2\chi + 2. \tag{2.20}$$

Then we have that E_1 *is a stable vector bundle on* C_1 *and* E_2 *is a semistable vector bundle on* C_2 *in the sense of Definition 1.1.3.*

Relation between stability of vector bundles and stability of the triples in which they appear

We now consider when the converse of Theorem 2.4.3 holds. Recall that assuming that $\chi_{C_1}(E_1)$ is odd implies that E_1 cannot be a strictly semistable vector bundle.

We further study a case which is suggested by the statement of [10], Lemma 2.3.

Remark 2.4.4. Let $(E_1, E_2, \overrightarrow{A}^{(q)})$ be a triple admitting a subtriple of the form $(E_1, L_2, \overrightarrow{A}^{(q)})$, for a subbundle L_2 of E_2 such that

$$\chi_{C_2}(L_2) = \chi_{C_2}(E_2)/2. \tag{2.21}$$

If $(E_1, E_2, \overrightarrow{A}^{(q)})$ is α-semistable, $\overrightarrow{A}^{(q)}$ has rank 1 and (2.21) yields the following stability check

$$\frac{\chi_{C_1}(E_1) + \chi_{C_2}(L_2) - 1}{2\alpha_1 + \alpha_2} \leq \frac{\chi}{2}.$$

By Definition 2.2.1 and (2.21), this implies

$$\frac{\chi_{C_1}(E_1) + \chi_{C_2}(L_2) - 1}{1 + \alpha_1} = \frac{\chi_{C_1}(E_1) + \chi_{C_2}(E_2)/2 - 1}{1 + \alpha_1} \leq \frac{\chi}{2}$$

yielding by (2.10)

$$\frac{\chi_{C_1}(E_1) + \chi}{2(1 + \alpha_1)} \leq \frac{\chi}{2},$$

thus

$$\chi_{C_1}(E_1) \leq \alpha_1\chi. \tag{2.22}$$

However, since (2.20) requires $\chi_{C_1}(E_1) > \alpha_1\chi$, triples $(E_1, E_2, \overrightarrow{A}^{(q)})$ admitting subtriples of the form $(E_1, L_2, \overrightarrow{A}^{(q)})$ are unstable. Hence there is no point of $\mathcal{M}_{12}^V$ associated to them.

Remark 2.4.5. On the other hand, recall the α-semistability condition (2.12), in particular (2.10). Since E_1 is stable and E_2 is semistable, in all cases which do not involve strictly semistable vector bundles E_2, the semistability conditions of E_1 and E_2 as vector bundles on C_1 and C_2, respectively, imply directly that triples of the form $(E_1, E_2, \overrightarrow{A}^{(q)})$ are α-stable.

Lemma 2.4.6. *Let E_1 be a rank-2 stable vector bundle on C_1 having odd Euler characteristic, let E_2 be a rank-2 semistable vector bundle on C_2 having even Euler characteristic and assume that $\chi_{C_i}(E_i)$ satisfy inequalities* (2.20) *for $i = 1, 2$. Then two cases are possible:*

(a) if E_2 does not admit any subbundles having its same slope, then for any nonzero linear map $\overrightarrow{A}^{(q)} : E_1^{(q)} \to E_2^{(q)}$ the triple $(E_1, E_2, \overrightarrow{A}^{(q)})$ is α-stable.

(b) if E_2 admits a subbundle having its same slope, then for any isomorphism $\overrightarrow{A}^{(q)} : E_1^{(q)} \to E_2^{(q)}$, the triple $(E_1, E_2, \overrightarrow{A}^{(q)})$ is α-stable.

Proof. Every triple $(E_1, E_2, \overrightarrow{A}^{(q)})$ which is not as the ones in point (a) is a α-stable triple by Remark 2.4.5.

Moreover, if $\overrightarrow{A}^{(q)}$ is an isomorphism, then $(E_1, L_2, \overrightarrow{A}^{(q)})$ is not a subtriple of $(E_1, E_2, \overrightarrow{A}^{(q)})$, so we exclude triples of the form discussed in Remark 2.4.4 and we have the claim. □

Remark 2.4.7. The map

$$\psi : \mathcal{M}_{12}^V \to \mathcal{M}_1^V \times \mathcal{M}_2^V$$

is surjective by Lemma 2.4.6 since, given any (E_1, E_2) in $\mathcal{M}_1^V \times \mathcal{M}_2^V$ and considering maps $\overrightarrow{A}^{(q)}$ of rank 2, we always have that $(E_1, E_2, \overrightarrow{A}^{(q)})$ is a triple of $\mathcal{M}_{12}^V$.

We now consider a special case of a result by Teixidor i Bigas [14], Proposition 2.1, yielding the dimension of $\mathcal{M}_{12}^V$.

Proposition 2.4.8. *Let C be a one-nodal curve of compact type having genus g. Then the dimension of $\mathcal{M}_{12}^V$ (respectively, the dimension of $\mathcal{M}_{21}^V$) is $4g-3$.*

Proof. In order to compute the dimension of $\mathcal{M}_{12}^V$, we consider the open dense subvariety which is given by α-stable triples in which both E_1 and E_2 are stable vector bundles on C_1 and C_2, respectively, and the map $\overrightarrow{A}^{(q)}$ has rank 2. Since for stable bundles we have $\mathrm{Aut}(E_i) \cong \mathbb{C}^*$, if two triples $(E_1, E_2, \overrightarrow{A}^{(q)})$ and $(E_1, E_2, (\overrightarrow{A}^{(q)})')$ satisfy, for $\lambda \in \mathrm{Aut}E_1$ and $\rho \in \mathrm{Aut}E_2$ the diagram

$$\begin{tikzcd} E_1^{(q)} \arrow[r, "\overrightarrow{A}^{(q)}"] \arrow[d, "\lambda"'] & E_2^{(q)} \arrow[d, "\rho"] \\ E_1^{(q)} \arrow[r, "(\overrightarrow{A}^{(q)})'"'] & E_2^{(q)} \end{tikzcd}, \tag{2.23}$$

then we have $(\overrightarrow{A}^{(q)})' = \frac{\rho}{\lambda}\overrightarrow{A}^{(q)}$. Conversely, if two triples $(E_1, E_2, \overrightarrow{A}^{(q)})$ and $(E_1, E_2, (\overrightarrow{A}^{(q)})')$ are such that $(\overrightarrow{A}^{(q)})' = \gamma\overrightarrow{A}^{(q)}$, for $\gamma \in \mathbb{C}^*$, then $(E_1, E_2, \overrightarrow{A}^{(q)})$ and $(E_1, E_2, (\overrightarrow{A}^{(q)})')$ belong to the same isomorphism class. Hence two triples are in the same isomorphism class if and only if $(\overrightarrow{A}^{(q)})'$ is a scalar multiple of $\overrightarrow{A}^{(q)}$. Fixing bases of $E_1^{(q)}$ and $E_2^{(q)}$, $\overrightarrow{A}^{(q)}$ can be identified with an element of $\mathrm{GL}(2, \mathbb{C})$. By the argument above, we have the following equivalence

$$[(E_1, E_2, \overrightarrow{A}^{(q)})] = [(E_1, E_2, \gamma\overrightarrow{A}^{(q)})]$$

for $\gamma \in \mathbb{C}^*$, hence each distinct isomorphism class of triples is in one to one correspondence with a unique element in $\mathrm{PGL}(2, \mathbb{C})$.

Thus we have, considering (1.3):

$$\dim \mathcal{M}_{12}^V = \dim \mathcal{M}_1^V + \dim \mathcal{M}_2^V + \dim \mathrm{PGL}(2, \mathbb{C}) = (4g_1-3)+(4g_2-3)+3 = 4g-3.$$

□

Remark 2.4.9. Since $\mathcal{N}^V$ is a divisor in $\mathcal{M}_{12}^V$ (respectively in $\mathcal{M}_{21}^V$), we have

$$\dim \mathcal{N}^V = 4g-4.$$

2.5 Triples with $\overrightarrow{A}^{(q)}$ of rank 1 in terms of vector bundles with parabolic structure

Recalling Theorem 2.4.2, we can also interpret the intersection

$$\mathcal{N}^V = \mathcal{M}_{12}^V \cap \mathcal{M}_{21}^V$$

in terms of vector bundles on the components C_i having parabolic structure only at the preimages of q via the normalization map $\nu : C_1 \sqcup C_2 \to C$. We recall the main definitions for the case of our interest, the general case being presented in [47].

Definition 2.5.1. Let Y be a smooth curve, let p be a point of Y and let V be a vector bundle of rank 2 on Y. A *p-parabolic bundle* is the datum of:

- a flag $\mathcal{F}$ of subspaces of the fibre of V at p:

$$0 \subset F^2V^{(p)} \subset F^1V^{(p)} = V^{(p)},$$

- constants $\beta := (\beta_1, \beta_2)$, called the *weights* of the parabolic structure, such that

$$0 < \beta_1 < \beta_2 < 1,$$

  where β_i is associated to $F^iV^{(p)}$.

We simply denote p-parabolic bundles by $(V, 0 \subset F^2V^{(p)} \subset V^{(p)})$.

Definition 2.5.2. Let V be a p-parabolic bundle. We say that W is a *p-parabolic subbundle* of V if W is a subbundle of V having the following parabolic structure at p:

- a flag

$$0 = F^2W^{(p)} \subset F^1W^{(p)} = W^{(p)},$$

  where $F^iW^{(p)} = W^{(p)} \cap F^iV^{(p)}$

- a weight β_2, since $i = 2$ is the maximum index such that we have the inclusion $W^{(p)} \subset F^2V^{(p)}$, while $W^{(p)} \not\subset F^3V^{(p)} = 0$.

Definition 2.5.3. Let V be a p-parabolic bundle (respectively, let W be a p-parabolic subbundle). Then the *parabolic degree* of V, respectively W, is defined as

$$\text{pardeg}\,(V) = \deg\,(V) + \beta_1 + \beta_2,$$

respectively

$$\text{pardeg}\,(W) = \deg\,(W) + \beta_2.$$

The *parabolic slope* of V is defined as

$$\text{par}\mu(V) = \frac{\text{pardeg}\,(V)}{\text{rank}\,(V)},$$

respectively

$$\text{par}\mu(W) = \frac{\text{pardeg}\,(W)}{\text{rank}\,(W)}.$$

Definition 2.5.4. Let V be a p-parabolic bundle on Y. Then V is *parabolic semistable* (respectively *parabolic stable*) if, for any p-parabolic subbundle W of V, we have par $\mu(W) \leq$ par $\mu(V)$ (respectively par$\mu(W) <$ par$\mu(V)$).

We now apply the discussion above to our case. Let C be a one-nodal curve of compact type having polarization α such that $\alpha_1 < \alpha_2$. We obtain the following, from [40], Theorem 6.1.

Theorem 2.5.5. *Consider a polarization $\alpha = (\alpha_1, \alpha_2)$ on C such that we have $\alpha_1 < \alpha_2$ and that $\alpha_1\chi$ is not an integer. Let q be the node of C. Let P_1^V be the moduli space of semistable q-parabolic bundles of rank 2 on C_1 given by $(E_1, 0 \subset F^2E_1^{(q)} \subset E_1^{(q)})$ with parabolic weights $(\alpha_1/2, \alpha_2/2)$. Assume that the degree of E_1 equals $\chi_{C_1}(E_1)$, in particular that it is odd and that it satisfies* (2.20)*. Let P_2^V be the moduli space of semistable q-parabolic bundles of rank 2 on C_2 given by $(E_2, 0 \subset F^2E_2^{(q)} \subset E_2^{(q)})$ with parabolic weights $(\alpha_1/2, \alpha_2/2)$. Assume that the degree of E_2 equals $\chi_{C_2}(E_2)$, in particular that it is even and that it satisfies* (2.20).

Then P_1^V and P_2^V are smooth. Moreover, we have an isomorphism

$$\gamma : \mathcal{N}^V \xrightarrow{\sim} P_1^V \times P_2^V. \tag{2.24}$$

given by:

$$\mathcal{N}^V \ni (E_1, E_2, \overrightarrow{A}^{(q)}) \mapsto (E_1, 0 \subset F^2E_1^{(q)} \subset E_1^{(q)}) \times (E_2, 0 \subset F^2E_2^{(q)} \subset E_2^{(q)}) \in P_1^V \times P_2^V. \tag{2.25}$$

where $F^2E_1^{(q)} = Ker\overrightarrow{A}^{(q)}$ *and* $F^2E_2^{(q)} = Im\overrightarrow{A}^{(q)}$.

Remark 2.5.6. Let E_2 be a vector bundle appearing in unstable triples as in Remark 2.4.4. Then q-parabolic vector bundles having underlying vector bundles E_2 are also unstable as parabolic vector bundles, thus they do not appear in P_2^V. In fact, consider the q-parabolic bundle

$$(E_2, 0 \subset L_2^{(q)} \subset E_2^{(q)}),$$

where L_2 is such that $\chi_{C_2}(L_2) = \chi_{C_2}(E_2)/2$ and $L_2^{(q)}$ is associated to the weight $\frac{\alpha_2}{2}$, in the same hypotheses of Theorem 2.5.5. Then we have

$$\text{par}\mu(L_2) = \chi_{C_2}(L_2) + \frac{\alpha_2}{2} = \frac{\chi_{C_2}(E_2)}{2} + \frac{\alpha_2}{2}$$

$$\text{par}\mu(E_2) = \frac{\chi_{C_2}(E_2)}{2} + \frac{1}{4}.$$

Asking $\text{par}\mu(L_2) \leq \text{par}\mu(E_2)$ is equivalent to asking

$$\frac{\alpha_2}{4} \leq \frac{\alpha_1}{4},$$

which is not possible, since $\alpha_2 > \alpha_1$.

In particular, there are no strictly semistable q-parabolic vector bundles.

We also consider an alternative computation of the dimension of $\mathcal{N}^V$, which agrees with Proposition 2.4.8 and Remark 2.4.9.

Remark 2.5.7. By Mehta-Seshadri [34], Theorem 4.1, referring to the flag variety $\mathcal{F}$ in Definition 2.5.1, we have

$$\dim P_i^V = r^2(g_i - 1) + 1 + \dim \mathcal{F}.$$

Since we have $\dim \mathcal{F} = 1$, in our hypotheses this implies

$$\dim P_1^V = \dim P_2^V = 4g_i - 2,$$

yielding, once again,

$$\dim \mathcal{M}_{12}^V = (\dim P_1^V + \dim P_2^V) + 1 = 4g - 3.$$

Chapter 3

Higgs data on a one-nodal curve of compact type

We now consider how the tools introduced in Chapter 1 and Chapter 2 adapt, endowing torsion-free sheaves on the curve C of compact type with the Higgs datum.

3.1 Torsion free Hitchin pairs on a one-nodal curve of compact type

We first give an equivalence of categories corresponding to Lemma 2.2.7.

Equivalence of categories: torsion free Hitchin pairs and Hitchin triples

We fix a line bundle L on C such that the degree of its restrictions to C_i is greater or equal to the degree of K_{C_i}. Recall the definition of torsion-free Hitchin pairs Definition 1.4.9. Since C is of compact type, Remark 2.1.5 yields a canonical identification, which we call l, between the fibres $L_1^{(q)}$ and $L_2^{(q)}$ of the restrictions L_1 and L_2 of L at q. So we associate to L the triple (L_1, L_2, l), for

$$l : L_1^{(q)} \xrightarrow{\sim} L_2^{(q)} \tag{3.1}$$

a canonical isomorphism.

Definition 3.1.1. A *Hitchin triple* is $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)} \otimes l)$, which we denote by $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, where $\hat{E}_i := (E_i, \phi_i)$ are locally free sheaves E_i on C_i together with Higgs fields $\phi_i : E_i \to E_i \otimes L_i$, $i = 1, 2$, such that the following diagram commutes:

$$\begin{tikzcd} E_1^{(q)} \arrow[r, "\phi_1^{(q)}"] \arrow[d, "\overrightarrow{A}^{(q)}"'] & E_1^{(q)} \otimes L_1^{(q)} \arrow[d, "\overrightarrow{A}^{(q)} \otimes l"] \\ E_2^{(q)} \arrow[r, "\phi_2^{(q)}"'] & E_2^{(q)} \otimes L_2^{(q)} \end{tikzcd} \tag{3.2}$$

Remark 3.1.2. Commutativity of (3.2) is equivalent to

$$(\overrightarrow{A}^{(q)} \otimes l)\ \phi_1^{(q)} = \phi_2^{(q)}\ \overrightarrow{A}^{(q)}. \tag{3.3}$$

Considering Lemma 2.2.7 and endowing E with the datum of the morphism ϕ, we have the following.

Lemma 3.1.3 ([7], Lemma 3.1.3, [13], Theorem 6.5(2))**.** *Fix an orientation of C such that C_1 is the first component and C_2 is the second component. There is an equivalence between the category of torsion-free Hitchin pairs and the category of Hitchin triples $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$.*

Proof. Given a torsion-free sheaf E we obtain, by Lemma 2.2.7, a triple $(E_1, E_2, \overrightarrow{A}^{(q)})$. As discussed above, we can associate the line bundle L to (L_1, L_2, l), for l as in (3.1).

As in (2.7), the morphism ϕ induces homomorphisms of $\mathcal{O}_{C_i}$-modules $\phi_i : E_i \to E_i \otimes L_i$, $i = 1, 2$, which make diagram (3.2) commute. Thus, the Hitchin pair $\hat{E}$ gives rise to the Hitchin triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)} \otimes l)$.

Conversely, a Hitchin triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)} \otimes l)$ gives rise to a Hitchin pair $\hat{E}$: a triple $(E_1, E_2, \overrightarrow{A}^{(q)})$ gives rise to a torsion-free sheaf E as in Lemma 2.2.7 and, as in (2.7), ϕ_i together with the identification l yield a homomorphism $\phi : E \to E \otimes L$. $\square$

Example 3.1.4. Consider the torsion-free sheaf $\mathcal{H}$ from Example 2.2.8. Endowing it with $\phi : \mathcal{H} \to \mathcal{H} \otimes L$, we obtain a Hitchin pair and the construction in the proof of Lemma 3.1.3 associates $\hat{\mathcal{H}}$ to the triple $(\hat{H}, 0, \overrightarrow{0})$.

Example 3.1.5. Let $L_i \cong K_{C_i}$. Given any semistable Higgs bundles $\hat{E}_1$ on C_1, $\hat{E}_2$ on C_2, there is at least an associated torsion-free Hitchin pair, since we can always consider the Hitchin triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ in which the map $\overrightarrow{A}^{(q)}$ is the zero map. However, as we see in Section 3.2, we will focus on maps $\overrightarrow{A}^{(q)}$ having rank at least one by a stability requirement (Theorem 3.2.10). This implies extra-requirements on $\phi_i^{(q)}$ that we will discuss in Section 4.2.

Notation 3.1.6. From now on, we simply call "Higgs bundles" the data that we should call (paralleling what we did in Definition 2.2.5) "locally free sheaves on C_i with Higgs datum".

We now define the Hecke modification for Hitchin triples, which enables us to extend the correspondence in (2.9) including Higgs fields at q. Consider again (3.2) in the following form:

$$\begin{tikzcd} E_2^{(q)} \arrow[d, "\phi_2^{(q)}"] & E_1^{(q)} \arrow[l, "\overrightarrow{A}^{(q)}"'] \arrow[d, "\phi_1^{(q)}"] & \mathrm{Ker}(\overrightarrow{A}^{(q)}) \arrow[l, hook'] \arrow[d] \\ E_2^{(q)} \otimes L_2^{(q)} & E_1^{(q)} \otimes L_1^{(q)} \arrow[l, "\overrightarrow{A}^{(q)} \otimes l"] & \mathrm{Ker}(\overrightarrow{A}^{(q)} \otimes l) \arrow[l, hook'] \end{tikzcd} \quad . \tag{3.4}$$

We have that $\phi_1^{(q)}$ restricts to $\phi_1^{(q)} : \mathrm{Ker}(\overrightarrow{A}^{(q)}) \to \mathrm{Ker}(\overrightarrow{A}^{(q)} \otimes l)$. Considering the Hecke modification $i : E_1 \to \dot{E}_1$ such that $\mathrm{Ker}(\overrightarrow{A}^{(q)}) = \mathrm{Ker}(i^{(q)})$, diagram (3.4) yields the following commutative diagram:

$$\begin{tikzcd} \mathrm{Ker}(i^{(q)}) \arrow[r, hook] \arrow[d, "\phi_1^{(q)}"] & E_1^{(q)} \arrow[r, "i^{(q)}"] \arrow[d, "\phi_1^{(q)}"] & \dot{E}_1{}^{(q)} \arrow[d] \\ \mathrm{Ker}(i^{(q)} \otimes \mathrm{id}) \arrow[r, hook] & E_1{}^{(q)} \otimes L_1^{(q)} \arrow[r, "i^{(q)} \otimes \mathrm{id}"'] & \dot{E}_1{}^{(q)} \otimes L_1^{(q)} \end{tikzcd} \quad . \tag{3.5}$$

So the map $\phi_1^{(q)}$ restricts to $\dot{\phi}_1^{(q)} : \dot{E}_1{}^{(q)} \to \dot{E}_1{}^{(q)} \otimes L_1^{(q)}$. Since, by the definition of the Hecke modification, at any point $x \in C_1 \setminus \{q\}$ we have $\dot{E}_1{}^{(x)} \cong E_1^{(x)}$, the morphism ϕ_1 induces the morphism $\dot{\phi}_1$.

