Alma Mater Studiorum - Università di Bologna

DOTTORATO DI RICERCA IN

SCIENZE STATISTICHE

Ciclo 34

Settore Concorsuale: 13/D2 - STATISTICA ECONOMICA

Settore Scientifico Disciplinare: SECS-S/03 - STATISTICA ECONOMICA

SMALL AREA ESTIMATION OF ECONOMIC SECURITY

Presentata da: Mario Marino

Coordinatore Dottorato

Monica Chiogna

Supervisore

Silvia Pacei

Esame finale anno 2022

Contents

1 Introduction

1.1 Overview

Economic security is a complex expression that carries a variety of meanings. Although there is no formal unambiguous definition, it is possible to provide a general definition by characterising it as a condition of well-being, absence of problems and also wealth. It has strongly influenced the institutional political debate of the last decade in the West, especially since the financial crisis of 2008 (see D'Ambrosio and Rohde, 2014 [1]), but even more in the last years after the economic consequences due to the Covid-19 pandemic (for example, Dvoryadkina *et al.*, 2021 [2]).

This research aims to study economic security through an indicator that takes into account the income levels of italian houseolds, from 2014 to 2016. The indicator used is the counterpart, with regard to economic security, of the economic insecurity indicator proposed by Bossert *et al.* (2019 [3]), which has two previous formulations: Bossert and D'Ambrosio (2013 [4]), D'Ambrosio and Rohde (2014 [1]). The main difference between this indicator and the previous ones (e.g. Osberg and Sharpe, 2002 [5]) is that this one has been constructed through an axiomatic approach and it benefits from some properties that will be explained later in this work. This indicator is based on a comparison of time, to capture the ability to overcome a crisis and to measure confidence in one's ability to recover after this crisis.

The objective of this work is to estimate the economic security for groups of Italians provinces, using EU-SILC longitudinal data. We notice that the sample size is too low to obtain reliable estimates for our target areas. Therefore we consider the possibility of resorting to some Small Area Estimation models to improve the reliability of the results. EU-SILC stands for European Union Statistics on Income and Living Conditions and is one of the main sources of data for the periodic reports of the European Union on the social situation and the spread of poverty in the member countries. The indicator suggested by Bossert *et al.* (2019 [3]), is calculated at individual level. We consider the weighted average obtained by group of provinces as indicator of economic security at province level. The major contributions when it comes to small area estimation are those of Rao (2003, [6]) and his updated version by Rao and Molina (2015, [7]), but also those of Jiang and Lahiri (2006,[8]) and the review by Pfeffermann (2013, [9]). Often the small sample size is a major obstacle to overcome, for instance for policy makers who need to make locally targeted policies or for local authorities who want to know precisely how to allocate resources and funds more efficiently. The advantage of small area estimation is that it allows to improve the reliability of small area estimates, by borrowing information from auxiliary variables available from administrative and census sources. Furthermore, it is not only possible to borrow information from census and administrative sources, but also from time (see Esteban *et al.*, 2016 [10]). We consider small area models specified at area level. Besides the basic Fay-Herriot area-level model (see Fay and Herriot, 1979 [11]), given the nature of the indicator, we propose to consider some longitudinal extensions of the Fay-Herriot model. These longitudinal models include time-specific random effects by taking into account autoregressive processes of order 1 (AR1; see Esteban *et al.*, 2012 [12]) and moving average of order 1 (MA1; see Esteban *et al.*, 2016 [10]). The results show a significant improvement in small area estimates obtained for the Italian groups of provinces between 2014 and 2016, expecially in the case of MA1 model. Finally, we carry out a simulation to investigate the design-based properties of the estimates obtained from the small area models considered. The simulation results further highlight that the MA1 model performs best.

The thesis is structured as follows. Section 2 presents the concept of economic security, with various possible definitions. Section 3 presents the data used, EUSILC. Section 4 explains the economic insecurity indicators, their properties and the corresponding economic security indicator employed in this work. Section 5 shows how the economic security indicator was calculated for the Italian provinces over the period 2014 to 2016. In Section 6 the basic models for small area estimation, area-level models and unit-level models, are introduced and explained. Sections 7 and 8 focus on the problem of estimating the variance of direct estimators when dealing with complex indicators and complex sample designs. In Section 9 the covariates employed for the small area models are presented. The small area models used are described in detail in Section 10. Section 11 shows the results achieved for the Italian groups of provinces and for the three years 2014, 2015 and 2016. In Section 12 the simulation study is presented.