Definition 3.1.7. The *Hecke modification of a Higgs bundle* $\hat{E}_1 = (E_1, \phi_1)$ is the Higgs bundle $\widehat{\dot{E}_1} = (\dot{E}_1, \dot{\phi}_1)$ such that diagram (3.5) commutes.

Similarly, we obtain the Hecke modification $\widehat{\dot{E}_2}$ of $\hat{E}_2$.

Hence Remark 2.2.10 extends as follows.

Remark 3.1.8 ([7], Section 3.4). To the Hitchin pair $\hat{E}$ is also associated the triple $(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})$ and the triples are related as in (3.6), where $j : \dot{E}_2 \to E_2$ (respectively $i : E_1 \to \dot{E}_1$) is the canonical Hecke modification such that $\mathrm{Im}(j^{(q)}) = \mathrm{Im}\overrightarrow{A}^{(q)}$ (respectively $\mathrm{Ker}(i^{(q)}) = \mathrm{Ker}\overrightarrow{A}^{(q)}$) and $\mathrm{Ker}(j^{(q)}) = \mathrm{Ker}\overleftarrow{B}^{(q)}$ (respectively $\mathrm{Im}(i^{(q)}) = \mathrm{Im}\overleftarrow{B}^{(q)}$) and $\dot{\phi}_1$, $\dot{\phi}_2$ are the Hecke modifications of ϕ_1 and ϕ_2, which are constructed as in Definition 3.1.7.

$$\begin{tikzcd}
\dot{E}_2^{(q)} \arrow[rr, "\overleftarrow{B}^{(q)}"] \arrow[dr, "\dot{\phi}_2^{(q)}"'] \arrow[dd, "j^{(q)}"'] & & \dot{E}_1^{(q)} \arrow[dr, "\dot{\phi}_1^{(q)}"] & \\
& \dot{E}_2^{(q)} \otimes L_2^{(q)} \arrow[rr, "\overleftarrow{B}^{(q)}\otimes l^{-1}"] \arrow[dd, "j^{(q)}\otimes\mathrm{id}"' near start] & & \dot{E}_1^{(q)} \otimes L_1^{(q)} \\
E_2^{(q)} \arrow[dr, "\phi_2^{(q)}"'] & & E_1^{(q)} \arrow[ll, "\overrightarrow{A}^{(q)}" near start] \arrow[uu, "i^{(q)}"' near end] \arrow[dr, "\phi_1^{(q)}"] & \\
& E_2^{(q)} \otimes L_2^{(q)} & & E_1^{(q)} \otimes L_1^{(q)} \arrow[ll, "\overrightarrow{A}^{(q)}\otimes l"] \arrow[uu, "i^{(q)}\otimes\mathrm{id}"']
\end{tikzcd} \tag{3.6}$$

Definition 3.1.9. We define the *Euler characteristic of a Hitchin triple* $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ as the Euler characteristic of the underlying triple $(E_1, E_2, \overrightarrow{A}^{(q)})$, i.e.

$$\chi_C(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)}) := \chi_C(E_1, E_2, \overrightarrow{A}^{(q)}) \tag{3.7}$$

Remark 3.1.10. Definition 3.1.9 naturally yields, as in Remark 2.2.15,

$$\chi_C(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)}) = \chi_C(\hat{E}) = \chi_C(\widehat{\dot{E}}_1, \widehat{\dot{E}}_2, \overleftarrow{B}^{(q)}).$$

3.2 Moduli spaces of semistable Hitchin triples on a one-nodal curve of compact type

We now define semistability for rank-2 torsion-free Hitchin pairs and for the associated Hitchin triples on the curve C having polarization α as in Definition 2.2.1.

Semistability for Hitchin pairs and for Hitchin triples

Definition 3.2.1. We define the *α-slope* of a torsion-free Hitchin pair (E, ϕ) as

$$\mu_\alpha(E) = \frac{\chi_C(E)}{r_\alpha(E)}.$$

Definition 3.2.2. We say that (E, ϕ) is *α-semistable* (respectively *α-stable*) if $\mu_\alpha(F) \leq \mu_\alpha(E)$ (respectively $\mu_\alpha(F) < \mu_\alpha(E)$) for all ϕ-invariant subsheaves $F \subset E$.

As in the classical case Definition 1.2.3, stability of Hitchin pairs is only checked on ϕ-invariant subsheaves, that is subsheaves F of E satisfying $\phi(F) \subset F \otimes L$. Moreover, it is enough to check stability on ϕ-invariant subbundles F of E (in the sense of Definition 2.3.6).

Lemma 3.2.3 ([7], Lemma 2.1.3)**.** *A torsion-free Hitchin pair (E, ϕ) is α-semistable (respectively α-stable) if and only if $\mu_\alpha(F) \leq \mu_\alpha(E)$ (respectively $\mu_\alpha(F) < \mu_\alpha(E)$) for all ϕ-invariant subbundles $F \subset E$.*

Proof. If (E, ϕ) is a α-semistable (respectively, α-stable) torsion-free Hitchin pair, we have that $\mu_\alpha(F) \leq \mu_\alpha(E)$ (respectively, $\mu_\alpha(F) < \mu_\alpha(E)$) for all ϕ-invariant subsheaves F of E, so in particular we have $\mu_\alpha(F) \leq \mu_\alpha(E)$ (respectively $\mu_\alpha(F) < \mu_\alpha(E)$) for all ϕ-invariant subbundles F of E.

Conversely, assume that $\mu_\alpha(F) \leq \mu_\alpha(E)$ for all ϕ-invariant subbundles F of E. Let G be a ϕ-invariant subsheaf of E. Then we need to show that $\mu_\alpha(G) \leq \mu_\alpha(E)$ (respectively, that $\mu_\alpha(G) < \mu_\alpha(E)$). Similarly to [32], Definition 8.20, letting H be the subbundle of E generically generated by G, we have $\operatorname{rank} H = \operatorname{rank} G$ and $\chi_C(H) \geq \chi_C(G)$, hence $\mu_\alpha(G) \leq \mu_\alpha(H)$.

Let $q : E \to E/G$ be the quotient map and let $\overline{\phi} : E/G \to (E/G) \otimes L$ be the restriction of ϕ to E/G. Then the following diagram commutes.

$$\begin{tikzcd} E \arrow[r, "\phi"'] \arrow[d, "q"'] & E \otimes L \arrow[d, "q\otimes \mathrm{id}"] \\ E/G \arrow[r, "\overline{\phi}"'] & (E/G) \otimes L \end{tikzcd}$$

Since we have that the torsion subsheaf of E/G maps into the torsion subsheaf of $(E/G) \otimes L$ under $\overline{\phi}$, we have that $\phi(H) \subset H \otimes L$. Thus H is a

ϕ-invariant subbundle of E, so by assumption we have that $\mu_\alpha(H) \leq \mu_\alpha(E)$, which implies that $\mu_\alpha(G) \leq \mu_\alpha(E)$ (respectively, that $\mu_\alpha(G) < \mu_\alpha(E)$). $\square$

We now consider torsion-free Hitchin pairs on C in terms of Hitchin triples.

Notation 3.2.4. From now on, we focus on Hitchin triples of type $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ and refer to Hitchin triples of type $(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})$ only when necessary.

Recall that Remark 3.1.8 provides a way to pass from Hitchin triples of the form $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ to Hitchin triples of the form $(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})$.

Definition 3.2.5. A Hitchin triple $(\hat{F}_1, \hat{F}_2, \overrightarrow{D}^{(q)})$ is said to be a *Hitchin subtriple* of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ if $\hat{F}_i$ is a Higgs subbundle of $\hat{E}_i$, $i = 1, 2$, and the following diagram commutes

$$\begin{tikzcd} F_1^{(q)} \arrow[r, hook] \arrow[d, "\overrightarrow{D}^{(q)}"'] & E_1^{(q)} \arrow[d, "\overrightarrow{A}^{(q)}"] \\ F_2^{(q)} \arrow[r, hook] & E_2^{(q)} \end{tikzcd} \quad . \tag{3.8}$$

We say that $(\hat{F}_1, \hat{F}_2, \overrightarrow{D}^{(q)})$ is a *proper Hitchin subtriple* if at least one subundle F_i is strictly contained in E_i.

Definition 3.2.6. Let $\hat{E}$ be a torsion-free Hitchin pair of rank (r_1, r_2) on C. We define the *slope* μ_α *of the triple* $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ associated to $\hat{E}$ as

$$\mu_\alpha(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)}) := \mu_\alpha(E_1, E_2, \overrightarrow{A}^{(q)}) = \frac{\chi_C(E_1, E_2, \overrightarrow{A}^{(q)})}{r_\alpha(E)}.$$

Definition 3.2.7. A Hitchin triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ is said to be α-*semistable* (respectively α-*stable*) if

$$\mu_\alpha(\hat{F}_1, \hat{F}_2, \overrightarrow{D}^{(q)}) \leq \mu_\alpha(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)}) \quad (\text{respectively } \mu_\alpha(\hat{F}_1, \hat{F}_2, \overrightarrow{D}^{(q)}) < \mu_\alpha(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})) \tag{3.9}$$

for all nontrivial proper Hitchin subtriples $(\hat{F}_1, \hat{F}_2, \overrightarrow{D}^{(q)})$ of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$.

Notation 3.2.8. From now on, unless otherwise stated, we assume that $\hat{E}$ is a torsion-free Hitchin pair on C of rank $(2, 2)$.

Remark 3.2.9. Let $S(2, \chi, \alpha)$ denote the set of all isomorphism classes of α-semistable torsion-free Hitchin pairs on C of rank $(2, 2)$. Let $\hat{E}$ be a torsion-free Hitchin pair such that $[\hat{E}] \in S(2, \chi, \alpha)$. Then the triple corresponding to it is α-semistable and conversely, because of our definition of Euler characteristic Definition 3.1.9.

Moreover, the Euler characteristics of the bundles appearing in a α-semistable Hitchin triple have to satisfy the following bounds.

Theorem 3.2.10 ([7], Theorem 3.1.9(1))**.** *Let $\chi \neq 0$ and let $\alpha = (\alpha_1, \alpha_2)$ be a polarization on C such that $\alpha_1\chi$ is not an integer. Let (b_1, b_2) be the unique tuple satisfying*

$$\alpha_1\chi < b_1 < \alpha_1\chi+1, \qquad \alpha_2\chi+1 < b_2 < \alpha_2\chi+2, \qquad b_1+b_2 = \chi+2. \tag{3.10}$$

Let $[\hat{E}] \in S(2, \chi, \alpha)$ and let $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ be the triple corresponding to $\hat{E}$. Then we must have rank $\overrightarrow{A}^{(q)} \geq 1$ *and there are two possibilities: either*

$$\chi_{C_1}(E_1) = b_1, \chi_{C_2}(E_2) = b_2 \tag{3.11}$$

or

$$\chi_{C_1}(E_1) = b_1 + 1, \chi_{C_2}(E_2) = b_2 - 1. \tag{3.12}$$

Moreover, if rank $\overrightarrow{A}^{(q)} = 1$ *only* (3.11) *is possible.*

Proof. First suppose that $\overrightarrow{A}^{(q)}$ has rank 2. Since the canonical map

$$(E_1 \otimes \mathcal{O}_{C_1}(-q))^{(q)} \to E_1^{(q)}$$

is zero, $(E_1 \widehat{\otimes \mathcal{O}_{C_1}}(-q), 0, \overrightarrow{0})$ is a Hitchin subtriple of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, which is α-semistable, so we must have $\dfrac{\chi_{C_1}(E_1) - 2}{2\alpha_1} \leq \dfrac{\chi}{2}$, hence

$$\chi_{C_1}(E_1) \leq \alpha_1\chi + 2. \tag{3.13}$$

On the other hand, considering $(0, \hat{E}_2, \overrightarrow{0})$ as a Hitchin subtriple of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, we must have

$$\chi_{C_2}(E_2) \leq \alpha_2\chi + 2. \tag{3.14}$$

We obtain

$$\alpha_1\chi \leq \chi_{C_1}(E_1)$$

by recalling that we have $\chi = \chi_{C_1}(E_1) + \chi_{C_2}(E_2) - 2$, that the polarization $\alpha = (\alpha_1, \alpha_2)$ is such that $\alpha_1 + \alpha_2 = 1$ and using this in (3.14). This, together with (3.13), yields

$$\alpha_1 \chi \leq \chi_{C_1}(E_1) \leq \alpha_1 \chi + 2 \tag{3.15}$$

and, similarly, we obtain

$$\alpha_2 \chi \leq \chi_{C_2}(E_2) \leq \alpha_2 \chi + 2. \tag{3.16}$$

Hence the only choices which are compatible with (3.15), (3.16) and with the condition $b_1 + b_2 = \chi + 2$ are either $\chi_{C_1}(E_1) = b_1, \chi_{C_2}(E_2) = b_2$ or $\chi_{C_1}(E_1) = b_1 + 1, \chi_{C_2}(E_2) = b_2 - 1$.

Now suppose that $\overrightarrow{A}^{(q)}$ has rank 1. Let $(\dot{E}_1, \dot{\phi}_1) \to (E_1, \phi_1)$ be the Hecke modification such that $\mathrm{Im}\dot{E}_1^{(q)} = \mathrm{Ker}\overrightarrow{A}^{(q)}$ obtained as in Remark 2.2.11, together with the modified Higgs datum, as in Definition 3.1.7. This implies that the triple $(\widehat{\dot{E}_1}, 0, \overrightarrow{0})$ is a Hitchin subtriple of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ and, since $\chi_{C_1}(\dot{E}_1) = \chi_{C_1}(E_1) - 1$, the α-semistability of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ implies

$$\chi_{C_1}(E_1) \leq \alpha_1 \chi + 1.$$

On the other hand, considering $(0, \hat{E}_2, \overrightarrow{0})$ as a subtriple of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ and arguing as in the case in which $\overrightarrow{A}^{(q)}$ has rank 2, we must have

$$\chi_{C_2}(E_2) \leq \alpha_2 \chi + 2.$$

Recalling the definition of Euler characteristic of a Hitchin triple (3.7), together with that of polarization (2.5), we obtain

$$\alpha_1 \chi \leq \chi_{C_1}(E_1) \leq \alpha_1 \chi + 1, \qquad a_2 \chi + 1 \leq \chi_{C_2}(E_2) \leq \alpha_2 \chi + 2, \tag{3.17}$$

which forces $\chi_{C_1}(E_1) = b_1, \chi_{C_2}(E_2) = b_2$.

Finally, we show that the case in which the rank of $\overrightarrow{A}^{(q)}$ is zero does not yield any b_1, b_2. In fact, considering the subtriples $(\hat{E}_1, 0, \overrightarrow{0})$ and $(0, \hat{E}_2, \overrightarrow{0})$ of $(\hat{E}_1, \hat{E}_2, \overrightarrow{0})$, we obtain $\chi_{C_1}(E_1) = \alpha_1 \chi$ and $\chi_{C_2}(E_2) = \alpha_2 \chi + 2$, which are not allowed by (3.10), preventing the existence of b_i. So (3.10) implies that the rank of $\overrightarrow{A}^{(q)}$ is at least one. □

Corollary 3.2.11 ([6], Section 9, [7], Corollary 3.1.10)**.** *In the hypotheses of Theorem 3.2.10, if* $[\hat{E}] \in S(2, \chi, \alpha)$*, then either* E *is locally free or it is* *a torsion-free $\mathcal{O}_C$-module such that $E_q \cong \mathcal{O}_{C,q} \oplus \mathfrak{m}_{C,q}$. The choice of the polarization α on C prevents the existence of torsion-free $\mathcal{O}_C$-modules whose stalk at the node q is of the form $\mathfrak{m}_{C,q} \oplus \mathfrak{m}_{C,q}$.*

Remark 3.2.12. Since the arguments that we give in the proof of Theorem 3.2.10 only involve the fibres of E_i at the node q, the characterizations given in Remark 2.3.15 and Example 2.3.16 also hold for α-semistable Hitchin triples.

Notation 3.2.13. Let $\mathcal{M}_{12}(2,\chi)$ be the moduli space of α-semistable Hitchin triples $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ of rank 2 and Euler characteristic $\chi = \chi_C(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, which we simply denote by $\mathcal{M}_{12}$, satisfying

$$\alpha_1\chi < \chi_{C_1}(E_1) < \alpha_1\chi + 1, \qquad \alpha_2\chi + 1 < \chi_{C_2}(E_2) < \alpha_2\chi + 2. \tag{3.18}$$

Let $\mathcal{M}_{21}(2,\chi)$ be the moduli space of α-semistable Hitchin triples $(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})$ of rank 2 and Euler characteristic $\chi = \chi_C(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})$, which we simply denote by $\mathcal{M}_{21}$, satisfying

$$\alpha_1\chi + 1 < \chi_{C_1}(\dot{E}_1) < \alpha_1\chi + 2, \qquad \alpha_2\chi < \chi_{C_2}(\dot{E}_2) < \alpha_2\chi + 1. \tag{3.19}$$

Remark 3.2.14. The existence of the moduli spaces $\mathcal{M}_{12}$ and $\mathcal{M}_{21}$ is given by the construction of the moduli space of α-semistable Hitchin pairs in [7], Chapter 2 (or [13], Appendix B) and the equivalence Lemma 3.1.3. Another construction of the moduli space of α-semistable Hitchin pairs can be obtained by Simpson's construction Remark 1.4.10.

Notation 3.2.15. We denote by $\mathcal{M}_1$ the moduli space of semistable Higgs bundles of rank 2 on C_1. We denote by $\mathcal{M}_2$ the moduli space of semistable Higgs bundles of rank 2 on C_2.

3.3 Hitchin triples having odd Euler characteristic

Notation 3.3.1. From now on, we assume that $\chi = \chi_C(E)$ is odd, in particular we assume that $\chi_{C_1}(E_1)$ is odd and that $\chi_{C_2}(E_2)$ is even. As in the case of vector bundles, we focus on the Euler characteristic of E_1, E_2 rather than on their degree.

Considering [7], Theorem 3.1.9 and assuming that χ is odd, we obtain the following.

Theorem 3.3.2. *Let χ be odd and let $\alpha = (\alpha_1, \alpha_2)$ be a polarization on C such that $\alpha_1\chi$ is not an integer. Then the moduli space $\mathcal{M}(2, \chi, \alpha)$ of α-semistable rank-2 torsion-free Hitchin pairs on C with Euler characteristic χ is given by the union of two smooth components $\mathcal{M}_{12}$ and $\mathcal{M}_{21}$ intersecting transversally along a smooth divisor $\mathcal{N}$, which parametrizes Hitchin triples of the form $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, where $\overrightarrow{A}^{(q)}$ has rank 1.*

Proof. We first notice that, since E has rank 2 and since we assume that χ is odd and $\alpha_1\chi \notin \mathbb{Z}$, every α-semistable Hitchin pair is α-stable, thus $\mathcal{M}_{12}$ is smooth. By Theorem 3.2.10, if $\hat{E}$ is a α-stable Hitchin pair of rank 2 and Euler characteristic χ, then there exists a unique α-stable triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ such that $\hat{E}_1, \hat{E}_2$ are rank-2 Higgs bundles on C_1, C_2 respectively, satisfying

$$\alpha_1\chi < \chi_{C_1}(E_1) < \alpha_1\chi+2, \qquad \alpha_2\chi < \chi_{C_2}(E_2) < \alpha_2\chi+2, \qquad \chi_{C_1}(E_1)+\chi_{C_2}(E_2) = \chi+2.$$

(respectively a unique α-stable triple $(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})$ such that $\widehat{\dot{E}_1}, \widehat{\dot{E}_2}$ is a rank-2 Higgs bundle on C_1, C_2 respectively, satisfying

$$\alpha_1\chi < \chi_{C_1}(\dot{E}_1) < \alpha_1\chi+2, \qquad \alpha_2\chi < \chi_{C_2}(\dot{E}_2) < \alpha_2\chi+2, \qquad \chi_{C_1}(\dot{E}_1)+\chi_{C_2}(\dot{E}_2) = \chi+2)$$

with $\overrightarrow{A}^{(q)}$ (respectively $\overleftarrow{B}^{(q)}$) a nonzero linear map. Moreover,

(a) if E is locally free at q, then $\overrightarrow{A}^{(q)}$ is invertible and we have $E_i \cong \dot{E}_i$, $i = 1, 2$, $\overleftarrow{B}^{(q)} = (\overrightarrow{A}^{(q)})^{-1}$,

(b) if E is not locally free at q, then $\overrightarrow{A}^{(q)}$ has rank 1 and

$$\alpha_1\chi < \chi_{C_1}(E_1) < \alpha_1\chi + 1, \qquad \alpha_2\chi + 1 < \chi_{C_2}(E_2) < \alpha_2\chi + 2 \quad (3.20)$$

(respectively

$$\alpha_1\chi + 1 < \chi_{C_1}(\dot{E}_1) < \alpha_1\chi + 2, \qquad \alpha_2\chi < \chi_{C_2}(\dot{E}_2) < \alpha_2\chi + 1).$$

The triples $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, $(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})$ are related as in (3.6).