1.2 Main Contributions of the Thesis

The aim of this thesis is to investigate how economic security has changed in Italian provinces between 2014 and 2016. To do this, an indicator of economic security was used, constructed through an axiomatic approach such as the one of economic insecurity presented in [3], [4], [1]. Here, longitudinal EUSILC data were used, which is the most comprehensive survey in Europe and a reference for much of the literature dealing with the analysis of inequality measurement. This recently proposed indicator is used here for the first time in a small area model estimation context, while poverty and inequality have been considered until now in literature. In addition, small area-level longitudinal models are compared and it is shown that the MA1 model proposed fairly recently in [10] provides better estimates for MSE and MSE efficiency gain than the basic Fay-Herriot model. Thus, it is shown that it is possible to borrow strenght from temporal correlation that helps to produce more reliable estimates of the parameter of interest.

2 The concept of Economic Security

Economic security has always been an object of investigation by researchers of different disciplines. In fact, this topic interests both those who deal with humanistic subjects, such as sociology, history, anthropology, psychology, and those who deal with scientific subjects, such as finance, statistics and obviously economics (e.g. Bossert and D'Ambrosio, 2013 [4]).

This interest, on the one hand, is purely academic, aimed at investigating what are the dynamics that lead to states and movements of security and insecurity; on the other hand, it is motivated by the need for policy makers to know about it, in order to have a picture that allows them to implement more effective economic policies. Therefore, in this sense it is possible to observe a common and integrated interest on the part of the academic and political worlds, with the former attempting to provide, through research, increasingly up-to-date and adequate tools to allow policy makers to have a complete picture of the context in which it is most appropriate to employ resources (e.g. Dynan, 2016, [13]).

Economic security is also only one of two sides of a coin, which is inevitably accompanied by its negative meaning, namely insecurity. If the first recalls the idea of stability, well-being, tranquility and in a certain sense, wealth, the second recalls a condition of precariousness, difficulty, obstacle to overcome, poverty. Economic insecurity has strongly characterized the institutional political debate of the last decade in the West, especially since the financial crisis that hit mainly Europe and the United States of America between 2008 and 2011 (see for example Hacker *et al.* 2014, [14]), and whose slow reabsorption has been swept away by the recent one started in 2020 caused by the Covid-19 pandemic (that instead had and still has a global spread; e.g. Lin *et al.*, 2021 [15]). Although security and insecurity are complementary and hard to treat separately, in this thesis we will deal mainly with economic security.

Unfortunately, it is impossible to find an unambiguous definition of economic security, because it is a very wide topic that cannot be standardized. Not only, but economic security is also closely linked to the context of space and time within which it is studied, so it varies with the variation of these as well as its possible definitions. In addition, those definitions of economic security have different meanings depending on the discipline with which it is studied and, often, since there is no common standard, even researchers from the same discipline use different definitions very often (see D'Ambrosio and Rohde, 2014 [1]).

Economic security can be seen indeed as a concept closely linked to the geo-spatial dimension and its peculiar perception. The concepts of economic security of the early 20th century, for example after the Great Depression of 1929 (e.g. Edgerton, 1931 [16]), are different from those developed during the Cold War and, likewise, these become outdated when compared with those of the globalised world of the 21st century. Moreover, many studies have focused precisely on the changing of economic security after the Cold War (see for example: Sperling and Kirchner, 1998 [17]) Similarly, the definitions of economic security developed today differ between countries such as China, USA and Russia, and in turn differ from those developed in the European Union (Nesadurai, 2004 [18]). In this sense we can also see how the concept of economic security is evolving. In fact, a population that has recently emerged from a war, for example, will be looking first for a level of security that is more linked to survival itself, and then for one that is firstly linked to social and political autonomy, then to economic autonomy and economic security(Pinder, 1985 [19]). So this can be seen, more broadly, as collective economic security (Nye, 1974 [20]).
A further definition of economic security is that provided by the International Committee of the Red Cross (ICRC):

> The ICRC defines economic security as the ability of individuals, households or communities to cover their essential needs sustainably and with dignity. This can vary according to an individual's physical needs, the environment and prevailing cultural standards. Food, basic shelter, clothing and hygiene qualify as essential needs, as does the related expenditure; the essential assets needed to earn a living, and the costs associated with health care and education also qualify.

To overcome the obstacle of the formal definition, two paths can be followed: the first is to carry out an etymological analysis of the words that compose it: 1. **Economy:** from Greek ***οικονομία*** (***οἶκος***: home; ***νόμος***: law, standard, rule, administration) literally housekeeping, or rather, in a contemporary sense, management of available resources; 2. **Security:** from Latin **se cura**, hence **securus** (**se:** without, absence, lack, deprivation; **cura:** worry, thought, troubles) indicates the absence of worries, thoughts, restlessness, then a condition of quiet, calm tranquility and, in a wider sense, a situation of well-being; 3. **Economic security**: condition characterized by the presence of a set of economic or financial assets that help to hold good living conditions

Starting from the etymological definition we can develop the concept in contemporary terms as done above, thus providing an idea of the meanings behind economic security, namely the idea of robustness, well-being, tranquility, wealth. We can then understand economic security also as a condition characterized by the presence of a set of economic or financial assets that help to maintain good living conditions, appropriate, at least decent.

The second problem is to choose the context in which to study economic insecurity. In fact, it is possible to analyze it both at the international level (e.g. Kahler, 2004 [21]) and at the national (e.g. Zhengyi, 2004 [22]) level, at the level of territorial divisions such as regions, provinces and municipalities (e.g. Murias *et al.*,2012 [23]), but also going into more detail, for example taking into consideration specific urban areas, neighborhoods and districts(e.g. Svetkina, 2021 [24]).

The literature is wide and there is no convergence among researchers on what are the most suitable indicators to study economic security. For example, some very commonly considered in the literature are wealth, income, consumption, but also employment, activity rate, or even the comparison of the same indicators of poverty between countries with different levels of well-being (e.g. Osberg and Sharpe, 2014 [25]).

This thesis aims to study economic insecurity through an indicator that takes into account the income levels of the year under consideration and those of the previous two years, for each reference period. The methodology with which the analysis will be carried out falls within those known as Small Area Estimation, which in our case will be Italian groups of provinces.

3 Data

The data sources available for the purposes of our research could have been numerous, but the choice fell on EU-SILC data concerning Italy, where EU-SILC stands for European Union Statistics on Income and Living Conditions. All EU countries participate in this survey and, in the case of our interest, Italy has been participating since 2004 with an annual survey dedicated to household income and living conditions, both longitudinal and cross-sectional.

EU-SILC was chosen because for years it has been one of the main sources of data for the periodic reports of the European Union on the social situation and the spread of poverty in the member countries, with indicators focused on income and social exclusion, in a multidimensional approach to the problem of poverty, and with particular attention to aspects of material deprivation. Therefore, within the EU-SILC databases, there are all the elements needed to conduct surveys regarding economic security, especially when it is necessary to know the income flows of families and individuals. It must be said that the survey is not limited to this, but within it there is a large amount of information concerning aspects inherent in the labor market, education, availability of primary livelihood (e.g. availability and quantity of meals per week), etc.

A further reason for using EU-SILC data is the fact that this survey, since it was born, has been a reference point for much of the literature dealing with the analysis of measures of inequality (e.g. Whelan *et al.*, 2008 [26]), within whose class economic insecurity can also be identified.

Remark: The data we will use are those longitudinal from waves of 2014 (2012, 2013, 2014), 2015 (2013, 2014, 2015) and 2016 (2014, 2015, 2016).

EU-SILC is a complex survey consisting of a highly articulated sample design. In terms of the survey sample, this is constructed with a rotating panel in which in each successive year of the survey only 25% of the sample is replaced, with 75% remaining within the survey, so that each unit remains within it for exactly 4 years. As an example, a unit that joined the sample in 2012 will also remain for the next 3 years, thus being surveyed in 2013, 2014, and 2015 as well. The longitudinal sample of households that are introduced in each successive year is selected according to a two-stage stratified sampling plan. The first-stage units are municipalities, stratified by region and population size, and within each region, municipalities are divided into self-representative and non-self-representative according to their population size. The first ones are always included in the sample, while the second ones are sampled in a stratified random fashion with probability proportional to population size.

The reference population of EU-SILC consists of all private households and their current members residing in the territory of the EU Member States at the time of data collection, with the exclusion of persons living in collective households and institutions, who are therefore usually excluded from the target population.

For each year, or wave in the case of longitudinal data, 4 files are compiled named respectively:

- **D-FILE or HOUSEHOLD REGISTER:** must contain every household selected, including those where the address could not be contacted or households that could not be interviewed;
- **R-FILE or PERSONAL REGISTER:** must contain a record for each person currently living in the household or who is temporarily absent. In the longitudinal component, it must also contain a record for each person recorded in the previous year's R-File or who lived in the household for at least three months during the income reference method;
- **H-FILE or HOUSEHOLD DATA:** In the other files, records for a household exist only if the household has been contacted and has a complete household interview in the household data file (H), and at least one member has complete data in the personal data file (P). This member must be the selected respondent if this selection mode is used;
- **P-FILE or PERSONAL DATA:** must contain a record for each eligible person for whom information may be supplemented by interviews and/or records.

4 The measure of economic insecurity

For this study, the choice fell on a new indicator first proposed by Walter Bossert and Conchita D'Ambrosio in 2013 [4]. Many indicators have been proposed before, e.g. by Osberg and Sharpe (2002 [27]) and Hacker *et al.* (2010 [28]), but none of those presents the innovative element of [4], i.e. the construction of the indicator through an axiomatic approach.