Let $\mathcal{M}_{12}$ and $\mathcal{M}_{21}$ be as in Notation 3.2.13.

The closed subschemes

$$\mathcal{N}_{12} = \{[(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})] \in \mathcal{M}_{12} : \operatorname{rank} \overrightarrow{A}^{(q)} = 1\}$$

and

$$\mathcal{N}_{21} = \{[(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})] \in \mathcal{M}_{21} : \operatorname{rank} \overleftarrow{B}^{(q)} = 1\}$$

are isomorphic because of Remark 3.1.8, in particular because of the commutativity of the diagram (3.6).

This, together with Lemma 3.1.3, yields that $\mathcal{M}(2, \chi, \alpha)$ is isomorphic to $\mathcal{M}_{12} \cup \mathcal{M}_{21}$ with the natural identification $\mathcal{N} := \mathcal{N}_{12} \xrightarrow{\sim} \mathcal{N}_{21}$, which is a divisor in $\mathcal{M}_{12}$ (respectively in $\mathcal{M}_{21}$), since it is defined by the vanishing of the determinant of $\overrightarrow{A}^{(q)}$. □

Stability of Hitchin triples on C implies stability of the Higgs bundles in the triples

Remark 3.3.3. Recall Notation 3.2.15, together with Notation 3.3.1. By Theorem 1.2.9, the moduli space $\mathcal{M}_1$ is smooth since Higgs bundles are assumed to have rank 2 and odd Euler characteristic. On the other hand, the moduli space $\mathcal{M}_2$ is singular, its singularities being as in Remark 1.4.11 (if we assume, for simplicity, that ϕ_2 has trace zero).

Theorem 3.3.4 ([7], Theorem 3.1.11)**.** *Let $\alpha = (\alpha_1, \alpha_2)$ be a polarization on C such that $\alpha_1\chi$ is not an integer. Let $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ be a triple of $\mathcal{M}_{12}$. Then $\hat{E}_1$ is a stable Higgs bundle on C_1 and $\hat{E}_2$ is a semistable Higgs bundle on C_2 in the sense of Definition 1.2.3.*

Proof. If E_2 has no ϕ_2-invariant subbundles, then $\hat{E}_2$ is a stable Higgs bundle on C_2. Otherwise, assume that there exists a ϕ_2-invariant line subbundle $L_2 \subset E_2$. Then $(0, \hat{L}_2, \overrightarrow{0})$ is a Hitchin subtriple of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, which is α-semistable, so we must have

$$\chi_{C_2}(L_2) - 1 \leq \alpha_2 \frac{\chi}{2}, \tag{3.21}$$

which implies

$$\chi_{C_2}(L_2) \leq \alpha_2 \frac{\chi}{2} + 1 < \frac{\chi_{C_2}(E_2) - 1}{2} + 1 \tag{3.22}$$

by (3.18). Hence

$$\chi_{C_2}(L_2) \leq \frac{\chi_{C_2}(E_2)}{2}.$$

So $\hat{E}_2$ is a semistable Higgs bundle on C_2.

We now show that also $\hat{E}_1$ is semistable. If E_1 has no ϕ_1-invariant subbundles, then $\hat{E}_1$ is automatically a stable Higgs bundle on C_1. Otherwise, let $L_1 \subset E_1$ be a ϕ_1-invariant line subbundle. Then we have that $L_1 \otimes \mathcal{O}_{C_1}(-q) \subset E_1 \otimes \mathcal{O}_{C_1}(-q)$, so $(\widehat{L_1 \otimes \mathcal{O}_{C_1}(-q)}, 0, \overrightarrow{0})$ is a Hitchin subtriple of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, so we must have

$$\chi_{C_1}(L_1) - 1 \leq \alpha_1 \frac{\chi}{2},$$

hence

$$\chi_{C_1}(L_1) \leq \alpha_1 \frac{\chi}{2} + 1 < \frac{\chi_{C_1}(E_1)}{2} + 1 \tag{3.23}$$

by (3.18). By the hypothesis that $\chi_{C_1}(E_1)$ is odd, (3.23) implies

$$\chi_{C_1}(L_1) \leq \frac{\chi_{C_1}(E_1)}{2} + \frac{1}{2}. \tag{3.24}$$

We claim that

$$\chi_{C_1}(L_1) \leq \frac{\chi_{C_1}(E_1)}{2}. \tag{3.25}$$

Assume by contradiction that there is a ϕ_1-invariant line subbundle L_1 of E_1 such that

$$\chi_{C_1}(L_1) > \frac{\chi_{C_1}(E_1)}{2}.$$

This, together with (3.24), implies that for any such subbundle we have

$$\chi_{C_1}(L_1) = \frac{\chi_{C_1}(E_1)}{2} + \frac{1}{2}. \tag{3.26}$$

Since $(\hat{L}_1, \hat{E}_2, \overrightarrow{A}^{(q)}_{|L_1})$ is a Hitchin subtriple of the α-semistable Hitchin triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, we have

$$\frac{\chi_{C_1}(L_1) + \chi_{C_2}(E_2) - 2}{\alpha_1 + 2\alpha_2} \leq \frac{\chi}{2}.$$

Using $\alpha_1 + \alpha_2 = 1$ and (3.26), we obtain

$$\chi_{C_1}(E_1) + 2\chi_{C_2}(E_2) - 3 \leq \chi + \alpha_2 \chi,$$

yielding, by the definition of Euler characteristic of a triple,

$$\chi + \chi_{C_2}(E_2) - 1 \leq \chi + \alpha_2\chi,$$

which implies

$$\chi_{C_2}(E_2) \leq \alpha_2\chi + 1,$$

which is in contrast with (3.18). Thus we have the claim (3.25), yielding semistability of $\hat{E}_1$. Moreover, since $\chi_{C_1}(E_1)$ is assumed to be odd, we have $\chi_{C_1}(L_1) < \frac{\chi_{C_1}(E_1)}{2}$, so $\hat{E}_1$ is a stable Higgs bundle on C_1. □

Relation between stability of Higgs bundles and stability of the triples in which they appear

We now consider when the converse of Theorem 3.3.4 holds. First, we study a case which is suggested by the statement of [7], Lemma 3.1.12.

Remark 3.3.5. Let $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ be a Hitchin triple admitting a Hitchin subtriple of the form $(\hat{E}_1, \hat{L}_2, \overrightarrow{A}^{(q)})$, for a ϕ_2-invariant subbundle L_2 of E_2 such that

$$\chi_{C_2}(L_2) = \chi_{C_2}(E_2)/2. \tag{3.27}$$

If $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ is α-semistable, $\overrightarrow{A}^{(q)}$ has rank 1 and (3.27) yields the following stability check

$$\frac{\chi_{C_1}(E_1) + \chi_{C_2}(L_2) - 1}{2\alpha_1 + \alpha_2} \leq \frac{\chi}{2},$$

which, by Definition 2.2.1 and (3.27), implies

$$\frac{\chi_{C_1}(E_1) + \chi_{C_2}(L_2) - 1}{1 + \alpha_1} = \frac{\chi_{C_1}(E_1) + \chi_{C_2}(E_2)/2 - 1}{1 + \alpha_1} \leq \frac{\chi}{2}$$

yielding by (2.10)

$$\frac{\chi_{C_1}(E_1) + \chi}{2(1 + \alpha_1)} \leq \frac{\chi}{2},$$

thus

$$\chi_{C_1}(E_1) \leq \alpha_1\chi. \tag{3.28}$$

However, since (3.18) requires $\chi_{C_1}(E_1) > \alpha_1\chi$, triples $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ admitting subtriples of the form $(\hat{E}_1, \hat{L}_2, \overrightarrow{A}^{(q)})$ do not belong to the space $\mathcal{M}_{12}$.

Remark 3.3.6. On the other hand, recall the α-semistability condition (3.9), in particular the definition of the Euler characteristic of a Hitchin triple (3.7). Since $\hat{E}_1$ is stable and $\hat{E}_2$ is semistable, in all cases which do not involve strictly semistable Higgs bundles $\hat{E}_2$, the semistability conditions of $\hat{E}_1$ and $\hat{E}_2$ as Higgs bundles on C_1 and C_2, respectively, imply that triples of the form $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ are α-stable.

Lemma 3.3.7. *Let $\hat{E}_1$ be a rank-2 stable Higgs bundle on C_1 having odd Euler characteristic, let $\hat{E}_2$ be a rank-2 semistable Higgs bundle on C_2 having even Euler characteristic and assume that $\chi_{C_i}(E_i)$ satisfy inequalities* (3.18) *for $i = 1, 2$. Moreover, assume that the Higgs fields ϕ_i have the same characteristic polynomial over q. Then two cases are possible:*

(a) if $\hat{E}_2$ does not admit any ϕ_2-invariant subbundles having its same slope, then, for any nonzero linear map $\overrightarrow{A}^{(q)} : E_1^{(q)} \to E_2^{(q)}$, the triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ is α-stable.

(b) if $\hat{E}_2$ admits ϕ_2-invariant subbundles having its same slope, then, for any isomorphism $\overrightarrow{A}^{(q)} : E_1^{(q)} \to E_2^{(q)}$, the triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ is α-stable.

Proof. Every Hitchin triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ as in (a) is a α-stable triple by Remark 3.3.6. Moreover, if $\overrightarrow{A}^{(q)}$ is an isomorphism, then $(\hat{E}_1, \hat{L}_2, \overrightarrow{A}^{(q)})$ is not a Hitchin subtriple of $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, so we exclude triples of the form discussed in Remark 3.3.5 and we have the claim. $\square$

While our approaches in this section mirror the ones in Section 2.4, we do not have an analogue of Remark 2.4.7.

Remark 3.3.8. The map

$$\psi : \mathcal{M}_{12} \to \mathcal{M}_1 \times \mathcal{M}_2$$

is not surjective.

For example, assume that E_1 is a stable vector bundle on C_1 and that $\phi_1^{(q)}$ is an isomorphism. Assume that E_2 is a stable vector bundle on C_2 and that $\phi_2^{(q)} = 0$. Then $(\hat{E}_1, \hat{E}_2)$ is a couple of stable Higgs bundles in $\mathcal{M}_1 \times \mathcal{M}_2$, but there is no point of $\mathcal{M}_{12}$ corresponding to it, since the only solution to

(3.3) in these hypotheses is $\overrightarrow{A}^{(q)}$ being the zero map, which is not allowed by Theorem 3.2.10.

Using the correspondence Lemma 3.1.3, together with Remark 3.2.9 and Theorem 3.3.4, we now give an example of a class of α-stable Hitchin pairs $\hat{E}$, in which $\hat{E}_2$ is a stable Higgs bundle having underlying strictly semistable vector bundle.

Example 3.3.9. Let $\hat{E}$ be a α-stable torsion-free Hitchin pair such that E has fixed determinant of odd degree. Assume that $\hat{E}$ corresponds to a α-stable Hitchin triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)} \otimes l)$, where $\overrightarrow{A}^{(q)}$ is an isomorphism. Moreover, assume that $L_1 \cong K_{C_1}$, $L_2 \cong K_{C_2}$, so that $l = \mathrm{id}$. We assume that E_1 has fixed determinant of odd degree and that E_2 has trivial determinant. We consider a stable Higgs bundle $\hat{E}_1$ having underlying stable vector bundle E_1. Then, as argued in Remark 1.2.4, $\hat{E}_1$ is stable for any $\phi_1 \in H^0(C_1, \mathrm{End}E_1 \otimes K_{C_1})$.
On the other hand, let $\hat{E}_2$ be a stable Higgs bundle having underlying strictly semistable vector bundle $E_2 = L_2 \oplus L_2^{-1}$, for $L_2 \in J^0(C_2)$ such that we have $L_2^2 \ncong \mathcal{O}_{C_2}$. Let

$$\phi_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix},$$

for $a_2 \in H^0(C_2, K_{C_2}), b_2 \in H^0(C_2, L_2^2 K_{C_2}), c_2 \in H^0(C_2, L_2^{-2} K_{C_2})$. Then (E_2, ϕ_2) is a stable Higgs bundle if and only if we have $b_2 \neq 0$ and $c_2 \neq 0$, so that L_2 and L_2^{-1} are not ϕ_2-invariant.
Moreover, as in Lemma 3.3.7, the Higgs fields ϕ_1, ϕ_2 need to have the same characteristic polynomial over the node q of C.

3.4 Hitchin triples with $\overrightarrow{A}^{(q)}$ of rank 1 in terms of Higgs bundles with parabolic structure

We now consider how the statements of Section 2.5 adapt when we also consider the Higgs fields ϕ_1, ϕ_2 over the node and see that the morphism corresponding to (2.24) is not surjective in this case.

Definition 3.4.1. Let Y be a smooth curve, let p be a point of it and let V be a vector bundle of rank 2 on Y. A *p-parabolic Higgs bundle* (V, ψ) is the datum of

- a p-parabolic bundle as in Definition 2.5.1,
- a morphism $\psi : V \to V \otimes L$ satisfying $\psi^{(p)}(F^j(V^{(p)})) \subset F^j(V^{(p)}) \otimes L^{(p)}$ for $j = 1, 2$.

We simply denote p-parabolic Higgs bundles by $(\hat{V}, 0 \subset F^2V^{(p)} \subset V^{(p)})$.

Definition 3.4.2. Let (V, ψ) be a p-parabolic Higgs bundle. Then the *parabolic degree* of V is defined as

$$\operatorname{pardeg}(V) = \deg(V) + \beta_1 + \beta_2,$$

where the weights β_i are as in Definition 2.5.1. The *parabolic slope* of V is defined as

$$\operatorname{par}\mu(V) = \frac{\operatorname{pardeg}(V)}{\operatorname{rank} V}.$$

Definition 3.4.3. Let (V, ψ) be a p-parabolic Higgs bundle on Y. Then V is *parabolic semistable* (respectively *parabolic stable*) if, for any nontrivial p-parabolic subbundle W of V, we have $\operatorname{par}\,\mu(W) \leq \operatorname{par}\,\mu(V)$ (respectively $\operatorname{par}\mu(W) < \operatorname{par}\mu(V)$).

We now apply the discussion above to our case. Let C be a one-nodal curve of compact type having polarization α such that $\alpha_1 < \alpha_2$. Theorem 6.1 in [40] adapts as follows.

Theorem 3.4.4 ([7], Lemma 3.4.2)**.** *Consider the polarization $\alpha = (\alpha_1, \alpha_2)$ on C such that $\alpha_1 < \alpha_2$ and that $\alpha_1\chi$ is not an integer. Let q be the node of C. Let P_1 be the moduli space of semistable q-parabolic Higgs bundles of rank 2 on C_1, given by $(\hat{E}_1, 0 \subset F^2E_1^{(q)} \subset E_1^{(q)})$ with parabolic weights $(\alpha_1/2, \alpha_2/2)$. Assume that the degree of E_1 equals $\chi_{C_1}(E_1)$, in particular that it is odd and that it satisfies* (3.18)*. Let P_2 be the moduli space of semistable q-parabolic Higgs bundles of rank 2 on C_2, given by $(\hat{E}_2, 0 \subset F^2E_2^{(q)} \subset E_2^{(q)})$ with parabolic weights $(\alpha_1/2, \alpha_2/2)$. Assume that the degree of E_2 equals $\chi_{C_2}(E_2)$, in particular that it is even and that it satisfies* (3.18)*.*

Then P_1 and P_2 are smooth. Moreover, we have an injective morphism

$$\gamma : \mathcal{N} \to P_1 \times P_2. \tag{3.29}$$

obtained by sending

$$\mathcal{N} \ni (\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)}) \mapsto (\hat{E}_1, 0 \subset F^2E_1^{(q)} \subset E_1^{(q)}) \times (\hat{E}_2, 0 \subset F^2E_2^{(q)} \subset E_2^{(q)}) \in P_1 \times P_2. \tag{3.30}$$

where $F^2E_1^{(q)} = Ker\overrightarrow{A}^{(q)}$ and $F^2E_2^{(q)} = Im\overrightarrow{A}^{(q)}$.

Proof. Let $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)}) \in \mathcal{N}$ be a α-stable Hitchin triple. We prove that

$$(\hat{E}_1, 0 \subset F^2E_1^{(q)} \subset E_1^{(q)})$$

is a q-parabolic stable Higgs bundle on C_1 with respect to the weights $(\alpha_1/2, \alpha_2/2)$, where $\alpha_1/2$ is associated to $E_1^{(q)}$ and $\alpha_2/2$ is associated to $F^2E_1^{(q)}$. By Theorem 3.3.4, we have that E_1 is semistable on C_1, so, by the assumptions on α_i we have that, for any ϕ_1-invariant subbundle L_1 of E_1,

$$\chi_{C_1}(L_1) + \alpha_2/2 < \frac{\chi_{C_1}(E_1)}{2} + \frac{1}{4} = \frac{\chi_{C_1}(E_1)}{2} + \frac{\alpha_2/2 + \alpha_1/2}{2}.$$

So, if either $L_1^{(q)} \neq F^2E_1^{(q)}$ or $\chi_{C_1}(L_1) < \chi_{C_1}(E_1)/2$, Definition 3.4.2 yields

$$\text{par deg}\,(L_1) = \text{par}\ \mu(L_1) < \text{par}\ \mu(E_1).$$

Since $\chi_{C_1}(E_1)$ is odd, there are no strictly semistable q-parabolic Higgs bundles in P_1. Thus all parabolic Higgs bundles in P_1 are stable and P_1 is smooth.

When we consider E_2, we still have

$$\text{par deg}\,(L_2) = \text{par}\ \mu(L_2) < \text{par}\ \mu(E_2)$$

if either $L_2^{(q)} \neq F^2E_2^{(q)}$ or $\chi_{C_2}(L_2) < \chi_{C_2}(E_2)/2$. By Remark 3.4.5, there are no strictly semistable parabolic Higgs bundles. Hence all parabolic Higgs bundles in P_2 are stable and P_2 is smooth.

Hence we obtain the injective morphism γ from the moduli space $\mathcal{N}$ to $P_1 \times P_2$. □

Remark 3.4.5. Let $\hat{E}_2$ be a Higgs bundle appearing in unstable Hitchin triples as in Remark 3.3.5. Then q-parabolic Higgs bundles having underlying Higgs bundles $\hat{E}_2$ are also unstable as parabolic Higgs bundles, thus they do not appear in P_2. In fact, consider the q-parabolic Higgs bundle

$$(\hat{E}_2, 0 \subset L_2^{(q)} \subset E_2^{(q)}),$$

with L_2 such that $\chi_{C_2}(L_2) = \chi_{C_2}(E_2)/2$ and with $L_2^{(q)}$ associated to the weight $\frac{\alpha_2}{2}$, in the same hypotheses of Theorem 3.4.4. Then we have

$$\text{par}\mu(L_2) = \chi_{C_2}(L_2) + \frac{\alpha_2}{2} = \frac{\chi_{C_2}(E_2)}{2} + \frac{\alpha_2}{2}$$

$$\text{par}\mu(E_2) = \frac{\chi_{C_2}(E_2)}{2} + \frac{1}{4}.$$

Asking $\text{par}\mu(L_2) \leq \text{par}\mu(E_2)$ is equivalent to asking

$$\frac{\alpha_2}{4} \leq \frac{\alpha_1}{4},$$

which is not possible, since $\alpha_2 > \alpha_1$.