In addition, this indicator has been further developed in two more papers, D'Ambrosio and Rohde (2014 [1]) and Bossert *et al.* (2019 [3]). Now we will see how the indicator has developed through these three papers and after we will use the latest version of the indicator for analysis. The 2013 Economic Insecurity Indicator is:

$$\mathcal{V}^{(T)}(w) = \sum_{\substack{t\in\{1,...,T\}: \\ w_{-t}>w_{-(t-1)}}} \alpha_{-t}(w_{-t} - w_{-(t-1)}) + \sum_{\substack{t\in\{1,...,T\}: \\ w_{-t}<w_{-(t-1)}}} \beta_{-t}(w_{-t} - w_{-(t-1)}) - w_0 \tag{1}$$

with:

$\mathcal{V}^{(T)}(w)$	economic insecurity indicator
T	number of years taken into account (e.g. 2013-2015 $\Rightarrow T = 3$)
t	single year $\quad t = 1, ..., T$
w	wealth flows
w_0	flow of wealth at current time (last year taken into consideration)
α, β	such that:

$$\alpha_{-t} = \frac{1}{2t-1} \qquad \beta_{-t} = \frac{\alpha_{-t}}{2}$$

i.e., they are two sequences, which allow to obtain a measure of loss aversion, that are constructed as follows:

$$[\alpha_{-t} > \alpha_{-(t+1)} > 0; \beta_{-t} > \beta_{-(t+1)} > 0], \qquad \forall t \in \mathbb{N}$$

This indicator has certain properties:

Difference Monotonicity: $\forall \mathrm{T} \in \mathbb{N}$, $\forall w \in \mathbb{R}^{(T-1)}$, $\forall \gamma \in \mathbb{R}$

$$\mathcal{V}^T(w_{-(T-1)} + \gamma, w) \geq \mathcal{V}^{T-1}(w) \Longleftrightarrow \gamma \geq 0$$

This property tells us that if we introduce an additional period in which there has been a loss in the flow of wealth the insecurity will increase, on the other hand it will decrease if the additional period will be characterized by a gain in wealth, it will not change at all if the level of wealth of the additional period will be identical.

Proximity Monotonicity: $\forall \mathrm{T} \in \mathbb{N}$, $\forall w \in \mathbb{R}^{(T)}$, $\forall t \in \{1, ..., T-1\}$

$$\begin{aligned}\mathcal{V}^T(w_{-T}, ..., w_{-(t+1)}, w_{-(t+1)}, w_{-(t-1)}, ..., w_0) \geq \mathcal{V}^T(&w_{-T}, ..., w_{-(t+1)}, w_{-(t-1)},\\ &w_{-(t-1)}, ..., w_0)\\ \Longleftrightarrow w_{-(t+1)} &\geq w_{-(t-1)}\end{aligned}$$

This property can be interpreted in this way: the closer one gets to current time, the more a loss of wealth increases insecurity; conversely, a gain in wealth decreases the insecurity index. In a certain sense, therefore, losses and gains have a memory effect on security, so that more recent gains will positively influence security, but less and less the further back one goes from the current time. Similarly, a loss which occurred, for example, 3 years ago will have less of an impact than one recorded in the previous year.

Homogeneity: $\forall \mathrm{T} \in \mathbb{N}$, $\forall w \in \mathbb{R}^{(T)}$, $\forall \lambda \in \mathbb{R}_{++}$

$$\mathcal{V}^T(\lambda w) = \lambda \mathcal{V}^T(w)$$

This property ensures us that changes in levels of economic security/insecurity are commensurate with changes in levels of wealth.

Translatability: $\forall \mathrm{T} \in \mathbb{N}$, $\forall w \in \mathbb{R}^{(T)}$, $\forall \delta \in \mathbb{R}$

$$\mathcal{V}^T(w + \delta \mathbf{1}_{T+1}) = \mathcal{V}^T(w) - \delta$$

This property has been stated because it ensures to establish an inverse relationship between insecurity and wealth (or direct relationship between security and wealth), so whenever wealth increases (or decreases) by a certain amount δ then security also increases by δ or in this case insecurity decreases by δ, so in the case of security we will have to modify equation (7) as follows:

$$-\mathcal{V}^T(w + \delta \mathbf{1}_{T+1}) = -\mathcal{V}^T(w) + \delta$$

where $-\mathcal{V}^T(w)$ is used to denote the security index.

Loss Priority: $\forall \mathrm{T} \in \mathbb{N}$, $\forall w \in \mathbb{R}^{(T-1)}$, $\forall \gamma \in \mathbb{R}_{++}$

$$\mathcal{V}^T(w_{-(T-1)} + \gamma, w) - \mathcal{V}^T(w_{-(T-1)}, w) \geq \mathcal{V}^T(w_{-(T-1)}, w) - \mathcal{V}^T(w_{-(T-1)} - \gamma, w)$$

This property states that the measure of economic security should have an element of risk aversion, as a loss of wealth affects economic insecurity more than a gain.

Temporal Aggregation Property: $\forall \mathrm{T} \in \mathbb{N}$, $\exists$ a function $\Phi^T : \mathbb{R}^2 \to \mathbb{R}$:

$$\mathcal{V}^T(w) = \Phi^T(w_{-T} - w_{-(T-1)}, \mathcal{V}^{T-1}(w_{-(T-1)}, ..., w_0))$$

$\forall w \in \mathbb{R}^T$

This property allows the economic insecurity index to be calculated from the present period to a desired number of past periods, recursively.

All these properties allow Bossert and D'Ambrosio to state the formulation of the following Theorem:

Theorem 1. *A measure of individual insecurity V satisfies difference monotonicity, proximity monotonicity, homogeneity, translatability, the temporal aggregation property, and loss priority if and only if V is a two-sequences loss-averse Gini measure of insecurity.*

For the proof, you can see [4]. In the same paper, an additional property is stated that is complementary to that of **Proximity Monotonicity**.

Proximity Indifference: $\forall \mathrm{T} \in \mathbb{N}$, $\forall w \in \mathbb{R}^{(T)}$, $\forall t \in (1, ..., T-1)$:

$$\begin{gathered}\mathcal{V}^T(w_{-T}, ..., w_{-(t+1)}, w_{-(t+1)}, w_{-(t-1)}, ..., w_0) = \\ \mathcal{V}^T(w_{-T}, ..., w_{-(t+1)}, w_{-(t-1)}, w_{-(t-1)}, ..., w_0)\end{gathered}$$

The key difference here is that a loss of wealth affects insecurity regardless of when it was recorded and likewise a gain in security. Take the three-year period 2013-2015 as an example. For **Proximity Monotonicity** a wealth gain in 2014 has more weight on the 2015 security index than the same one that occurred in 2013, in contrast for **Proximity Indifference** a gain that occurred in 2013 has exactly the same weight as one in 2014. Substituting the second property for the first the index becomes:

$$\mathcal{V}^{(T)}(w) = \alpha \sum_{\substack{t \in \{1,...,T\}: \\ w_{-t} > w_{-(t-1)}}} (w_{-t} - w_{-(t-1)}) + \beta \sum_{\substack{t \in \{1,...,T\}: \\ w_{-t} < w_{-(t-1)}}} (w_{-t} - w_{-(t-1)}) - w_0 \quad (2)$$

where α and β are a pair of real numbers, two parameters, and no longer sequences, so:

$$\alpha_{-(t-1)} = \alpha_{-t} = ... = \alpha_{-1} := \alpha > 0$$

The theorem then becomes:

Theorem 2. *A measure of individual insecurity V satisfies difference monotonicity, proximity indifference, homogeneity, translatability, the temporal aggregation property, and loss priority if and only if V is a two-parameters loss-averse Gini measure of insecurity.*

This is with respect to the early formulations of the index in the 2013 article.

In [1] D'Ambrosio and Rohde provide a generalized version of the index and apply it empirically to two case studies, one for Italy and one for the United States of America:

$$\mathcal{V}^T_{(\alpha,\beta)}(w) = \sum_{\substack{t\in\{1,...,T\}: \\ w_{-t}>w_{-(t-1)}}} \alpha_{-t}(w_{-t}-w_{-(t-1)}) + \sum_{\substack{t\in\{1,...,T\}: \\ w_{-t}<w_{-(t-1)}}} \beta_{-t}(w_{-t}-w_{-(t-1)}) - w_0 \tag{3}$$

where we have an expression that is identical to (1), where w denotes a flow of wealth, but which is more general since in this case the inverse of the Gini social evaluation function is of the type:

$$\alpha_{-t} = \frac{\gamma}{2t-1} \qquad \beta_{-t} = \frac{\alpha_{-t}}{2}$$

where γ is a parameter that allows to weigh current wealth against its historical fluctuations, and thus allows to establish a level of insecurity.

The paper [3] (2019) relates economic insecurity to Brexit and the rise of right-wingers in the Western world, specifically is considered the 2016 election of Trump. The properties of this indicator are the following:

Gain-loss monotonicity: $\forall \mathrm{t} \in \mathbb{N}$, $\forall p \in \mathbb{R}$, $\forall q \in \mathbb{R}_{++}$:

$$\mathcal{I}^t(p+q, p\mathbf{1}_t) > \mathcal{I}^t(p, p\mathbf{1}_t) > \mathcal{I}^t(p-q, p\mathbf{1}_t) \tag{4}$$

where p denotes some level of resource, q on the other hand represents a gain or loss (e.g. of wealth or income) that occurs when moving from one time to the next, and of course still present is the:

Proximity monotonicity: $\forall \mathrm{t} \in \mathbb{N}$, $\forall p \in \mathbb{R}$, $\forall q \in \mathbb{R}_{++}$:

$$\begin{split}\mathcal{I}^{t+2}(p, p, p+q, p\mathbf{1}_t) > \mathcal{I}^{t+2}(p, p+q, p, p\mathbf{1}_t) > \mathcal{I}^{t+2}(p, p, p, p\mathbf{1}_t) > \\ \mathcal{I}^{t+2}(p, p-q, p, p\mathbf{1}_t) > \mathcal{I}^{t+2}(p, p, p-q, p\mathbf{1}_t)\end{split} \tag{5}$$

In analogy to [4], **Proximity monotonicity** is a condition that assures that the closer we get to the current time the more a wealth gain weighs on security or a wealth loss affects insecurity.