In particular, there are no strictly semistable q-parabolic Higgs bundles.

Remark 3.4.6. The morphism γ in (3.29) is not surjective. In fact, in the hypotheses of Theorem 3.4.4, we need diagram (3.2) to be commutative and, by Definition 3.4.1, we have that $F^2E_1^{(q)} = \text{Ker}\overrightarrow{A}^{(q)}$ is $\phi_1^{(q)}$-invariant and $F^2E_2^{(q)} = \text{Im}\overrightarrow{A}^{(q)}$ is $\phi_2^{(q)}$-invariant.

Let $\{u_1\}$ be a basis of $F^2E_1^{(q)}$, let $\{u_1, v_2\}$ a basis of $E_1^{(q)} \otimes L_1^{(q)}$, respectively let $\{t_1\}$ be a basis of $F^2E_2^{(q)}$, let $\{t_1, w_2\}$ be a basis of $E_2^{(q)} \otimes L_2^{(q)}$.

By the assumptions above, the matrices of $\phi_1^{(q)}$ and $\phi_2^{(q)}$ have the following form

$$\phi_1^{(q)} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad \phi_2^{(q)} = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

and the assumptions on $\overrightarrow{A}^{(q)}$ involving the subbundles $F^2E_i^{(q)}$ imply that

$$\overrightarrow{A}^{(q)} = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}.$$

Condition (3.3) implies $e = 0$ and $c = f$. Thus, if we take $e \neq 0$ or $c \neq f$, this forces $\overrightarrow{A}^{(q)}$ to be the zero map, which is not allowed by Theorem 3.2.10. Hence the morphism γ is not surjective.

Chapter 4

The Hitchin map adapted to Hitchin triples

We now generalize Section 1.3 to the case in which C is of compact type, adapting the definition of Hitchin map to Hitchin triples.

4.1 The adapted Hitchin map

Recall that we assume that E is a torsion-free sheaf of rank $(2, 2)$ and let

$$\mathbb{A}_i := H^0(C_i, L_i) \oplus H^0(C_i, L_i^2), \qquad i = 1, 2.$$

The commutativity condition on Higgs data Definition 3.1.1, in particular the identification l in (3.1), implies restricting the target $\mathbb{A}_1 \times \mathbb{A}_2$ of the product of the classical Hitchin maps $h_1 : \mathcal{M}_1 \to \mathbb{A}_1$ and $h_2 : \mathcal{M}_2 \to \mathbb{A}_2$ to the locus

$$\mathbb{A}^{(q)} := \{(a_1, a_2) \in \mathbb{A}_1 \times \mathbb{A}_2 : a_1^{(q)} = a_2^{(q)}\}. \tag{4.1}$$

We define the adapted Hitchin map as follows.

$$h^{ad} : \mathcal{M}_{12} \to \mathbb{A}^{(q)} \qquad (\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)}) \mapsto (a_1, a_2). \tag{4.2}$$

Recall that the torsion-free Hitchin pair $\hat{E}$ is also associated to the triple $(\widehat{\dot{E}_1}, \widehat{\dot{E}_2}, \overleftarrow{B}^{(q)})$, which is related to the triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ as in (3.6).

Lemma 4.1.1 ([7], Lemma 3.2.2). *The map h^{ad} is well defined.*

Proof. Recall Definition 3.1.7. The Hecke modification is an isomorphism over smooth points of C, thus we have that on $C_1 \setminus \{q\}$ the characteristic polynomial of ϕ_1 and that of $\dot{\phi}_1$ coincide. Thus (E_1, ϕ_1) and $(\dot{E}_1, \dot{\phi}_1)$ have the same characteristic polynomial on C_1. Similarly, (E_2, ϕ_2) and $(\dot{E}_2, \dot{\phi}_2)$ have the same characteristic polynomial on C_2.

So, taking into account the canonical identification (3.1), the Hitchin triples $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ and $(\hat{\dot{E}}_1, \hat{\dot{E}}_2, \overleftarrow{B}^{(q)})$ define the same point in $\mathbb{A}^{(q)}$. □

Moreover, also when the base curve is one-nodal of compact type, the Hitchin map is proper.

Theorem 4.1.2 ([7], Theorem 3.3.1, Lemma 5.1.1, [6], Theorem 9.2)**.** *The adapted Hitchin map* (4.2) *is proper.*

4.2 Gluing condition for Higgs fields in α-semistable Hitchin triples

We now describe a gluing condition of the Higgs fields ϕ_i over the node q of C in terms of the identification l from (3.1) and of Theorem 3.2.10.

As in Remark 1.3.14, considering Higgs fields $\phi_i^{(q)} : E_i^{(q)} \to E_i^{(q)} \otimes L_i^{(q)}$, $i = 1, 2$, over the node, their characteristic polynomial is given by (1.15):

$$x_i^2 + a_{i,1}^{(q)}\, x_i + a_{i,2}^{(q)} = 0, \tag{4.3}$$

where x_i is the tautological coordinate on Tot (L_i). In particular, we have $a_{i,1}^{(q)} = \text{trace}\, \phi_i^{(q)} \in H^0(C_i, L_i^{(q)})$ and $a_{i,2}^{(q)} = \det \phi_i^{(q)} \in H^0(C_i, (L_i^{(q)})^2)$.

Remark 4.2.1. Assume that $L_1 \cong K_{C_1}$ and $L_2 \cong K_{C_2}$, so that the identification l in (3.1) is the identity. Then $\phi_1^{(q)}$ and $\phi_2^{(q)}$ need to have the same trace and the same determinant. Referring to (4.3), let $a_1^{(q)} := (a_{1,1}^{(q)}, a_{1,2}^{(q)})$ and let $a_2^{(q)} := (a_{2,1}^{(q)}, a_{2,2}^{(q)})$.

In these hypotheses the diagram (3.2) defining Hitchin triples commutes if and only if

$$(a_1, a_2) \in \mathbb{A}^{(q)}. \tag{4.4}$$

Since we want to generalize the classical description given in Sections 1.2 and 1.3 to curves of compact type, we also need the characterization of α-stability of Hitchin triples given by Theorem 3.2.10. It implies that when a Hitchin triple is α-semistable, diagram (3.2) has to commute for $\overrightarrow{A}^{(q)}$ having nonzero rank.

Definition 4.2.2. We say that a α-semistable Hitchin triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ is *associated to a* if we have $a = (a_1, a_2)$ and (4.4) holds.

As we did in Remark 1.3.14, we pull-back the vector bundles E_1, E_2 to the spectral curves X_{a_1}, X_{a_2} respectively and we give an explicit form of maps $\overrightarrow{A}^{(q)}$ making diagram (3.2) commute.

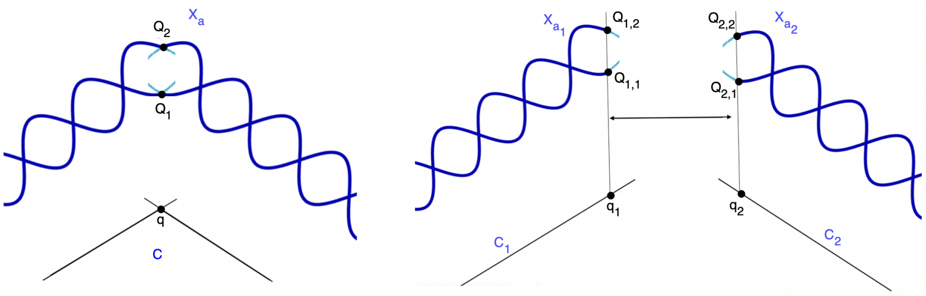


Figure 4.1: Relation between the spectral cover X_a of C and spectral covers X_{a_i} of C_i.

Remark 4.2.3 ([7], Remark 5.1.3). Assume that we have that the spectral curves $\pi_{a_i} : X_{a_i} \to C_i$ intersect transversally at two distinct points over the node q, as in Figure 4.1.

Let $\nu : C_1 \sqcup C_2 \to C$ be the normalization map and let $q_i \in C_i$ be the preimages of q via ν. Pulling back E_i to X_{a_i}, letting x_i be the tautological sections of $\pi_{a_i}^* L_i$ and letting $Q_{i,j}$ $(i = 1, 2, j = 1, 2)$ be the distinct preimages of q_i via the covers π_{a_i}, we have $\det(x_i \cdot I_2 - \phi_i^{(q_i)}) = 0$ and

$$\phi_1^{(q_1)} = \begin{pmatrix} c_1 & 0 \\ 0 & d_1 \end{pmatrix} \quad \phi_2^{(q_2)} = \begin{pmatrix} c_2 & 0 \\ 0 & d_2 \end{pmatrix}, \tag{4.5}$$

where $c_i = x_i(Q_{i,1})$, $d_i = x_i(Q_{i,2})$.

Remark 4.2.4. Let $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ be a Hitchin triple associated to a, in the sense of Definition 4.2.2 and assume, for simplicity, that $L_1 \cong K_{C_1}$ and that $L_2 \cong K_{C_2}$. Then the commutativity condition (3.3) simplifies to:

$$\overrightarrow{A}^{(q)} \phi_1^{(q_1)} = \phi_2^{(q_2)} \overrightarrow{A}^{(q)}. \tag{4.6}$$

Let

$$\overrightarrow{A}^{(q)} = \begin{pmatrix} k & y \\ z & t \end{pmatrix},$$

for $k, y, z, t \in \mathbb{C}$. Assume that $\overrightarrow{A}^{(q)}$ satisfies (4.6), which, by Remark 4.2.1, is equivalent to the requirement that ϕ_i have the same eigenvalues over q and that $\phi_i^{(q_i)}$, $i = 1, 2$, are as in (4.5). Then, if $\overrightarrow{A}^{(q)}$ has rank 2, it is of the form

$$\overrightarrow{A}^{(q)} = \begin{pmatrix} k & 0 \\ 0 & t \end{pmatrix} \text{ assuming that } c_1 = c_2 \text{ and } d_1 = d_2 \tag{4.7}$$

$$\overrightarrow{A}^{(q)} = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \text{ assuming that } c_1 = d_2 \text{ and } d_1 = c_2 \tag{4.8}$$

for nonzero k, t (respectively nonzero y, z). If $\overrightarrow{A}^{(q)}$ has rank one and it is of the form (4.7), then exactly one of k and t vanishes. If it is of the form (4.8), then exactly one of y and z vanishes.

Remark 4.2.5. Let $\hat{E}_1$ be any semistable Higgs bundle on C_1. We claim that there is always at least a semistable Higgs bundle $\hat{E}_2$ on C_2 and a linear map $\overrightarrow{A}^{(q)}$ such that diagram (3.2) commutes. In fact, asking (4.6) and solving for k, y, z, t always yields solutions in which $\overrightarrow{A}^{(q)}$ has nonzero rank. The same argument clearly holds if we fix a semistable Higgs bundle $\hat{E}_2$ and it yields $\hat{E}_1$ together with linear maps $\overrightarrow{A}^{(q)}$ of nonzero rank. Note that, since ϕ_1, ϕ_2 can also be seen as holomorphic one-forms with values in End$E$, the condition (4.4) is not restrictive for the behaviour of the Higgs fields ϕ_1, ϕ_2 over points which are different from q. In particular, for any $\hat{E}_1 \in \mathcal{M}_1, \hat{E}_2 \in \mathcal{M}_2$, we can find suitable $\phi_1^{(q)}$, $\phi_2^{(q)}$ such that (3.2) commutes for nonzero linear maps $\overrightarrow{A}^{(q)}$. Thus we have obtained a locus which is open and dense in $\mathcal{M}_1 \times \mathcal{M}_2$.

Note that, if (4.6) has nonzero solutions $\overrightarrow{A}^{(q)}$, then it has infinitely many. Thus we describe a way to indentify triples, which is motivated by the definition of the adapted Hitchin map for Hichin triples (4.2), which forgets the datum of the map $\overrightarrow{A}^{(q)}$.

Remark 4.2.6. We consider the following equivalence relation: we say that two Hitchin triples $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$, $(\hat{E}_1', \hat{E}_2', (\overrightarrow{A}^{(q)})')$, are *a-equivalent* if they are both associated to a. We obtain the moduli space $\mathcal{M}_{12}(2,\chi)$ of a-equivalent Hitchin triples of rank 2 and Euler characteristic χ as in Remark 3.2.14.

Remark 4.2.7. Assuming $L_1 \cong K_{C_1}$ and $L_2 \cong K_{C_2}$, Remark 4.2.5 and the dimension of $\mathcal{M}_1, \mathcal{M}_2$ in (1.7) yield

$$\dim \mathcal{M}_{12} = \dim \mathcal{M}_1 + \dim \mathcal{M}_2 = 8g_1 - 6 + 8g_2 - 6 = 8g - 12. \qquad (4.9)$$

In particular note that, even if the number of maps $\overrightarrow{A}^{(q)}$ making diagram (3.2) commute is not finite, Definition 4.2.2 provides a way to identify equivalent triples, mirroring the structure of $\mathcal{M}_1 \times \mathcal{M}_2$. We give an alternative way to compute the dimension of $\mathcal{M}_{12}$ via the fibres of the adapted Hitchin map in Section 4.4.

Remark 4.2.8. As we saw in Section 2.1, the existence of the identification l is a specific property of curves of compact type that enables us to define the Hitchin map over all points of C. On the other hand, for irreducible one-nodal curves, the Jacobian of the curve is not compact and we cannot define the characteristic polynomial of ϕ over the node, since there is not a unique way to identify the fibres of L at the preimages of the node on the normalization of the curve.

4.3 Adapted spectral curves

We define the adapted spectral curve X_a to be a ramified covering of degree 2 of C (recall that we assume that the rank of E is 2). Consider the canonical identification (3.1) between the fibres of L_1 and L_2 at the node q of C and let $\pi : \text{Tot}\ (L) \to C$ be the natural projection.

Definition 4.3.1. The *adapted spectral curve* X_a is the zero divisor in Tot (L) of a nonzero section in $H^0(\text{Tot}\ (L), \pi^* L^2)$.

Similarly to Remark 1.3.8, by *generic adapted spectral curve* we mean that, over points of C which are different from q, the components of the adapted spectral curve are smooth.

Spectral covers which are unramified over the node can be thought as the union of the spectral covers of C_i, meeting transversally over the node q.

We refer to [13], Section 4.1 for a discussion which parallels the discussion of spectral curves that we gave in Section 1.3.

4.4 Fibres of the adapted Hitchin map

Fibres of the adapted Hitchin map for spectral covers unramified over the node

Recall that, by Remark 4.2.1, we have $h^{ad}(\mathcal{M}_{12}) \subset \mathbb{A}^{(q)}$, for $\mathbb{A}^{(q)}$ as in (4.1).

Let $\mathbb{A}^{ur}$ be the open subset of $\mathbb{A}^{(q)}$ corresponding to adapted spectral curves which are not ramified over the node. We now let $a \in \mathbb{A}^{ur}$ and we relate spectral data on the components X_{a_i} (which we assume to be integral, as we did in Notation 1.3.16) to spectral data on X_a. The correspondence in Proposition 1.3.22 plays a key role.

Remark 4.4.1. Consider Figure 4.1. By Proposition 1.3.22, the datum of rank-1 torsion-free sheaves η_i on the components X_{a_i} is equivalent to the Higgs data $\hat{E}_i$ on the smooth components C_i, $i = 1, 2$. Moreover, consider Lemma 3.3.7 and assume that $\overrightarrow{A}^{(q)}$ is an isomorphism. Then, since C is of compact type, the total spaces of L_1 and L_2 are identified over q and any choice of identifications

$$\eta_1^{(Q_{1,1})} \xrightarrow{\sim} \eta_2^{(Q_{1,2})}, \eta_1^{(Q_{2,1})} \xrightarrow{\sim} \eta_2^{(Q_{2,2})}. \tag{4.10}$$

corresponds to the choice of a map $\overrightarrow{A}^{(q)} : E_1^{(q_1)} \to E_2^{(q_2)}$ making diagram (3.2) commute.

By (1.24) applied to the smooth components C_1 and C_2, we have

$$\deg \eta_i = \deg E_i + \deg L_i.$$

Assuming $L_i \cong K_{C_i}$, this yields

$$\deg \eta_1 + \deg \eta_2 = (\deg E_1 + 2g_1 - 2) + (\deg E_2 + 2g_2 - 2) = d + 2g - 4 = \gamma - 2, \tag{4.11}$$

for γ as in (1.25). Let $\gamma' = \gamma - 2$.

Remark 4.4.2. The assumption that $\overrightarrow{A}^{(q)}$ is an isomorphism in Remark 4.4.1 is without loss of generality. In fact, the Hitchin map forgets the datum of the map $\overrightarrow{A}^{(q)}$ and the space $\mathcal{N}$ is a divisor in $\mathcal{M}_{12}$, so we can assume that $\overrightarrow{A}^{(q)}$ has rank 2.

Moreover, the assumption that X_a is not ramified over the node implies that the eigenvalues of $\phi_1^{(q)}$, $\phi_2^{(q)}$ are different, so we cannot have the case described in Remark 3.3.8.

Remark 4.4.3 ([7], Remark 5.2.2). Consider the line bundle L associated to the line bundles L_1, L_2 on C_1, C_2 respectively and let Z be the projective completion of the total space of L, as in (1.12). Since the adapted spectral curve X_a can be realized as a closed subscheme $X_a \subset Z$, Remark 1.4.13 endows X_a with the polarization $\mathcal{O}_{X_a}(1)$ induced from (1.32).

By restricting the polarization $\mathcal{O}_{X_a}(1)$ to the components X_{a_1}, X_{a_2} of X_a and by arguing as in Remark 2.2.2, we obtain a polarization $\beta = (\beta_1, \beta_2)$ on X_a such that $\beta_1 + \beta_2 = 1$. Since the ample line bundles on X_{a_1}, X_{a_2} are the pullbacks of ample line bundles on C_1, C_2 yielding the polarization $\alpha = (\alpha_1, \alpha_2)$, we obtain $\alpha = \beta$.

We have the following analogue of Proposition 1.3.22 for base curves C of compact type.

Proposition 4.4.4 ([7], Theorem 5.3.1, [13], Lemma 4.6, Proposition 4.7)**.** *Let $a \in \mathbb{A}^{ur}$ and let $\pi_a : X_a \to C$ be an adapted spectral curve such that its components X_{a_i} are integral. Let $\alpha = (\alpha_1, \alpha_2)$ be a polarization of C such that $\alpha_1 \chi$ is not an integer. Then there is a bijective correspondence between*

- *isomorphism classes of α-stable rank-1 torsion-free sheaves η on X_a,*
- *isomorphism classes of α-stable Hitchin triples $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ associated to a, as in Definition 4.2.2.*

This yields an isomorphism

$$(h^{ad})^{-1}(a) \xrightarrow{\sim} \overline{J^{\gamma'}}(X_a),$$

where $\overline{J^{\gamma'}}(X_a)$ is Simpson's compactified Jacobian from Remark 1.4.15(b).

Proof. Remark 4.4.1 yields a description of torsion-free sheaves in the compactified Jacobian of the adapted spectral curve, whose degree is computed in (4.11). The equivalence between β-stability of rank-1 torsion-free sheaves on X_a and α-stability of Hitchin pairs on C (which is equivalent to α-stability of Hitchin triples on C, as in Remark 3.2.9) is given by Remark 4.4.3. $\square$

Dimension of $\mathcal{M}_{12}$

Proposition 4.4.4 yields an alternative way to Remark 4.2.7 to compute the dimension of $\mathcal{M}_{12}$, as we now see.

Proposition 4.4.5 ([7], Proposition 5.2.1)**.** *The adapted Hitchin map*

$$h^{ad} : \mathcal{M}_{12} \to \mathbb{A}^{ur} \tag{4.12}$$

is surjective.