Linear homogeneity: $\forall \mathrm{T} \in \mathbb{N}$, $\forall w \in \mathbb{R}^T$, $\forall b \in \mathbb{R}_{++}$:

$$\mathcal{I}^T(bw) = b\mathcal{I}^T(w) \tag{6}$$

Again, the analogy with **Homogeneity** in [4] is obvious; in fact, this property asserts that if one multiplies wealth or income by some positive constant quantity, then the economic insecurity index will also be multiplied by the same quantity.

Translation invariance: $\forall \mathrm{T} \in \mathbb{N}$, $\forall w \in \mathbb{R}^T$, $\forall c \in \mathbb{R}$:

$$\mathcal{I}^T(w + c\mathbf{1}_t) = \mathcal{I}^T(w) \tag{7}$$

So if the same amount c of wealth (or income) is added to to the wealth levels in each year, then the economic insecurity index will not change.

Quasilinearity: $\forall \mathrm{T} \in \mathbb{N}$, $\exists$ a function $F^T : \mathbb{R}^2 \rightarrow \mathbb{R}$:

$$\mathcal{I}^T(w) = \mathcal{I}^{T-1}(w_{-(T-1)}, ..., w_0) + F^T(w_{-T}, w_{-(T-1)}) \tag{8}$$

$\forall\ w \in \mathbb{R}^T$

i.e., the economic insecurity index is a quasilinear function that takes into account the insecurity created by more recent wealth levels and a F function that instead involves the wealth levels of earlier periods. Recalling the example of the 2013-2015 three-year period, we will therefore have w_{2014} and w_{2015} involved, respectively, on the one hand, w_{2013} on the other.

Stationarity: $\forall r \in \mathbb{N}_0$, $\exists$ an increasing function $G^r : \mathbb{R} \rightarrow \mathbb{R}$: $\forall t \in \mathbb{N}_0$ and $\forall p, p', s \in \mathbb{R}$:

$$\mathcal{I}^{t+2+r}(p, p', s\mathbf{1}_{t+1}, s\mathbf{1}_r) = G^r(\mathcal{I}^{t+2}(p, p', s\mathbf{1}_{t+1})) \tag{9}$$

This property is necessary because we are in a context where a particular stream of wealth is placed a certain number of years backwards, in which case this certain amount is represented by r.

All these properties lead to the formulation of the following theorem:

Theorem 3. *A measure of individual economic insecurity $\mathcal{I}$ satisfies gain-loss monotonicity, proximity monotonicity, linear homogeneity, translation invariance, quasilinearity and stationarity* $\Longleftrightarrow \exists\, l_0,\, g_0 \in \mathbb{R}_{++},\, \delta \in (0, \min\{\frac{l_0}{g_0}, \frac{g_0}{l_0}\})$*:* $\forall\, T \in \mathbb{N},\, \forall\, w \in \mathbb{R}^T$ *we have:*

$$\mathcal{I}^{(T)}(w) = l_0 \sum_{\substack{t\in\{1,\dots,T\}:\\ w_{-t}>w_{-(t-1)}}} \delta^{t-1}(w_{-t} - w_{-(t-1)}) + g_0 \sum_{\substack{t\in\{1,\dots,T\}:\\ w_{-t}<w_{-(t-1)}}} \delta^{t-1}(w_{-t} - w_{-(t-1)}) \tag{10}$$

l_0	weight on aggregate discounted losses
g_0	weight on aggregate discounted gains
δ	discount factor
t	single year $\quad t = 1, ..., T$
w	wealth stream

In our study, in order to simplify notation and calculation, we will deal with an economic security indicator. So, contrary to [3], for us positive values will indicate, in general, more economic security and negative values more insecurity, hence we start from this:

$$-\mathcal{S}^{(T)}(x) = \lambda_0 \sum_{\substack{t\in\{1,\dots,T\}:\\ x_{-t}>x_{-(t-1)}}} \delta^{t-1}(x_{-t} - x_{-(t-1)}) + \gamma_0 \sum_{\substack{t\in\{1,\dots,T\}:\\ x_{-t}<x_{-(t-1)}}} \delta^{t-1}(x_{-t} - x_{-(t-1)}) \tag{11}$$

with:

$-\mathcal{S}^{(T)}(x)$	economic insecurity indicator
T	number of years (ex. 2013-2015 $\Rightarrow T = 3$)
t	single year $\quad t = 1, ..., T$
x	equivalent disposable income
λ_0	loss function
γ_0	gain function
δ	discount factor

that becomes:

$$\mathcal{S}^{(T)}(x) = -\lambda_0 \sum_{\substack{t \in \{1,\dots,T\}: \\ x_{-t} > x_{-(t-1)}}} \delta^{t-1}(x_{-t} - x_{-(t-1)}) - \gamma_0 \sum_{\substack{t \in \{1,\dots,T\}: \\ x_{-t} < x_{-(t-1)}}} \delta^{t-1}(x_{-t} - x_{-(t-1)}) \tag{12}$$

The variable chosen was the equivalent disposable income, i.e. the variable HX090 present in the H-File of the EU-SILC databases. This variable is widely used in studies concerning inequality and the calculation of indices of social exclusion, poverty and distribution of assets. Equivalent disposable income is a very useful tool because it takes into account differences in household composition and size. This is why it is equivalized for each household composition and size. Moreover disposable means the fraction between the income that is available now for saving or spending and the number of household members converted into equalised adults. The equalisation process is carried out taking into account the age of the household members. In this case, it was also considered appropriate to deflate this income, in order to have the same unit of measurement deprived of the effect of inflation and deflation between one period and another, taking 2010 as the base year. The deflators provided by the Bank of Italy were used. Using this procedure we can compare the available equivalent incomes of different years by having a common unit of measurement which is not affected by changes in inflation, but is constant over time (after choosing a suitable base).

The choice of equivalent disposable income is motivated by many factors. In the first instance, it expresses how much money people actually have available for spending and saving, after taxation (for example for food, rent, etc.). Another variable often used in the literature on economic security is wealth, that measures all assets owned by a person or a household and it is accumulated over time, which has both disadvantages and advantages over income. One advantage of wealth is that, in general, it provides a more comprehensive view of people's financial status. On the other hand, a person with greater wealth may be worse off in terms of economic security than someone with a higher income. For example, an individual living in a house worth 350,000 €, that he has inherited, but with an income of 1,000 €, is likely to be more economically insecure than someone living in a rented house but with an income of 6,000 €, also because of the difficulty to quickly convert assets in the corresponding monetary value. Another aspect that led to the choice of income is that wealth is often much more complicated to calculate comprehensively. Indeed, some features of wealth are often difficult to reconstruct and derive from a survey, for example funds accumulated in insurance or pension plans, or other secondary sources of wealth that are difficult to identify. Furthermore, another problem with wealth (in the perspective of future studies on the subject) is that countries do not calculate it considering the same variables (e.g. some of them take pensions into account, while others exclude them). Finally, the biggest advantage of income is that it is collected annually (also in EUSILC), which makes it possible to construct indicators that are based on consecutive years, whereas wealth often has gaps of years (e.g. data from the Bank of Italy that are bi-annual).

Now for the calculation of the economic security indicator, it is necessary to choose values for the loss function, for the gain function and finally choose a discount factor that meet the following conditions:

$$\lambda_0 > \gamma_0 \qquad \Longrightarrow \qquad \delta \in (0, \frac{\gamma_0}{\lambda_0}) \tag{13}$$

so:

$$\frac{\gamma_0}{\lambda_0} < 1 < \frac{\lambda_0}{\gamma_0} \tag{14}$$

As was done in [3], the following values were chosen:

$$\begin{aligned} \lambda_0 &= 1 \\ \gamma_0 &= \frac{15}{16} = 0.9375 \\ \delta &= 0.9 \end{aligned} \tag{15}$$

From which it can be seen that these values meet the conditions above:

$$\lambda_0 = 1 > 0.9375 = \gamma_0 > 0.9 = \delta$$

$$\delta \in (0, 0.9375) \qquad \text{so} \quad \delta \in (0, \gamma_0) \quad \text{because} \quad \lambda_0 = 1$$

$$\frac{\gamma_0}{\lambda_0} = \gamma_0 = 0.9375 < 1 < 1.0\overline{6} = \frac{\lambda_0}{\gamma_0}$$

Finally, it is worth noting that the discount factor δ gives a decreasing weight to the past values of wins and losses, i.e. it decreases as we go back in time; in fact we will have:

$$\delta^0 = 1$$

$$\delta^1 = 0.9 = \delta$$

$$\delta^2 = 0.81$$

and so on.