Proof. Given any generic $a = (a_1, a_2) \in \mathbb{A}^{ur}$, by Proposition 1.3.22 there are stable Higgs bundles $\hat{E}_1, \hat{E}_2$ corresponding to a. Considering $\hat{E}_1, \hat{E}_2$ associated to a and considering a map $\overrightarrow{A}^{(q)}$ of rank 2 making diagram (3.2) commute (which always exists by Remark 4.2.4), we have that the triple $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)})$ is α-stable by Lemma 3.3.7, yielding the claim. $\square$

Thus we have

$$\dim \mathcal{M}_{12} = \dim \mathbb{A}^{ur} + \dim (h^{ad})^{-1}(a) = \dim \mathbb{A}^{(q)} + \dim \overline{J^{\gamma'}}(X_a). \tag{4.13}$$

Proposition 4.4.6. *Let L_1 and L_2 be the canonical bundles K_{C_1} and K_{C_2}, respectively, of the smooth components C_1 and C_2 of C, which has arithmetic genus g. Then we have*

$$\dim \mathcal{M}_{12} = 8g - 12.$$

Proof. As discussed above, the generic fibre of h^{ad} is $\overline{J^{\gamma'}}(X_a)$, whose dimension equals the arithmetic genus of X_a (e.g. by Theorem 1.1.19 in [44], see also Caporaso [18]). Since X_a is associated to the union of the smooth spectral curves X_{a_i}, each having genus $4g_i - 3$, $i = 1, 2$, intersecting transversally at two nodes, its arithmetic genus is given by (2.3), yielding

$$g_{X_a} = g_{X_{a_1}} + g_{X_{a_2}} + (\delta - \gamma + 1) = (4g_1 - 3) + (4g_2 - 3) + 1 = 4g - 5.$$

Thus the dimension of the Hitchin fibre is $4g - 5$.

Since the locus $\mathbb{A}^{(q)}$ has codimension 1 in $\mathbb{A}_1 \times \mathbb{A}_2$, we have:

$$\dim \mathbb{A}^{(q)} = \dim \mathbb{A}_1 + \dim \mathbb{A}_2 - 1 = (4g_1 - 3) + (4g_2 - 3) - 1 = 4g - 7.$$

Hence (4.13) yields

$$\dim \mathcal{M}_{12} = 8g - 12. \tag{4.14}$$

□

An example of fibres of the adapted Hitchin map for spectral covers ramified over the node

Consider a generic adapted spectral curve X_a. Assume that it is ramified over the node q and that the ramification point Q over q is a nodal singularity, as in Figure 4.2. We show that, also in this case, the fibre of the Hitchin map is the compactified Jacobian of X_a.

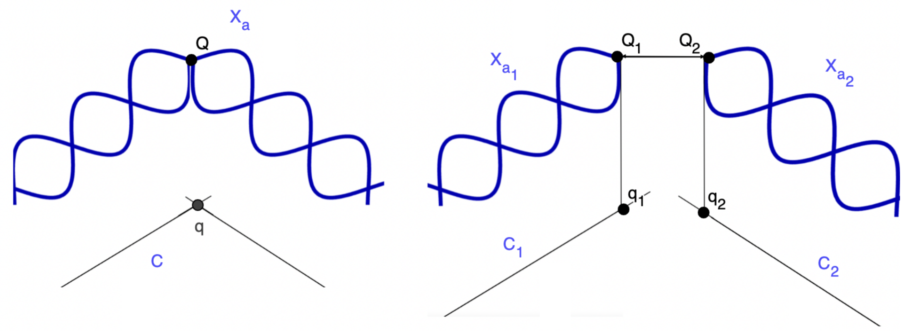


Figure 4.2: BNR correspondence for adapted spectral curve ramified over q

Proposition 4.4.7. *Let X_a be a generic adapted spectral curve which is ramified over the node q of C, as above. Then there is a one-to-one correspondence between rank-1 torsion-free sheaves on X_a and Hitchin triples on C. We have*

$$(h^{ad})^{-1}(a) \cong \overline{J}(X_{a_1}) \times \overline{J}(X_{a_2}).$$

Proof. Using the notation from (2.3), we have

$$g_{X_a} = g_{X_{a_1}} + g_{X_{a_2}} + (\delta - \gamma + 1) = g_{X_{a_1}} + g_{X_{a_2}}, \tag{4.15}$$

thus X_a is of compact type by Remark 2.1.5.

Consider Figure 4.2. By Lemma 3.1.3, the datum of a rank-2 torsion-free Hitchin pair $\hat{E}$ on C is equivalent to the datum of its restrictions $\hat{E}_1$ on C_1 and $\hat{E}_2$ on C_2, together with a linear map $\overrightarrow{A}^{(q)}$ making diagram (3.2) commute.

By the classical BNR correspondence Proposition 1.3.22 between η_i on X_{a_i} and $\hat{E}_i$ on C_i, $i = 1, 2$, we obtain a correspondence between rank-1 torsion-free sheaves on X_{a_i} and Higgs bundles on C_i, yielding

$$h_i^{-1}(a_i) \cong \overline{J}(X_{a_i}).$$

Moreover, since the spectral curve X_a is of compact type, we have an equivalence at the level of spectral curves: the datum of a rank-1 torsion-free sheaf η on X_a is equivalent to the datum of its restrictions η_i on X_{a_i} as in Remark 2.1.5. This yields

$$(h^{ad})^{-1}(a) \cong h_1^{-1}(a_1) \times h_2^{-1}(a_2),$$

which yields the claim. □

Remark 4.4.8. Consider the canonical map

$$\Pi : \overline{J}(X_a) \to \overline{J}(X_{a_1}) \times \overline{J}(X_{a_2}).$$

Then:

- if the adapted spectral curve X_a is not ramified over the node, then Π is the map $(\eta_1, \eta_2, f) \mapsto (\eta_1, \eta_2)$ forgetting the identifications between the fibres of the torsion-free sheaves η_i at the preimages of q, which yield the map $\overrightarrow{A}^{(q)}$.

- If X_a is ramified over the node, then Π is an isomorphism.

Chapter 5

The G-Hitchin map

Let G be an affine reductive group over $\mathbb{C}$ such that its associated Lie algebra is $\mathfrak{so}(4,\mathbb{C})$, i.e. it is isogenous to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. We apply the results presented in the first four chapters to describe the fibre of the G-Hitchin map for a smooth base curve of genus $g \geq 2$ and the fibre of the adapted G-Hitchin map for a one-nodal base curve of compact type such that the genus of both components is at least 2.

This builds on the study of $\mathrm{SL}(2,\mathbb{C})$-Higgs data, which we now recall.

5.1 $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles on a smooth base curve

We first consider the case in which the base curve C is smooth, then we consider the case in which C is one-nodal of compact type in Section 5.2.

Definition 5.1.1. The datum of a $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle on C corresponds to the datum of (E,ϕ), where:

- E is a vector bundle on C having fixed determinant Λ,
- $\phi : E \to E \otimes L$ is a L-twisted endomorphism having trace zero.

Notation 5.1.2. We assume, unless otherwise stated, that $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles have fixed determinant Λ of odd degree.

This follows the convention of the work by de Cataldo-Hausel-Migliorini [23], Section 1.2.2 and enables us to use the characterization from the previous chapters, in which we assume that χ is odd. On the other hand, the usual assumption on $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles, e.g. in the works [16], [20], [45], is that they have trivial determinant.

Remark 5.1.3. Notation 5.1.2 corresponds, via (1.34), to asking

$$\mathrm{Nm}_\pi(\eta) \xrightarrow{\sim} \Lambda \otimes \det(\pi_* \mathcal{O}_X)^{-1} =: \Lambda'. \tag{5.1}$$

Similarly to Remark 1.1.9, tensoring by a fixed line bundle of suitable degree induces a noncanonical isomorphism between line bundles on the spectral curve, having fixed norm, and line bundles in the Prym variety Pr_π of a cover as in Definition 1.5.4.

Let $\mathcal{M}(2,d)$ be the moduli space of semistable Higgs bundles of rank 2 and degree d on C. Let L be a line bundle on C whose degree is greater or equal to the degree of K_C and consider the map

$$\psi : \mathcal{M}(2,d) \to J(C) \times H^0(C,L), \qquad (E,\phi) \mapsto (\det E, \mathrm{tr}\phi).$$

Definition 5.1.4. Let C be a smooth base curve. We define the moduli space of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles as

$$\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})} = \psi^{-1}(\Lambda, 0).$$

Notation 5.1.5. We denote by $\mathbb{A}^0 \subset \mathbb{A}$ be the locus of characteristics such that $a_1 = 0$. Moreover, from now on, we denote the spectral curve $\pi_a : X_a \to C$ simply by $\pi : X_a \to C$.

Remark 5.1.6. Referring to (1.9) and to Notation 5.1.5, we call by a the section $a_2 \in H^0(C, L^2)$ and we define the $\mathrm{SL}(2,\mathbb{C})$-Hitchin map as:

$$h_{\mathrm{SL}(2,\mathbb{C})} : \mathcal{M}_{\mathrm{SL}(2,\mathbb{C})} \to \mathbb{A}^0 \subset \mathbb{A} \qquad (E,\phi) \mapsto (0,a), \tag{5.2}$$

where $a = \det \phi$. In particular, $\mathrm{SL}(2,\mathbb{C})$-spectral curves X_a have equation

$$x^2 - a = 0, \tag{5.3}$$

where x is the tautological coordinate on Tot (L). The curve in (5.3) has an involution $\iota(x) = -x$, having the zeros of a as its only fixed points.

Remark 5.1.7 ([37], Section I.3). Let $\pi : X \to C$ be a smooth double cover and assume that X has an involution ι. Then we can define Pr_π as the locus of line bundles M on X such that

$$M^\vee \cong \iota^* M.$$

More generally, by Remark 1.3.6, spectral curves are Gorenstein curves. The dual of a torsion-free sheaf on a Gorenstein curve is a torsion-free sheaf of the same rank, so we can characterize torsion-free sheaves η in the compactification $\overline{\mathrm{Pr}}_\pi$ of Pr_π in terms of torsion-free sheaves in $\overline{J}(X_a)$ satisfying

$$\eta^\vee \cong \iota^* \eta.$$

Remark 5.1.8 ([30], Section 4). More generally, for an affine reductive group G over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ of rank n, the characteristic coefficients $a_1, \cdots, a_n$ in (1.9) are naturally associated to a homogeneous basis of polynomials which are invariant under the adjoint action of G.

In particular, as we have seen above and as we see in Section 5.3, for groups G having underlying Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, the basis is simply given by a_2 in (1.9). On the other hand, for groups G having underlying Lie algebra

$$\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}),$$

which we consider in Sections 5.4, 5.6, 5.8, the basis is given by $\{a_2, p_4\}$ and we have $a_4 = p_4^2$.

Fibres of the $\mathrm{SL}(2, \mathbb{C})$-Hitchin map

We now describe the fibre of the Hitchin map $h_{\mathrm{SL}(2,\mathbb{C})}$.

Remark 5.1.9. Since $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C})$, we can endow $\mathrm{SL}(2, \mathbb{C})$-Higgs bundles with a non-degenerate symplectic form

$$\omega : E \otimes E \to \det E \cong \Lambda \tag{5.4}$$

satisfying the condition

$$\omega(\phi v, w) = -\omega(v, \phi w).$$

Thus, by slightly adapting the proof of Proposition 4.3.1 in [20] (for $r = 1$), we obtain a proof of Proposition 5.1.10.

Proposition 5.1.10 ([20], Proposition 4.1.2, [26], Theorem 6.1)**.** *Let $a \in \mathbb{A}^0$ be such that $\pi : X_a \to C$ is an integral spectral curve. Then we have*

$$h^{-1}_{SL(2,\mathbb{C})}(a) \cong \overline{Pr}_\pi.$$

Proof. By Remark 1.3.25, the rank-1 torsion free sheaf η on X_a associated to (E,ϕ) by Proposition 1.3.22 fits into the exact sequence

$$0 \to \eta \otimes \pi^* L^{-1} \to \pi^* E \xrightarrow{\pi^*\phi - x} \pi^*(E \otimes L) \to \eta \otimes \pi^* L \to 0. \tag{5.5}$$

Considering the dualized sequence and tensoring by $\pi^*(L \otimes \Lambda)$ we have the sequence

$$0 \to \eta^\vee \otimes \pi^*\Lambda \to \pi^*(E^\vee \otimes \Lambda) \xrightarrow{\pi^*\phi^t - x} \pi^*(E^\vee \otimes \Lambda) \otimes \pi^* L,$$

which is exact on the left.

Applying ι^* to the exact sequence (5.5) we have

$$0 \to \iota^*\eta \otimes \pi^* L^{-1} \to \pi^* E \xrightarrow{\pi^*\phi + x} \pi^*(E \otimes L) \to \iota^*\eta \otimes \pi^* L \to 0.$$

The form ω in (5.4) induces an isomorphism $\omega_E : E \xrightarrow{\sim} E^\vee \otimes \Lambda$, hence a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \eta^\vee \otimes \pi^*\Lambda \arrow[r] & \pi^*(E^\vee \otimes \Lambda) \arrow[r, "\pi^*\phi^t - x"] & \pi^*(E^\vee \otimes \Lambda) \otimes \pi^* L \; . \\
0 \arrow[r] & \iota^*\eta \otimes \pi^* L^{-1} \arrow[r] & \pi^* E \arrow[u, "\omega_E"] \arrow[r, "\pi^*\phi + x"] & \pi^*(E \otimes L) \arrow[u, "\omega_E"]
\end{tikzcd}$$

This yields an isomorphism

$$\eta^\vee \otimes \pi^*\Lambda \xrightarrow{\sim} \iota^*\eta \otimes \pi^* L^{-1}, \text{ thus } \eta^\vee \xrightarrow{\sim} \iota^*\eta \otimes \pi^*(L^{-1} \otimes \Lambda^{-1}) =: \iota^*\eta \otimes \pi^*\widetilde{L}^{-1}.$$

Hence we have

$$\eta_0 = \eta \otimes \pi^*\widetilde{L}^{-1/2} \in \overline{\mathrm{Pr}}_\pi \tag{5.6}$$

by Remark 5.1.7. □

We also consider an approach which refers to the characterization of Nm_π in (5.1).

Remark 5.1.11. By the classical BNR correspondence, the datum of a $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle is equivalent to that of a rank-1 torsion-free sheaf η on X_a. Let $B := \det(\pi_* \mathcal{O}_{X_a})^{-1} \in J(C)$. Then (1.34) yields

$$\mathrm{Nm}_\pi(\eta) = \det(\pi_* \eta) \otimes B,$$

so giving a trivialization $\det(\pi_*\eta) \xrightarrow{\sim} \Lambda$ is the same as giving an isomorphism $\mathrm{Nm}_\pi(\eta) \xrightarrow{\sim} \Lambda'$. Hence, arguing as in [20], Proposition 4.1.1, the datum of a $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle (E,ϕ) is the same as the datum of a torsion-free sheaf η in $\overline{\mathrm{Pr}}_\pi$ together with the datum of the isomorphism (5.1), i.e. it is a constrained torsion-free sheaf as anticipated in Remark 1.5.8.

Remark 5.1.12. If we choose $L \cong K_C$, spectral curves are ramified double covers, thus the associated Prym variety Pr_π yielding the generic fibre of the $\mathrm{SL}(2,\mathbb{C})$-Hitchin map is connected (recall Remark 1.5.5), e.g. by Mumford [37], Section I.3 for smooth spectral curves. We refer to Gothen-Oliveira [26], Theorem 6.3 for integral spectral curves.

We now consider the behaviour for different choices of L and discuss endoscopic loci.

Endoscopic loci for the $\mathrm{SL}(2,\mathbb{C})$-Hitchin map

Normalizing the spectral curve does not change the number of connected components of the associated Prym variety, as we see in the following lemma, since our case is a special case of the one which is studied by Hausel-Pauly [29], Lemma 4.1(4).

Lemma 5.1.13. *Let $\pi : X_a \to C$ be a reduced spectral curve, let the map $\nu : \widetilde{X}_a \to X_a$ be its normalization and let $\widetilde{\pi} : \widetilde{X}_a \to C$ be the natural projection. Then we have*

$$\mathcal{G}_{conn}(Pr_\pi) \cong \mathcal{G}_{conn}(Pr_{\widetilde{\pi}}),$$

where $\mathcal{G}_{conn}$ denotes the group of connected components.

Definition 5.1.14. We say that a $\mathrm{SL}(2,\mathbb{C})$-cover $\pi : X_a \to C$ is *endoscopic* if the natural projection $\widetilde{\pi}$ factors through an étale double cover of C.

We also say that the characteristic a is *endoscopic* and we call *endoscopic loci*, denoted $\mathbb{A}^0_{\mathrm{SL}(2,\mathbb{C}),e}$, the loci of characteristics $a \in \mathbb{A}^0$ such that the associated spectral curve X_a is endoscopic.

Notation 5.1.15. We denote by Γ the group $J^0(C)[2] \cong \mathbb{Z}_2^{2g}$ of line bundles $\gamma \in J^0(C)$ such that $\gamma^2 \cong \mathcal{O}_C$.

We will refer to the natural projection $\widetilde{\pi}$ as the normalization of π.

Remark 5.1.16. We can interpret Definition 5.1.14 by considering a two-torsion point $\gamma \in \Gamma^*$ and taking a as a section of

$$(K_C \otimes \gamma)^2 \cong K_C^2. \tag{5.7}$$

As in de Cataldo-Hausel-Migliorini [23], Section 4, there is a "squaring" map

$$i_\gamma : H^0(C, K_C \otimes \gamma) \to H^0(C, K_C^2) = \mathbb{A}^0 \qquad i_\gamma(a') = a' \otimes a', \tag{5.8}$$

with image $\mathbb{A}^0_\gamma \subset \mathbb{A}^0$.

By Proposition 1.5.7, together with Lemma 5.1.13, endoscopic loci are characterized by the fact that the number of connected components of Pr_π is larger than the number of connected components that Pr_π has when we consider $L \cong K_C$. More precisely, from [29], Theorem 5.3, we obtain the following characterization of endoscopic $\mathrm{SL}(2,\mathbb{C})$-characteristics.

Theorem 5.1.17. *Let $\pi : X_a \to C$ be a $SL(2,\mathbb{C})$-spectral curve. The group $\mathcal{G}_{conn}(\mathrm{Pr}_\pi)$ is nontrivial if and only if the characteristic a belongs to*

$$\mathbb{A}^0_{SL(2,\mathbb{C}),e} = \cup_{\gamma \in \Gamma^*} \mathbb{A}^0_\gamma.$$

We now discuss endoscopic loci by considering the total space Tot (L) of the line bundle L.

Remark 5.1.18. Assume that X_a is a $\mathrm{SL}(2,\mathbb{C})$-spectral curve. By (5.7), we can see X_a either in Tot (K_C) or in Tot $(K_C \otimes \gamma)$. Let $\pi : X_a \to C$ be a standard $\mathrm{SL}(2,\mathbb{C})$-spectral curve, let π_γ be the étale double cover of C associated to γ.

If we consider the generic $\mathrm{SL}(2,\mathbb{C})$-spectral curve $\pi : X_a \to C$, then it has equation $x^2 - a = 0$, where $a \in H^0(C, K_C^2)$ and Pr_π is connected by Remark 5.1.12. Hence we have

$$\mathcal{G}_{conn}(\mathrm{Pr}_\pi) \cong \{\mathcal{O}_C\}.$$

If we consider $\gamma \in \Gamma^*$ and we consider the cover $X_a \to C$ in Tot $(K_C \otimes \gamma)$, then it is reducible and it has equation

$$x^2 - (a')^2 = 0. \tag{5.9}$$

In particular, it has double points at the $2g-2$ zeros of $a' \in H^0(C, K_C \otimes \gamma)$. Its normalization $\widetilde{\pi}$ factors via the étale cover π_γ. In this case, Proposition 1.5.7, together with Lemma 5.1.13, yields:

$$\mathcal{G}_{conn}(\mathrm{Pr}_\pi) \cong \mathbb{Z}/2\mathbb{Z}.$$

Notation 5.1.19. Let $a \in \mathbb{A}$ be a characteristic such that the associated spectral curve X_a is endoscopic. We say that X_a is *generic* (also, that a is generic) if X_a has ramification points as its only singularities.

Remark 5.1.20. Let C be a smooth base curve. Then there are no $\mathrm{GL}(n, \mathbb{C})$-endoscopic loci because, in our assumptions Notation 1.3.16, the fibre of the Hitchin map is irreducible.

5.2 $\mathrm{SL}(2, \mathbb{C})$-Higgs bundles on a one-nodal curve of compact type

Notation 5.2.1. Let $\mathcal{M}_{12}^{\mathrm{SL}(2,\mathbb{C})}$ be the moduli space of Hitchin triples associated to α-stable Hitchin pairs (E, ϕ), where E has fixed determinant Λ of odd degree (so, in particular, E has odd Euler characteristic) and ϕ has trace zero. In particular, as we did in Section 3.3, we assume that $\chi_{C_1}(E_1)$ is odd and that $\chi_{C_2}(E_2)$ is even. By these assumptions, the study of α-stability of $\mathrm{SL}(2, \mathbb{C})$-Hitchin triples reduces to the study of α-stability of Hitchin triples that we gave in Chapters 3 and 4, so in this chapter we assume that Hitchin triples are α-stable.

We first parallel Section 4.4, describing the fibres of the adapted $\mathrm{SL}(2, \mathbb{C})$-Hitchin map and computing the dimension of $\mathcal{M}_{12,\mathrm{SL}(2,\mathbb{C})}$ for $L_1 \cong K_{C_1}$ and $L_2 \cong K_{C_2}$, then we consider endoscopic loci.

Fibres of the adapted $\mathrm{SL}(2,\mathbb{C})$-Hitchin map for spectral covers unramified over the node

Notation 5.2.2. Referring to Section 4.2, in particular to Remark 4.2.1, we denote by

$$(\mathbb{A}^0)^{(q)} \subset \mathbb{A}^{(q)}$$

the locus of characteristics $a = (a_1, a_2)$, $a_1 = (0, a_{1,2})$, $a_2 = (0, a_{2,2})$ such that the Higgs fields $\phi_i : E_i \to E_i \otimes L_i$, $i = 1, 2$ have trace zero. In particular, they have trace zero over the node q, so in order to have a semistable $\mathrm{SL}(2,\mathbb{C})$-Hitchin triple it is enough to ask that $\phi_i^{(q)}$, which are given e.g. as in (4.5), with $d_i = -c_i$, have the same determinant.

We first adapt Proposition 4.4.4 to $\mathrm{SL}(2,\mathbb{C})$-spectral curves. Denote by $(\mathbb{A}^0)^{ur} \subset (\mathbb{A}^0)^{(q)}$ the locus of characteristics associated to adapted spectral covers which are not ramified over q.

Proposition 5.2.3. *Let C be a one-nodal curve of compact type, consider $a \in (\mathbb{A}^0)^{ur}$ and let $X_a \to C$ be an adapted $SL(2,\mathbb{C})$-spectral cover having integral components X_{a_1}, X_{a_2}. Let $\alpha = (\alpha_1, \alpha_2)$ be a polarization such $\alpha_1\chi$ is not an integer. Then there is a bijective correspondence between:*

- *isomorphism classes of α-stable rank-1 torsion free sheaves η on X_a such that there is an isomorphism $\lambda : \det(\pi_*\eta) \xrightarrow{\sim} \Lambda$,*
- *isomorphism classes of $SL(2,\mathbb{C})$-Hitchin triples $(\hat{E}_1, \hat{E}_2, \overrightarrow{A}^{(q)}) \in \mathcal{M}_{12,SL(2,\mathbb{C})}$ associated to a.*

Hence we have an isomorphism

$$(h^{ad}_{SL(2,\mathbb{C})})^{-1}(a) \xrightarrow{\sim} \overline{Pr}_\pi.$$

Proof. The correspondence between rank-1 torsion-free sheaves on X_a and Hitchin triples on C is given by Proposition 4.4.4.

Letting $h_i : \mathcal{M}_{\mathrm{SL}(2,\mathbb{C}),i} \to \mathbb{A}_i$, $i = 1, 2$, be the Hitchin maps for the smooth base curves C_i and arguing as in the proof of Proposition 5.1.10 for each C_i, we have $h_i^{-1}(a_i) \cong \overline{\mathrm{Pr}}_{\pi_i}$.

Thus, letting f be the identification between the fibres of η_i at the preimages of q_i via the spectral cover, as in Figure 4.1 and Remark 4.4.1, the triple $(\eta_1, \eta_2, f) \in \overline{\mathrm{Pr}}_\pi$ yields the claim. □

Dimension of $\mathcal{M}_{12,\mathbf{SL}(2,\mathbb{C})}$

Let L_1 and L_2 be the canonical bundles K_{C_1} and K_{C_2} of the smooth components C_1 and C_2 of C and consider an adapted $\mathrm{SL}(2,\mathbb{C})$-spectral curve which is not ramified over the node.

By the proof of Proposition 5.2.3 and the proof of Proposition 4.4.5, we obtain the following.

Proposition 5.2.4. *The adapted Hitchin map*

$$h^{ad}_{SL(2,\mathbb{C})} : \mathcal{M}_{12,SL(2,\mathbb{C})} \to (\mathbb{A}^0)^{ur} \tag{5.10}$$

is surjective.

Thus we have

$$\dim \mathcal{M}_{12,\mathrm{SL}(2,\mathbb{C})} = \dim(\mathbb{A}^0)^{ur} + \dim(h^{ad}_{\mathrm{SL}(2,\mathbb{C})})^{-1}(a) = \dim(\mathbb{A}^0)^{(q)} + \dim \overline{\mathrm{Pr}}_\pi. \tag{5.11}$$

Proposition 5.2.5. *Let C be a one-nodal base curve of compact type having arithmetic genus g and assume that $L_1 \cong K_{C_1}$ and $L_2 \cong K_{C_2}$. Then we have:*

$$\dim \mathcal{M}_{12,SL(2,\mathbb{C})} = 6g - 12. \tag{5.12}$$

Proof. As discussed above, the generic fibre of $h^{ad}_{\mathrm{SL}(2,\mathbb{C})}$ is $\overline{\mathrm{Pr}}_\pi \subset \overline{J}(X_a)$ and we have

$$\dim \overline{\mathrm{Pr}}_\pi = \dim \overline{J}(X_a) - g = 3g - 5.$$

On the other hand, the locus $(\mathbb{A}^0)^{(q)}$ has codimension 1 in $\mathbb{A}^0_1 \times \mathbb{A}^0_2$. Thus we have

$$\dim(\mathbb{A}^0)^{(q)} = (3g_1 - 3) + (3g_2 - 3) - 1 = 3g - 7.$$

Hence (5.11) yields:

$$\dim \mathcal{M}_{12,\mathrm{SL}(2,\mathbb{C})} = 6g - 12.$$

$\square$

Endoscopic loci for the adapted $\mathbf{SL}(2,\mathbb{C})$-Hitchin map

We consider Theorem 5.1.17.

Definition 5.2.6. We say that an adapted spectral cover $\pi : X_a \to C$ of a one-nodal curve of compact type is *endoscopic* if the associated Prym variety Pr_π has more connected components than in the case in which we take $L_1 \cong K_{C_1}$ and $L_2 \cong K_{C_2}$ (in this case, each Prym variety Pr_{π_i} is connected by Remark 5.1.12).

Lemma 5.1.13 yields the following description of $\mathrm{SL}(2,\mathbb{C})$-endoscopic loci.

Proposition 5.2.7. *Let $a \in (\mathbb{A}^0)^{ur}$ be associated to a generic adapted spectral curve. If at least a $SL(2,\mathbb{C})$-spectral cover X_{a_i} of the component C_i is endoscopic, then X_a is endoscopic.*

Proof. Since the normalization $\widetilde{\pi}$ of a generic adapted $\mathrm{SL}(2,\mathbb{C})$-spectral curve $\pi : X_a \to C$ is given by the disjoint union of its components, we have

$$\mathrm{Pr}_{\widetilde{\pi}} \cong \mathrm{Pr}_{\pi_1} \times \mathrm{Pr}_{\pi_2}.$$

Thus, if at least a spectral cover X_{a_i} is endoscopic, then Lemma 5.1.13 yields that the associated adapted spectral curve is endoscopic. □

So, if exactly one spectral cover X_{a_i} is endoscopic, then we have

$$\mathcal{G}_{conn}(\mathrm{Pr}_\pi) \cong \mathbb{Z}/2\mathbb{Z},$$

while, if both covers are endoscopic, we have

$$\mathcal{G}_{conn}(\mathrm{Pr}_\pi) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Remark 5.2.8. When we study endoscopic loci for adapted spectral curves, it will be useful to consider the endoscopy of each component of the adapted spectral curve separately.

5.3 PGL$(2,\mathbb{C})$-Higgs bundles

We now present the moduli space of PGL$(2,\mathbb{C})$-Higgs bundles on a smooth base curve, referring to our description in Section 5.1 and to the exact sequence:

$$0 \to \mu_2 \xrightarrow{\lambda \mapsto \lambda I_2} \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PGL}(2,\mathbb{C}) \to 0, \tag{5.13}$$

where μ_2 is the group of square roots of the unity.

Let Γ be as in Notation 5.1.15.

Remark 5.3.1. Let $\gamma \in \Gamma$ and assume that $\rho : \gamma^2 \xrightarrow{\sim} \mathcal{O}_C$ is a trivialization. Let (E,ϕ) be a SL$(2,\mathbb{C})$-Higgs bundle and consider the equivalence relation $\sim$ defined by

$$(E,\phi) \sim (E \otimes \gamma, \phi \otimes 1_\gamma).$$

Then $[(E,\phi)]$ is the orbit of (E,ϕ) under the action of Γ on $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}$ defined by:

$$\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})} \times \Gamma \to \mathcal{M}_{\mathrm{SL}(2,\mathbb{C})} \qquad ((E,\phi),\gamma) \mapsto (E \otimes \gamma, \phi \otimes 1_\gamma). \tag{5.14}$$

Recalling Notation 5.1.2, we have that

$$\det(E \otimes \gamma) \xrightarrow{\sim} \det(E) \otimes \gamma^2 \xrightarrow[\rho]{\sim} \Lambda.$$

Definition 5.3.2. We define the moduli space of PGL$(2,\mathbb{C})$-Higgs bundles as the quotient

$$\mathcal{M}_{\mathrm{PGL}(2,\mathbb{C})} := \mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}/\Gamma.$$

Remark 5.3.3. Moreover, since $H^2(C,\mu_2) = \mathbb{Z}/2\mathbb{Z}$, (5.13) induces the exact sequence

$$H^1(C,\mu_2) \to H^1(C,\mathrm{SL}(2,\mathcal{O}_C)) \twoheadrightarrow H^1(C,\mathrm{PGL}(2,\mathcal{O}_C)) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

A PGL$(2,\mathbb{C})$-Higgs bundle (E,ϕ) of odd degree $\deg \Lambda$ lifts to a SL$(2,\mathbb{C})$-Higgs bundle (E_0,ϕ_0) of the same degree; any other lifting differs by the action of a 2-torsion line bundle by tensor product as above.

Remark 5.3.4. The action (5.14) of Γ on the moduli space of SL$(2,\mathbb{C})$-Higgs bundles corresponds to an action of $\pi^*\Gamma$ on the SL$(2,\mathbb{C})$-Hitchin fibre $\overline{\mathrm{Pr}}_\pi$ from Proposition 5.1.10, which we now describe:

$$\overline{\mathrm{Pr}}_\pi \times \pi^*\Gamma \to \overline{\mathrm{Pr}}_\pi \qquad (\eta,\pi^*\gamma) \mapsto \eta \otimes \pi^*\gamma \tag{5.15}$$

and, by (5.1), together with Remark 1.5.3, we have

$$\mathrm{Nm}_\pi(\eta \otimes \pi^*\gamma) \xrightarrow{\sim} \mathrm{Nm}_\pi(\eta) \otimes \mathrm{Nm}_\pi(\pi^*\gamma) \xrightarrow{\sim} \mathrm{Nm}_\pi(\eta) \otimes \gamma^2 \xrightarrow{\sim} \Lambda'.$$

Fibres of the $\mathbf{PGL}(2,\mathbb{C})$-Hitchin map

Proposition 4.2.5 in [20] yields the following.

Proposition 5.3.5. *Consider the $PGL(2,\mathbb{C})$-Hitchin map*

$$h_{PGL(2,\mathbb{C})} : \mathcal{M}_{PGL(2,\mathbb{C})} \to \mathbb{A}^0, \qquad (E,\phi) \mapsto (0,a). \tag{5.16}$$

Let $a \in \mathbb{A}^0$ be such that the associated spectral curve $\pi : X_a \to C$ is integral. Consider the action of $\pi^*\Gamma$ on the $SL(2,\mathbb{C})$-Hitchin fibre as in Remark 5.3.4. Then we have

$$h^{-1}_{PGL(2,\mathbb{C})}(a) \cong \overline{Pr}_\pi / \pi^*\Gamma. \tag{5.17}$$

Proof. By Proposition 5.1.10, the datum of (E,ϕ) is equivalent to that of $\eta \in \overline{\mathrm{Pr}}_\pi$.

Let γ be any 2-torsion line bundle on C. By the projection formula, we have

$$\pi_*(\eta \otimes \pi^*\gamma) \cong \pi_*\eta \otimes \gamma$$

and, letting x be the tautological section of π^*L, the following diagram commutes

$$\begin{tikzcd}
\pi_*(\eta \otimes \pi^*\gamma) \arrow[r, "\sim"] \arrow[d, "\pi_*(\cdot x)"] & \pi_*\eta \otimes \gamma \arrow[d, "\phi"] \\
\pi_*(\eta \otimes \pi^*\gamma) \otimes L \arrow[r, "\sim"] & (\pi_*\eta \otimes \gamma) \otimes L
\end{tikzcd}.$$

Hence the spectral correspondence is equivariant with respect to the action of $\pi^*\Gamma$ by tensor product on $\overline{\mathrm{Pr}}_\pi$ as in (5.15) and by the action of Γ on the moduli space $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}$ as in (5.14).

Passing to the quotient on both sides, the datum of the Γ-equivalence class $[(E,\phi)]$ corresponds uniquely to the datum of the $\pi^*\Gamma$-equivalence class $[\eta] \in \overline{\mathrm{Pr}}_\pi/\pi^*\Gamma$, yielding the claim. □

Proposition 5.3.6. *There are no nontrivial $PGL(2,\mathbb{C})$-endoscopic loci.*

Proof. The quotient defining $\mathrm{PGL}(2,\mathbb{C})$-Higgs data identifies extra-components of Pr_π, forcing the $\mathrm{PGL}(2,\mathbb{C})$-endoscopic locus to be trivial. □

PGL$(2, \mathbb{C})$-Higgs bundles on a base curve of compact type

We define the moduli space of PGL$(2, \mathbb{C})$-Hitchin triples as follows:

$$\mathcal{M}_{12}^{\mathrm{PGL}(2,\mathbb{C})} := \mathcal{M}_{12}^{\mathrm{SL}(2,\mathbb{C})}/\Gamma,$$

where the equivalence relation associated to the quotient by Γ is obtained by considering the action of two-torsion line bundles associated to the restrictions of γ to C_i as in Remark 2.1.5.

The fibre of the adapted PGL$(2, \mathbb{C})$-Hitchin map is given by

$$(\eta_1, \eta_2, f) \in \overline{\mathrm{Pr}}_\pi/\pi^*\Gamma$$

and, arguing as in the proof of Proposition 5.3.6 for both components C_i, we have the following.

Proposition 5.3.7. *Let C be a curve of compact type. Then the adapted PGL$(2, \mathbb{C})$-spectral curve X_a is not endoscopic.*

5.4 Spin$(4, \mathbb{C})$-Higgs bundles on a smooth curve

Notation 5.4.1. From now on, we assume that the group G has underlying Lie algebra $\mathfrak{so}(4, \mathbb{C})$.

We define Spin$(4, \mathbb{C})$-Higgs data via the isomorphism

$$\mathrm{Spin}(4, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}). \tag{5.18}$$

Notation 5.4.2. Relating to the assumptions from Notation 5.1.2, we assume that E_1 has fixed determinant Λ of odd degree. Moreover, we assume that E_2 has trivial determinant.

Definition 5.4.3. We define $\mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})}$ as $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})} \times \mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}$, thus:

$$\mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})} = \{(E_1 \otimes E_2, \phi_1 \otimes 1 + 1 \otimes \phi_2), (E_i, \phi_i) \in \mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}\},$$

where:

- $E = E_1 \otimes E_2$ is a rank-4 holomorphic vector bundle having fixed determinant Λ, with the choice of a positively oriented local frame for E,

- $\varphi = \phi_1 \otimes 1 + 1 \otimes \phi_2$ is a Higgs field having trace zero.

Notation 5.4.4. Following [16], when we discuss Higgs bundles associated to $\mathfrak{so}(4, \mathbb{C})$ on a smooth base curve, we denote by b_i the data referring to the i-th copy of $\mathrm{SL}(2, \mathbb{C})$-Higgs bundles via the $\mathrm{SL}(2, \mathbb{C})$-Hitchin map, for $i = 1, 2$.

The choice of using b_i instead of a_i is to try to avoid misunderstandings with notation in the previous sections and chapters.

Remark 5.4.5 ([5], Section 5.2). Let (E_1, ϕ_1) be a $\mathrm{SL}(2, \mathbb{C})$-Higgs bundle on C having fixed determinant Λ of odd degree, let (E_2, ϕ_2) be a $\mathrm{SL}(2, \mathbb{C})$-Higgs bundle having trivial determinant. Arguing as in Remark 5.1.9, we can endow $\mathrm{SL}(2, \mathbb{C})$-Higgs bundles with symplectic forms given, for $i = 1, 2$, by $\omega_i : E_i \otimes E_i \to \det E_i$.

Let E and φ be as in Definition 5.4.3. Then the vector bundle E has rank 4 and a non-degenerate symmetric quadratic form Q associated to $\omega_1 \otimes \omega_2$ and defined by

$$Q : E \times E \to \det E \cong \Lambda \qquad (y_1^{(1)} \otimes y_1^{(2)}, y_2^{(1)} \otimes y_2^{(2)}) \mapsto (y_1^{(1)} \wedge y_2^{(1)}) \otimes (y_1^{(2)} \wedge y_2^{(2)}). \tag{5.19}$$

Let $\{e_1^{(1)}, e_2^{(1)}\}$ and $\{e_1^{(2)}, e_2^{(2)}\}$ be bases of E_1 and E_2 respectively, and let the Higgs fields be

$$\phi_1 = \begin{pmatrix} \phi_{1,1}^{(1)} & \phi_{1,2}^{(1)} \\ \phi_{2,1}^{(1)} & -\phi_{1,1}^{(1)} \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} \phi_{1,1}^{(2)} & \phi_{1,2}^{(2)} \\ \phi_{2,1}^{(2)} & -\phi_{1,1}^{(2)} \end{pmatrix}$$

in the bases of E_1 and E_2 respectively. Since E is a $\mathrm{Spin}(4, \mathbb{C})$-Higgs bundle, referring to Definition 5.4.3, we choose

$$\{e_1^{(1)} \otimes e_1^{(2)}, e_1^{(1)} \otimes e_2^{(2)}, e_2^{(1)} \otimes e_1^{(2)}, e_2^{(1)} \otimes e_2^{(2)}\} \tag{5.20}$$

as a positively oriented local frame for E. With respect to (5.20), φ is given by

$$\varphi = \begin{pmatrix} \phi_{1,1}^{(1)} + \phi_{1,1}^{(2)} & \phi_{1,2}^{(2)} & \phi_{1,2}^{(1)} & 0 \\ \phi_{2,1}^{(2)} & \phi_{1,1}^{(1)} - \phi_{1,1}^{(2)} & 0 & \phi_{1,2}^{(1)} \\ \phi_{2,1}^{(1)} & 0 & -\phi_{1,1}^{(1)} + \phi_{1,1}^{(2)} & \phi_{1,2}^{(2)} \\ 0 & \phi_{2,1}^{(1)} & \phi_{2,1}^{(2)} & -\phi_{1,1}^{(1)} - \phi_{1,1}^{(2)} \end{pmatrix} \tag{5.21}$$

while Q is given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and we have ${}^t\varphi Q + Q\varphi = 0$.

Hence φ is skew-symmetric and the determinant of the matrix (5.21) is the square of a polynomial, which is called the *Pfaffian* of φ. The choice of a sign for the Pfaffian is an equivalent condition to the choice of a positively oriented local frame for E.

Note that the datum of Q is classically associated to $\mathrm{SO}(4,\mathbb{C})$-Higgs bundles lifting to $\mathrm{Spin}(4,\mathbb{C})$-Higgs bundles, as in Remark 5.6.3 and Remark 5.6.5.

On the other hand, the approach that we have recalled is consistent with the discussion in [39], Section 2 and with the fact that spectral curves associated to G-Higgs bundles, where G has underlying Lie algebra $\mathfrak{so}(4,\mathbb{C})$, have equation of the same form (Remark 5.1.8).

We now describe generic $\mathrm{Spin}(4,\mathbb{C})$-spectral curves.

Remark 5.4.6. As we will see in Remark 5.4.10, G-spectral curves are always singular, their generic singularities being double points. Note that this is different from the characterization of generic $\mathrm{GL}(2,\mathbb{C})$-spectral curves or $\mathrm{SL}(2,\mathbb{C})$-spectral curves, which are smooth by Remark 1.3.8.

Remark 5.4.7. From now on, relying on the genericity conditions on $\mathrm{SL}(2,\mathbb{C})$-spectral curves and on the fact that G-spectral data are given on the desingularization of the spectral curve, we only consider line bundles in the fibre of the Hitchin map.

We now recall the approach in [16], especially Section 5.4, whose fibred-product construction yields the desingularization of the G-spectral curve and the generic fibre of the $\mathrm{Spin}(4,\mathbb{C})$-Hitchin map.

Remark 5.4.8. Let $\pi_i : X_{b_i} \to C$, $i = 1, 2$, be the $\mathrm{SL}(2,\mathbb{C})$-Higgs data associated to $b_i \in H^0(C, L^2)$, whose equation, letting x be the tautological section of π^*L, is $x^2 - b_i = 0$. The fibred product

$$\widetilde{X}_b = X_{b_1} \times_C X_{b_2} \tag{5.22}$$

is thus given, as a curve in the total space of $L \oplus L$, by

$$x_1^2 - b_1 = x_2^2 - b_2 = 0, \tag{5.23}$$

(x_1, x_2) denoting the tautological section on $L \oplus L$. The situation is represented in the following diagram

$$\begin{tikzcd} & \widetilde{X}_b \arrow[dl, "p_1"'] \arrow[d, "\pi"] \arrow[dr, "p_2"] \arrow[drr, "p"] & & \\ X_{b_1} \arrow[dr, "\pi_1"'] & {} & X_{b_2} \arrow[dl, "\pi_2"] & \widetilde{X}_b/\widetilde{\iota} \arrow[dll, "\overline{\pi}"] \\ & C. & & \end{tikzcd} \tag{5.24}$$

where $p_i : \widetilde{X}_b \to X_{b_i}$ are the natural projections from the fibred product to its factors and p is obtained as in Remark 5.4.9.

Remark 5.4.9. Since generic $\mathrm{SL}(2, \mathbb{C})$-spectral curves are smooth, the curve $\widetilde{X}_b$ in (5.24) is smooth and provides the desingularization of the curve X_b. We obtain the equation of the curve X_b via the map

$$L \oplus L \to L, \tag{5.25}$$

given by fiberwise addition. In fact, setting $x = x_1 + x_2$ gives X_b the equation

$$x^4 - 2(b_1 + b_2)x^2 + (b_1 - b_2)^2 = 0. \tag{5.26}$$

Generically, the curve X_b has singularities which are double points corresponding to the zeros of $(b_1 - b_2)$. The $\mathrm{SL}(2, \mathbb{C})$-curves X_{b_i} have involutions ι_i, having the zeros of b_i as fixed points. Since these zeros are generically different, the involution $\widetilde{\iota} := (\iota_1, \iota_2)$ on $\widetilde{X}_b$ has no fixed points. Thus the map

$$p : \widetilde{X}_b \to \widetilde{X}_b/\widetilde{\iota}. \tag{5.27}$$

is an étale double cover. Then by [38], Lemma 1, the Prym variety Pr_p has two connected components whose elements are of the form $M \otimes \widetilde{\iota}^* M^\vee$, for M a line bundle on $\widetilde{X}_b$, since $\deg M$ can either be zero or one.

The choice of $\mathrm{Spin}(4, \mathbb{C})$-data as in Definition 5.4.3 comes with a canonical choice of the connected component of Pr_p, as we see in Proposition 5.4.11, giving a further equivalent characterization to the choice of a positively oriented local frame for E.

Remark 5.4.10. The involution $\widetilde{\iota}$ descends to an involution $x \mapsto -x$ on X_b, having fixed points at the zeros of $(b_1 - b_2)$, which are generically double points.

Fibres of the Spin$(4, \mathbb{C})$-Hitchin map

Recall Notation 5.1.5, together with Remark 5.1.8. If G has underlying Lie algebra $\mathfrak{so}(4, \mathbb{C})$, the target of the G-Hitchin map is given by

$$H^0(C, K_C^2) \oplus H^0(C, K_C^2) \cong \mathbb{A}^0 \oplus \mathbb{A}^0.$$

Considering [16], Proposition 14, Proposition 16 and Section 7.1 we obtain the following.

Proposition 5.4.11. *Let $b_i \in \mathbb{A}^0$ and let $\pi : X_b \to C$ be the curve associated to the $SL(2, \mathbb{C})$-spectral curves X_{b_i} as in Remark 5.4.9. Assume that X_b is integral and that its equation is* (5.26)*. Consider the Spin$(4, \mathbb{C})$-Hitchin map*

$$h_{Spin(4,\mathbb{C})} : \mathcal{M}_{Spin(4,\mathbb{C})} \to \mathbb{A}^0 \oplus \mathbb{A}^0, \quad ((E_1, \phi_1), (E_2, \phi_2)) \mapsto (-2(b_1 + b_2), b_1 - b_2). \tag{5.28}$$

Then, referring to diagram (5.24)*, we have*

$$h^{-1}_{Spin(4,\mathbb{C})}(b) \cong Pr^1_p.$$

Proof. Recalling Notation 5.4.2, let M_1 be a line bundle from Pr_{π_1} and let M_2 be a line bundle from Pr_{π_2}. Then we have

$$M := p_1^* M_1 \otimes p_2^* M_2 \in \mathrm{Pr}^1_p. \tag{5.29}$$

In fact, the line bundle M_1 has fixed odd degree and it is isomorphic to its dual via the involution ι_1, while the line bundle M_2 has degree zero and it is isomorphic to its dual via the involution ι_2. In particular, by (5.1), we have that M has norm Λ'. □

Notation 5.4.12. From now on, we simply refer to the fibre of the G-Hitchin map as Pr_p.

Endoscopic loci for the Spin$(4, \mathbb{C})$-Hitchin map

We first parallel Remark 5.1.16.

Remark 5.4.13. If only X_{b_1} is endoscopic, we consider the map

$$H^0(C, K_C \otimes \gamma_1) \oplus H^0(C, K_C^2) \to \mathbb{A}^0 \oplus \mathbb{A}^0 \qquad (b_1', b_2) \mapsto (b_1' \otimes b_1', b_2),$$

where $\gamma_1 \in \Gamma^*$ is a two-torsion line-bundle on C, while, if only X_{b_2} is endoscopic, we consider the map

$$H^0(C, K_C^2) \oplus H^0(C, K_C \otimes \gamma_2) \to \mathbb{A}^0 \oplus \mathbb{A}^0 \qquad (b_1, b_2') \mapsto (b_1, b_2' \otimes b_2'),$$

where $\gamma_2 \in \Gamma^*$ is a two-torsion line-bundle on C. If both covers X_{b_i} are endoscopic, then we consider the map

$$H^0(C, K_C \otimes \gamma_1) \oplus H^0(C, K_C \otimes \gamma_2) \to \mathbb{A}^0 \oplus \mathbb{A}^0 \qquad (b_1', b_2') \mapsto (b_1' \otimes b_1', b_2' \otimes b_2'),$$

for γ_i distinct 2-torsion line bundles in Γ^*.

Consider the SL$(2, \mathbb{C})$-spectral curves $X_{b_i} \to C_i$ and (5.24). Considering Proposition 5.4.11, we have the following.

Proposition 5.4.14. *A Spin$(4, \mathbb{C})$-spectral curve is endoscopic if $b_i \in \mathbb{A}^0_{SL(2,\mathbb{C}),e}$ for at least one value of i.*

Proof. If exactly one spectral curve X_{b_i} is endoscopic, assume without loss of generality that it is X_{b_1}. Thus the normalization of the SL$(2, \mathbb{C})-$cover $X_{b_1} \to C$ factors through an étale double cover. Hence, considering (5.24), the associated Spin$(4, \mathbb{C})$-cover factors through this cover and we have the claim. In particular, we have that

$$\mathcal{G}_{conn}(\mathrm{Pr}_p) \cong \mathbb{Z}/2\mathbb{Z},$$

as in Remark 5.1.18.

If both spectral curves X_{b_i} are endoscopic, we can argue as above twice and we have

$$\mathcal{G}_{conn}(\mathrm{Pr}_p) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

$\square$

Remark 5.4.15. We can also consider the case in which both $\mathrm{SL}(2,\mathbb{C})$-covers associated to a $\mathrm{Spin}(4,\mathbb{C})$-spectral cover are endoscopic by referring to (5.26) and observing that

$$x^4 - 2(b_1+b_2)x^2 + (b_1-b_2)^2 = x^4 - 2(b_1+b_2)x^2 + (b_1+b_2)^2 - 4b_1b_2. \quad (5.30)$$

Assume that both covers X_{b_i} are endoscopic, having equation $x_i^2 - (b_i')^2 = 0$ in Tot $(K_C \otimes \gamma_i)$. Referring to Remark 5.1.18, we can rewrite (5.30) as

$$(x^2-((b_1')^2+(b_2')^2)-2b_1'b_2')(x^2-((b_1')^2+(b_2')^2)+2b_1'b_2') = (x^2-(b_1'+b_2')^2)(x^2-(b_1'-b_2')^2) \quad (5.31)$$

which further factors. Hence we have

$$\mathcal{G}_{conn}(\mathrm{Pr}_p) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

5.5 $\mathrm{Spin}(4,\mathbb{C})$-Higgs bundles on a base curve of compact type

Notation 5.5.1. When we discuss the Hitchin map for base curves of compact type, we use the following notation: we denote with the subscripts E_i, b_i the data referring to the components C_i, while we denote by the superscripts E^j, b^j the j-th copy of a datum.

For example, considering the torsion-free sheaf E^2 on C, we denote by E_1^2 the datum of the vector bundle which is associated to the restriction of E^2 to C_1.

Remark 5.5.2. Recalling the description of the moduli space $\mathcal{M}_{12,\mathrm{SL}(2,\mathbb{C})}$ from Notation 5.2.1 and that of the moduli space $\mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})}$ in Definition 5.4.3, we make the following assumptions. Let $\hat{E}^1, \hat{E}^2$ be distinct $\mathrm{SL}(2,\mathbb{C})$-Hitchin pairs on C.

- the $\mathrm{SL}(2,\mathbb{C})$-Hitchin pair $\hat{E}^1$ on C is such that E_1^1 has fixed determinant of odd degree, while the E_2^1 has trivial determinant.

- the $\mathrm{SL}(2,\mathbb{C})$-Hitchin pair $\hat{E}^2$ on C is such that both E_1^2 and E_2^2 have trivial determinant.

Moreover, we assume that all adapted $\mathrm{SL}(2,\mathbb{C})$-spectral curves are smooth over points which are different from the node q of C.

We define $\mathcal{M}_{12,\mathrm{Spin}(4,\mathbb{C})}$ as

$$\mathcal{M}_{12,\mathrm{Spin}(4,\mathbb{C})} = \{((E_1^1\otimes E_1^2, \phi_1^1\otimes 1+1\otimes\phi_1^2), (E_2^1\otimes E_2^2, \phi_2^1\otimes 1+1\otimes\phi_2^2), \overrightarrow{A}^{(q)})\}.$$

We now adapt Remark 5.4.6 to the case in which C is a one-nodal curve of compact type. Arguing as in the proof of Proposition 5.4.11 for each smooth base component C_i, we have the following.

Proposition 5.5.3. *Let C be a one-nodal base curve of compact type, let b be the characteristic associated to the adapted $Spin(4,\mathbb{C})$-spectral curve having components*

$$\widetilde{X}_{b_i} := (X_{b_i^1} \times_{C_i} X_{b_i^2}). \tag{5.32}$$

Then we have that

$$(h^{ad}_{Spin(4,\mathbb{C})})^{-1}(b) \cong Pr_p. \tag{5.33}$$

Remark 5.5.4. The form of the fibre (5.33) is consistent with the assumption in Remark 5.4.7. In particular, elements of Pr_p are line bundles M_i on $\widetilde{X}_{b_i}$ together with gluing data over the preimages of the node, mirroring Figure 4.1.

In the same assumptions as above, we have the following.

Proposition 5.5.5. *If at least a $SL(2,\mathbb{C})$-spectral cover $X_{b_i^j}$ of C_i is endoscopic, then the associated adapted $Spin(4,\mathbb{C})$-spectral curve is endoscopic. In particular, we have*

$$\mathcal{G}_{conn}(Pr_p) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus k},$$

where $1 \le k \le 4$ is the number of $X_{b_i^j}$ which are endoscopic.

Proof. By Proposition 5.2.7, if at least a $\mathrm{SL}(2,\mathbb{C})$-spectral cover of C_i is endoscopic, then the adapted spectral curve is endoscopic. Arguing as in the proof of Proposition 5.4.14, we have the claim. □

5.6 SO$(4,\mathbb{C})$-Higgs bundles on a smooth base curve

We now present the moduli space of SO$(4,\mathbb{C})$-Higgs bundles on a smooth base curve, referring to our description in Section 5.4, in particular to the choice of the degrees of the vector bundles E_1 and E_2, and to the exact sequence:

$$0 \to \mu_2 \to \mathrm{Spin}(4,\mathbb{C}) \to \mathrm{SO}(4,\mathbb{C}) \to 0, \tag{5.34}$$

where μ_2 is the group of square roots of the unity.

Remark 5.6.1. Let $\gamma \in \Gamma$ and assume that $\rho : \gamma^2 \xrightarrow{\sim} \mathcal{O}_C$ is a trivialization. Let (E,ϕ) be a Spin$(4,\mathbb{C})$-Higgs bundle and consider the equivalence relation $\sim$ defined by

$$(E,\phi) \sim (E \otimes \gamma, \phi \otimes 1_\gamma).$$

Then $[(E,\phi)]$ is the orbit of (E,ϕ) under the action of Γ on $\mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})}$ defined by:

$$\mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})} \times \Gamma \to \mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})} \qquad ((E,\phi),\gamma) \mapsto (E \otimes \gamma, \phi \otimes 1_\gamma). \tag{5.35}$$

Recalling Notation 5.1.2, we have that

$$\det(E \otimes \gamma) \xrightarrow{\sim} \det(E) \otimes \gamma^2 \xrightarrow[\rho]{\sim} \Lambda.$$

Definition 5.6.2. We define the moduli space of SO$(4,\mathbb{C})$-Higgs bundles as

$$\mathcal{M}_{\mathrm{SO}(4,\mathbb{C})} := \mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})}/\Gamma.$$

Remark 5.6.3. The exact sequence (5.34) yields the sequence

$$H^1(C,\mu_2) \to H^1(C,\mathrm{Spin}(4,\mathcal{O}_C)) \twoheadrightarrow H^1(C,\mathrm{SO}(4,\mathcal{O}_C)) \to \mathbb{Z}/2\mathbb{Z}. \tag{5.36}$$

A SO$(4,\mathbb{C})$-Higgs bundle (E,ϕ) having fixed determinant Λ of odd degree lifts to a Spin$(4,\mathbb{C})$-Higgs bundle having fixed determinant Λ of odd degree, any other lifting differs by the action of a 2-torsion line bundle by tensor product as above.

We now describe the action of $\pi^*\Gamma$ on $h^{-1}_{\mathrm{Spin}(4,\mathbb{C})}(b) \cong \mathrm{Pr}_p$.

Remark 5.6.4. Let $M \in \mathrm{Pr}_p$ be a line bundle on $\widetilde{X}_b$, let $\beta : \mathrm{Nm}_\pi(M) \xrightarrow{\sim} \Lambda'$ and $\rho : \gamma^2 \xrightarrow{\sim} \mathcal{O}_C$ be isomorphisms. Let $\pi^*\Gamma$ act on Pr_p as follows:

$$\mathrm{Pr}_p \times \pi^*\Gamma \to \mathrm{Pr}_p \quad (M, \pi^*\gamma) \mapsto M \otimes \pi^*\gamma. \tag{5.37}$$

Then we have

$$\mathrm{Nm}_\pi(M \otimes \pi^*\gamma) \xrightarrow{\sim} \mathrm{Nm}_\pi(M) \otimes \mathrm{Nm}_\pi(\pi^*\gamma) \xrightarrow[\beta\otimes\rho]{\sim} \Lambda'.$$

Remark 5.6.5. We go on following the description in [16], which relies on the isogeny between $\mathrm{SO}(4,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. We refer to [45], Section 2.2.5 for the classical description of $\mathrm{SO}(4,\mathbb{C})$-Higgs bundles by Hitchin, which relies on the form Q from Remark 5.4.5.

Fibres of the SO$(4,\mathbb{C})$-Hitchin map

Proposition 5.6.6. *Let $b_i \in \mathbb{A}^0$ and let $\pi : X_b \to C$ be the curve associated to the $SL(2,\mathbb{C})$-spectral curves X_{b_i} as in Remark 5.4.9. Assume that X_b is integral and that its equation is* (5.26). *Consider the $SO(4,\mathbb{C})$-Hitchin map*

$$h_{SO(4,\mathbb{C})} : \mathcal{M}_{SO(4,\mathbb{C})} \to \mathbb{A}^0 \oplus \mathbb{A}^0, \quad ((E_1,\phi_1),(E_2,\phi_2)) \mapsto (-2(b_1+b_2), b_1 - b_2). \tag{5.38}$$

Let $\pi^*\Gamma$ act on Pr_p as in Remark 5.6.4. Then we have

$$(h_{SO(4,\mathbb{C})})^{-1}(b) \cong Pr_p/\pi^*\Gamma, \tag{5.39}$$

where Pr_p is as in Proposition 5.4.11.

Proof. By Proposition 5.4.11, the datum of (E,ϕ) corresponds to the datum of $M \in \mathrm{Pr}_p$. Let γ be any 2-torsion line bundle on C. By the projection formula, we have

$$\pi_*(M \otimes \pi^*\gamma) \cong \pi_* M \otimes \gamma$$

and the following diagram commutes

$$\begin{tikzcd}
\pi_*(M \otimes \pi^*\gamma) \arrow[r, "\sim"] \arrow[d, "\pi_*(\cdot x)"] & \pi_* M \otimes \gamma \arrow[d, "\phi"] \\
\pi_*(M \otimes \pi^*\gamma) \otimes L \arrow[r, "\sim"] & (\pi_* M \otimes \gamma) \otimes L
\end{tikzcd}$$

The spectral correspondence is equivariant with respect to the action of $\pi^*\Gamma$ on Pr_p in (5.37) and by the action of Γ on the moduli space $\mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})}$ in (5.35).

Passing to the quotient on both sides, the datum of the Γ-equivalence class $[(E,\phi)]$ corresponds uniquely to the datum of the $\pi^*\Gamma$-equivalence class $[M] \in \mathrm{Pr}_p/\pi^*\Gamma$. □

Endoscopic loci for the SO$(4,\mathbb{C})$-Hitchin map

Proposition 5.6.6, in particular (5.39), implies the following.

Proposition 5.6.7. *A $SO(4,\mathbb{C})$-spectral curve is endoscopic if and only if both associated $SL(2,\mathbb{C})$-spectral covers appearing in* (5.24) *are endoscopic.*

Proof. Assume, without loss of generality, that X_{b_1} is the only endoscopic cover. Then the action of $\pi^*\Gamma$ identifies extra components of the associated Prym variety, thus the associated SO$(4,\mathbb{C})$-characteristic is not endoscopic.

Hence a SO$(4,\mathbb{C})$-spectral curve is endoscopic if and only if both associated SL$(2,\mathbb{C})$-spectral curves are endoscopic. □

5.7 SO$(4,\mathbb{C})$-Higgs bundles on a base curve of compact type

Recall Notation 5.5.1. We define $\mathcal{M}_{12,\mathrm{SO}(4,\mathbb{C})}$ as

$$\mathcal{M}_{12,\mathrm{SO}(4,\mathbb{C})} = \mathcal{M}_{12,\mathrm{Spin}(4,\mathbb{C})}/\Gamma.$$

Arguing as in the proof of Proposition 5.5.3, we have the following characterization of the fibre of the adapted SO$(4,\mathbb{C})$-Hitchin map.

Proposition 5.7.1. *The fibre of the adapted $SO(4,\mathbb{C})$-Hitchin map is given by:*

$$(h^{ad}_{SO(4,\mathbb{C})})^{-1}(b) \cong Pr_p/\pi^*\Gamma$$

Considering Proposition 5.5.3, together with Proposition 5.7.1, we have the following.

Proposition 5.7.2. *If at least three characteristics b_i^j are endoscopic, then the associated adapted $SO(4,\mathbb{C})$-spectral cover is endoscopic.*

In particular, if exactly three spectral covers $X_{b_i^j}$ are endoscopic, we have that

$$\mathcal{G}_{conn}((h^{ad}_{\mathrm{SO}(4,\mathbb{C})})^{-1}(b)) \cong \mathbb{Z}/2\mathbb{Z}.$$

If all spectral covers $X_{b_i^j}$ are endoscopic, we have that

$$\mathcal{G}_{conn}((h^{ad}_{\mathrm{SO}(4,\mathbb{C})})^{-1}(b)) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

5.8 PSO$(4,\mathbb{C})$-Higgs bundles

We now present the moduli space of PSO$(4,\mathbb{C})$-Higgs bundles on a smooth base curve, referring to our description in Sections 5.4 and 5.6.

We consider the exact sequence

$$0 \to \mu_2 \times \mu_2 \to \mathrm{Spin}(4,\mathbb{C}) \to \mathrm{PSO}(4,\mathbb{C}) \to 0, \tag{5.40}$$

where μ_2 is the group of square roots of the unity.

Remark 5.8.1. Let $\gamma, \zeta \in \Gamma$ and assume that $\rho : \gamma^2 \xrightarrow{\sim} \mathcal{O}_C, \tau : \zeta^2 \xrightarrow{\sim} \mathcal{O}_C$ are trivializations. Consider the equivalence relation $\sim$ defined by

$$(E,\phi) \sim (E \otimes \gamma \otimes \zeta, \phi \otimes 1_\gamma \otimes 1_\zeta), \quad \text{for any } (\gamma,\zeta) \in \Gamma \times \Gamma \tag{5.41}$$

Then $[(E,\phi)]$ is the orbit of (E,ϕ) under the action of $\Gamma \times \Gamma$ on $\mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})}$ defined by:

$$\mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})} \times (\Gamma \times \Gamma) \to \mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})} \qquad ((E,\phi),(\gamma,\zeta)) \mapsto (E \otimes \gamma \otimes \zeta, \phi \otimes 1_\gamma \otimes 1_\zeta). \tag{5.42}$$

Recalling Notation 5.1.2, we have that

$$\det(E \otimes \gamma \otimes \zeta) \xrightarrow{\sim} \det(E) \otimes \gamma^2 \otimes \zeta^2 \xrightarrow[\rho\otimes\tau]{\sim} \Lambda.$$

Definition 5.8.2. We define the moduli space of PSO$(4,\mathbb{C})$-Higgs bundles as the quotient

$$\mathcal{M}_{\mathrm{PSO}(4,\mathbb{C})} := \mathcal{M}_{\mathrm{Spin}(4,\mathbb{C})}/(\Gamma \times \Gamma).$$

Remark 5.8.3. The sequence (5.40) yields the sequence

$$H^1(C, \mu_2 \times \mu_2) \to H^1(C, \mathrm{Spin}(4,\mathcal{O}_C)) \twoheadrightarrow H^1(C, \mathrm{PSO}(4,\mathcal{O}_C)) \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{5.43}$$

A PSO$(4,\mathbb{C})$-Higgs bundle (E,ϕ) having fixed determinant Λ of odd degree lifts to a Spin$(4,\mathbb{C})$-Higgs bundle having fixed determinant Λ of odd degree, any other lifting differs by an action as in (5.42).

Remark 5.8.4. Let $\pi^*\Gamma \times \pi^*\Gamma$ act on $h^{-1}_{\mathrm{Spin}(4,\mathbb{C})}(b) \cong \mathrm{Pr}_p$ as follows:

$$\mathrm{Pr}_p \times (\pi^*\Gamma \times \pi^*\Gamma) \to \mathrm{Pr}_p \quad (M, (\pi^*\gamma, \pi^*\zeta)) \mapsto M \otimes \pi^*\gamma \otimes \pi^*\zeta, \tag{5.44}$$

and we have

$$\mathrm{Nm}(M \otimes \pi^*\gamma \otimes \pi^*\zeta) \xrightarrow{\sim} \mathrm{Nm}(M) \otimes \gamma^2 \otimes \zeta^2 \xrightarrow[\rho\otimes\tau]{\sim} \Lambda'.$$

Proposition 5.8.5. *Let $b_i \in \mathbb{A}^0$ and let $\pi : X_b \to C$ be the curve associated to the $SL(2,\mathbb{C})$-spectral curves X_{b_i} as in Remark 5.4.9. Assume that X_b is integral and that its equation is* (5.26)*. Consider the $PSO(4,\mathbb{C})$-Hitchin map*

$$h_{PSO(4,\mathbb{C})} : \mathcal{M}_{PSO(4,\mathbb{C})} \to \mathbb{A}^0 \oplus \mathbb{A}^0, \quad ((E_1,\phi_1),(E_2,\phi_2)) \mapsto (-2(b_1+b_2), b_1 - b_2). \tag{5.45}$$

Let $\pi^*\Gamma \times \pi^*\Gamma$ act on Pr_p as in Remark 5.8.4. Then we have

$$h^{-1}_{PSO(4,\mathbb{C})}(b) \cong Pr_p/(\pi^*\Gamma \times \pi^*\Gamma)$$

where Pr_p is as in Proposition 5.4.11.

Proof. By Proposition 5.4.11, the datum of (E,ϕ) corresponds to the datum of $M \in \mathrm{Pr}_p$. By the projection formula, we have

$$\pi_*(M \otimes \pi^*\gamma \otimes \pi^*\zeta) \cong \pi_* M \otimes \gamma \otimes \zeta$$

and the following diagram commutes

$$\begin{tikzcd}
\pi_*(M \otimes \pi^*\gamma \otimes \pi^*\zeta) \arrow[r, "\sim"] \arrow[d, "\pi_*(\cdot x)"] & \pi_* M \otimes \gamma \otimes \zeta \arrow[d, "\phi"] \\
\pi_*(M \otimes \pi^*\gamma \otimes \pi^*\zeta) \otimes L \arrow[r, "\sim"] & (\pi_* M \otimes \gamma \otimes \zeta) \otimes L
\end{tikzcd}$$

Passing to the quotient, the datum of the $(\Gamma \times \Gamma)$-equivalence class $[(E,\phi)]$ corresponds uniquely to the datum of the $(\pi^*\Gamma \times \pi^*\Gamma)$-equivalence class $[M] \in \mathrm{Pr}_p/(\pi^*\Gamma \times \pi^*\Gamma)$.

$\square$

Proposition 5.8.5 implies the following characterization of $\mathrm{PSO}(4,\mathbb{C})$-endoscopic loci.

Proposition 5.8.6. *There are no nontrivial PSO$(4,\mathbb{C})$-endoscopic loci.*

We can define the moduli space of $\mathrm{PSO}(4,\mathbb{C})$-Hitchin triples arguing similarly to Section 5.7, defining

$$\mathcal{M}_{12,\mathrm{PSO}(4,\mathbb{C})} := \mathcal{M}_{12,\mathrm{Spin}(4,\mathbb{C})}/(\Gamma \times \Gamma).$$

We have

$$(h^{ad}_{\mathrm{PSO}(4,\mathbb{C})})^{-1}(b) \cong \mathrm{Pr}_p/(\pi^*\Gamma \times \pi^*\Gamma),$$

which, applying Proposition 5.8.5 to both components C_i, yields the following.

Proposition 5.8.7. *Let C be a base curve of compact type. There are no nontrivial PSO$(4,\mathbb{C})$-endoscopic loci.*

Appendix A

Spectral curves in terms of moduli spaces of double covers

In this Appendix, we interpret generic spectral covers (as in Remark 1.3.8) of a smooth base curve C in terms of families of double covers and we interpret adapted spectral curves (as in Definition 4.3.1) in terms of a compactification of this family.

A.1 Double covers of smooth base curves as generic spectral curves

Remark A.1.1. Let C be a smooth curve and let $\pi : X \to C$ be a double cover of C. Then we have

$$\pi_* \mathcal{O}_X \cong \mathcal{O}_C \oplus L^{-1}, \tag{A.1}$$

where L is a line bundle on C such that $L^2 \cong \mathcal{O}_C(B)$, where B is the branch divisor of π.

Let $n = \deg B = 2\deg L$ be an even integer. Consider the moduli space $\mathcal{M}_{g,n}$ of smooth curves of genus g with n marked points. By an abuse of notation, we refer to points of $\mathcal{M}_{g,n}$ as (C, B) where C is a smooth curve having arithmetic genus g, on which we consider the divisor B given by the sum of n unordered marked points, which are smooth and distinct. Endowing (C, B) with the datum of L yields a point of the moduli space $\mathcal{D}_{g,n}$, parametrizing smooth double covers of C which are ramified over the divisor B. We refer to Harris-Morrison [27] for a description of the moduli spaces $\mathcal{D}_{g,n}$ and $\mathcal{M}_{g,n}$, together with their compactifications.

We will focus on the case in which the line bundle L on C has degree $2g-2$. Remark A.1.1 yields the following interpretation of generic spectral curves in terms of points of $\mathcal{D}_{g,4g-4}$. Recall the definition of spectral data Definition 1.3.24.

Proposition A.1.2. *Let (E, ϕ) be a rank-2 Higgs bundle on a smooth base curve C and let $\pi : X \to C$ be the associated spectral curve, which is assumed to be smooth and ramified at $4g-4$ distinct points p_i. Let x_i be their images on C. Consider $(C, B) \in \mathcal{M}_{g,4g-4}$ as a smooth curve which is pointed at $B = x_1 + \cdots + x_{4g-4}$. Let $L \in J^{2g-2}(C)$ be a square root of $\mathcal{O}_C(B)$. Then spectral data are also given by the datum of $(C, B, L) \in \mathcal{D}_{g,4g-4}$ and line bundles in $J^d(X)$ having suitable degree $d = \deg E + \deg L$ yield the Higgs bundle (E, ϕ) on C.*

Proof. Let $(C, B) \in \mathcal{M}_{g,4g-4}$ be a smooth base curve, which is pointed at the $4g-4$ points of B. Considering a square root L of $\mathcal{O}_C(B)$, we obtain a double cover X of C. Consider a line bundle M on X of suitable degree d, satisfying (1.24) for $n = 2$:

$$d = \deg E + \deg L.$$

Then the pushforward of M via π yields a vector bundle E which, together with the pushforward of multiplication by the tautological coordinate x of Tot (L), yields the Higgs datum (E, ϕ) on C. $\square$

A.2 Double covers of base curves of compact type as adapted spectral curves

We now consider a pointed one-nodal base curve (C, B) of compact type from the compactification $\overline{\mathcal{M}}_{g,4g-4}$ of $\mathcal{M}_{g,4g-4}$ having the points of B which distribute on the components C_i. We consider points from a compactification $\overline{\mathcal{D}}_{g,4g-4}$ of the moduli space $\mathcal{D}_{g,4g-4}$ of double covers of C.

Definition A.2.1. Let (C, B) be a one-nodal pointed curve of compact type. An *admissible double cover* of (C, B) is a pointed nodal curve X with a finite map $\pi : X \to C$ of degree 2 such that:

- the preimages of nonsingular points of C are nonsingular points of X and the restriction of π to the open set of nonsingular points is as follows: it is simply branched over the marked points and it is unramified otherwise,
- the preimage of the node of C consists of one or two nodes of X.

Similarly to the description by Cornalba-Harris [21], we obtain the following.

Proposition A.2.2. *Let C be a one-nodal pointed curve of compact type having genus g. Let X be its admissible double cover. Then only two cases are possible:*

(a) the cover X is unramified over the node q. This happens precisely when each component C_i carries an even number of marked points,

(b) the cover X is ramified over the node q. This happens precisely when each component C_i carries an odd number of marked points.

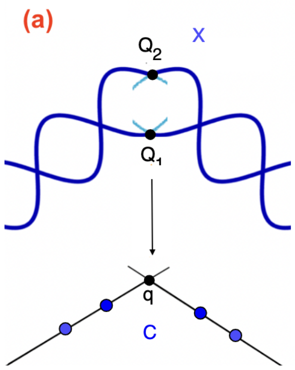
Proof. Recall that the line bundles L_1 and L_2 (given by restricting L to C_1 and C_2), yielding the double covers of C_1 and C_2, exist if and only if the degree of $\mathcal{O}_C(B_i)$ is even. Since we have patterns of $4g - 4$ smooth points which distribute on the two components C_i as $x_1 + \cdots + x_k, x_{k+1} + \ldots + x_{4g-4}$, the only two possible cases are the following ones (as in Figures A.1a and A.1b):

(a) the cover is not ramified over q and $k = 2s + 2$ is even. Boundary points have the form

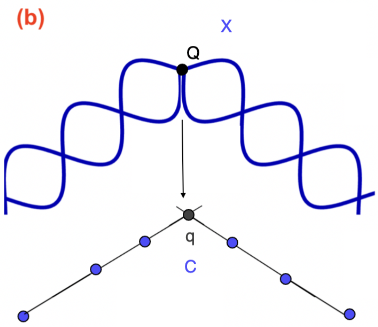
$$(C, B, L) : L_1^2 \cong \mathcal{O}_{C_1}(x_1 + \ldots + x_{2s+2}), L_2^2 \cong \mathcal{O}_{C_2}(x_{2s+3} + \ldots + x_{4g-4}), \tag{A.2}$$

(b) the cover π is ramified over q and $k = 2s + 1$ is odd. Boundary points have the form

$$(C, B, L) : L_1^2 \cong \mathcal{O}_{C_1}(x_1 + \ldots + x_{2s+1} + q), L_2^2 \cong \mathcal{O}_{C_2}(q + x_{2s+2} + \ldots + x_{4g-4}). \tag{A.3}$$



(a) Unramified over q



(b) Ramified over q

Figure A.1: Admissible double covers

□

Remark A.2.3. Let C be a one-nodal curve of compact type having genus g, which is given by the sum of the genera g_i of C_i. We have the following possibilities for the arithmetic genus g_X of the cover X by Riemann-Hurwitz applied to the smooth covers $X_i \to C_i$ and by (2.3).

(a) By (A.2) $X_1 \to C_1$ is ramified at $2s + 2$ points, hence its genus is given by $g_{X_1} = 2g_1 + s$. Similarly, $X_2 \to C_2$ is ramified at $4g - 2s - 6$ points, so we have $g_{X_2} = 2g_1 + 4g_2 - s - 4$, thus (2.3) yields that $g_X = g_{X_1} + g_{X_2} + (\delta - \gamma + 1) = 4g - 3$.

(b) The genera of X_i from part (b) are $g_{X_1} = 2g_1 + s$, $g_{X_2} = 2g_1 + 4g_2 - s - 3$, thus (2.3) yields $g_X = g_{X_1} + g_{X_2} = 4g - 3$, so X is of compact type by Remark 2.1.5.

Remark A.2.4. Since C is of compact type, there is an identification between the total space of the line bundle L and the product of the total spaces of the line bundles L_1 and L_2. This induces clutching maps

$$(a) \qquad \mathcal{D}_{g_1,2s+2,1} \times \mathcal{D}_{g_2,4g-2s-6,1} \to \overline{\mathcal{D}}_{g,4g-4}$$

for double covers unramified over the node, respectively

$$(b) \qquad \mathcal{D}_{g_1,2s+1,1} \times \mathcal{D}_{g_2,4g-2s-5,1} \to \overline{\mathcal{D}}_{g,4g-4},$$

for double covers ramified over the node.

Considering the top right hand side of Figure A.2 and Figure A.3, the last index '1' stands:

(a) if the cover is unramified over the node q, $\{Q_{1,1}, Q_{1,2}\}$ being its preimages on X_1 (respectively $\{Q_{2,1}, Q_{2,2}\}$ on X_2), for the gluing of $Q_{1,j}$ with $Q_{2,j}$, $j = 1, 2$,

(b) if the cover is ramified over the node q, Q_1 being its preimage on X_1 (respectively Q_2 on X_2), for the gluing of Q_1 with Q_2.

Remark A.2.5. Note that for admissible double covers of type (b), the node q of C cannot be a marked point because marked points of $\overline{\mathcal{M}}_{g,n}$ need to be smooth by definition.

Proposition A.2.6. *Let (E, ϕ) be a rank-2 Hitchin pair on a one-nodal curve C of compact type and let $\pi : X \to C$ be the associated generic adapted spectral curve such that B_1 is the branch divisor of $\pi_1 : X_1 \to C_1$ and B_2 is the branch divisor of $\pi_2 : X_2 \to C_2$. Consider $(C, B) \in \overline{\mathcal{M}}_{g,4g-4}$ as a one-nodal curve of compact type. Assume that it is pointed at the divisor $B = B_1 + B_2 = x_1 + \cdots + x_{4g-4}$. Let $L \in J^{2g-2}(C)$ be a square root of $\mathcal{O}_C(B)$. Then X is also given by the datum of $(C, B, L) \in \overline{\mathcal{D}}_{g,4g-4}$ as follows:*

(a) generic adapted spectral covers X of C which are unramified over q are admissible double covers of C that are not ramified over q. This happens when the components C_i carry an even number of marked points.

(b) generic adapted spectral covers X of C which are ramified over q are admissible double covers of C that are ramified over q. This happens when the components C_i carry an odd number of marked points.

Moreover, by a suitable gluing of line bundles M_i in the Jacobians $J^{d_i}(X_i)$, having degrees $d_i = \deg E_i + \deg L_i$, we obtain the Hitchin pair (E, ϕ) on C.

Proof. We first consider part (a). As on the left hand side of Figure A.2, we can build the unramified admissible double cover of C via the datum of L as in Proposition A.2.2 (a), which is canonically associated to L_1 and L_2 as in (A.2).

As on the lower part of Figure A.2, we can normalize C and keep track of the marked points and of the preimages q_1, q_2 of the node.

We can build a smooth double cover of C_i via the datum of L_i, $i = 1, 2$, which, by Proposition A.1.2, is also a smooth spectral cover X_i, which is contained in the total space of the line bundle L_i and is not ramified over the node. So we can consider a clutching map as in Remark A.2.4(a) from the smooth components X_i to the admissible double cover X on the top left hand side of Figure A.2.

We now obtain (E, ϕ) on C. By Proposition 1.3.22, torsion-free sheaves η_i on X_i yield Higgs bundles $\hat{E}_i$ on C_i, together with the map $\overrightarrow{A}^{(q)}$ which comes from the choice of the clutching. Thus we obtain, by Lemma 3.1.3, a Hitchin triple on C.

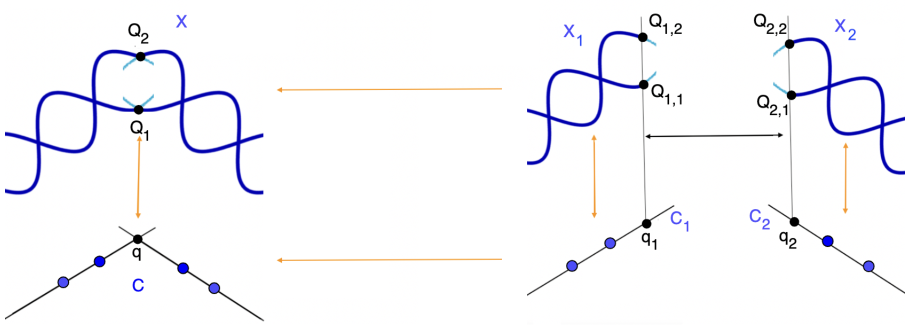


Figure A.2: One-nodal adapted spectral curve unramified over the node as an admissible double cover.

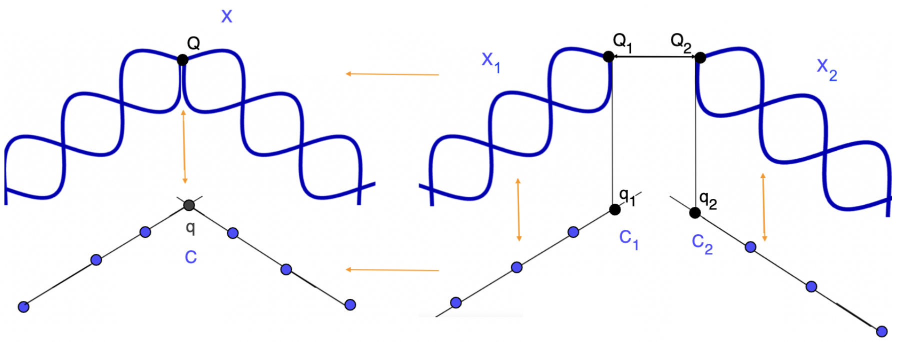


Figure A.3: One-nodal adapted spectral curve ramified over the node as an admissible double cover.

The steps of the proof of part (b) are similar, considering Figure A.3, clutching the two pieces of smooth double covers as in Remark A.2.4(b), where the double covers are ramified over the node. □

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