5 The estimation of Economic Security index in the Italian groups of provinces

The aim is to estimate the average of the indicator calculated at individual level for the Italian groups of provinces. We decided to use groups of neighbouring provinces, instead of provinces, because the number of observations for Nuts3 is, in general, too small in the longitudinal EU-SILC sample, to obtain reliable estimates even using a small area strategy. From now on, we will refer to these groups of provinces also by simply using the words "provinces" or areas (or domains). Specifically, we merge the Italian provinces into 60 groups of provinces. For the union, we decide to join provinces that are close to each other and that are within the same region. In this way we attempt to increase, albeit slightly, the number of observations available for the EU-SILC sample of each year, especially for those areas that have a very low number of such observations. The results of this procedure are shown below in Table 1. As can be seen, some provinces, especially those with the most populous metropolitan cities, such as Roma, Milano, Torino, have the highest number of observations. On the other hand, some provinces that have metropolitan cities as their capital city have also been merged. This occurred when a neighboring province either had no other neighboring provinces within the region or its merger with a neighboring province would have been irrelevant in terms of the number of observations. In addition, some provinces that continued to have excessively low numbers of observations, with differences of several units between the sample in one year and another, were left isolated. This was also done in order not to excessively reduce the number of areas available to us and to maintain a number of domains greater than at least half of the provinces.

As anticipated, the arising problem is that the EUSILC sample size for these areas is too small to produce reliable direct estimates, so we have to use small area estimation. The concept of small area or domain is closely related to the sample size associated with a given area. For example EU-SILC samples for regions (Nuts2) and territorial repartitions (Nuts1) are large enough to provide reliable direct estimates. Intuitively, a direct estimate is simply an estimate obtained from sample data using weights. Groups of provinces, on the other hand, do not have a sufficient sample size to obtain reliable direct estimates. Thus, we can define small areas as areas whose sample is not large enough to ensure reliable direct estimates, as also stated in Rao and Molina (2015, [7]).

Region	Groups of provinces	N. 14-16	N. 14	N. 15	N. 16
1-Piemonte	Biella, Vercelli	125	26	37	62
	Cuneo, Asti, Alessandria	726	237	185	304
	Novara, Verbano-Cusio-Ossola	208	59	52	97
	Torino	1371	425	444	502
2-Valle d'Aosta	Aosta	583	182	180	221
3-Lombardia	Bergamo	472	143	140	189
	Brescia	751	212	248	291
	Lecco, Monza e della Brianza, Sondrio	420	80	115	225
	Milano	1068	302	276	490
	Pavia, Lodi, Cremona, Mantova	389	102	92	195
	Varese, Como	864	235	274	355
4-Trentino-Alto Adige	Bolzano, Trento	496	128	135	233
5-Veneto	Belluno, Treviso,Padova	608	165	218	225
	Rovigo	200	72	61	67
	Venezia	397	112	107	178
	Verona, Vicenza	940	318	260	362
6-Friuli-Venezia Giulia	Gorizia, Trieste	557	176	169	212
	Udine, Pordenone	1411	442	455	514
7-Liguria	Genova, La Spezia	1256	421	380	455
	Imperia, Savona	521	173	186	162
8-Emilia-Romagna	Bologna	663	208	208	247
	Ferrara, Ravenna	402	132	108	162
	Forlì-Cesena, Rimini	390	89	109	192
	Parma, Piacenza	460	91	105	264
	Reggio nell'Emilia, Modena	718	206	227	285
9-Toscana	Arezzo	169	15	41	113
	Firenze	568	181	180	207
	Livorno, Pisa, Siena, Grosseto	519	152	156	211
	Massa-Carrara, Lucca, Pistoia	514	125	160	229
	Prato	131	57	19	55
10-Umbria	Perugia, Terni	1232	385	365	482
11-Marche	Ancona	606	200	142	264
	Macerata, Ascoli Piceno, Fermo	999	292	338	369
	Pesaro e Urbino	387	112	126	149
12-Lazio	Frosinone	375	118	99	158
	Latina	321	70	84	167
	Roma	1396	438	461	497
	Viterbo, Rieti	274	99	82	93
13-Abruzzo	L'Aquila	87	24	24	39
	Pescara, Chieti	589	182	172	235
	Teramo	140	39	42	59
14-Molise	Campobasso, Isernia	510	181	122	207
15-Campania	Avellino, Salerno	476	148	146	182
	Benevento	144	34	36	74
	Caserta	135	36	32	67
	Napoli	1129	380	334	415
16-Puglia	Bari, Taranto	893	285	232	376
	Barletta-Andria-Trani	110	27	35	48
	Brindisi, Lecce	466	180	115	171
	Foggia	123	28	42	53
17-Basilicata	Potenza, Matera	606	219	151	236
18-Calabria	Catanzaro, Vibo Valentia	398	105	138	155
	Cosenza, Crotone	662	226	183	253
	Reggio di Calabria	151	48	22	81
19-Sicilia	Agrigento, Caltanissetta, Enna, Ragusa	419	132	135	152
	Catania, Siracusa	456	141	130	185
	Trapani, Palermo, Messina	897	271	292	334
20-Sardegna	Cagliari, Carbonia-Iglesias, Medio Campidano	659	194	209	256
	Nuoro, Oristano, Olbia-Tempio,Ogliastra	60	21	19	20
	Sassari	110	36	24	50

Table 1: Groups of provinces (areas) and number of observations available for each EU-SILC sample considered and overall: 2014-2016, 2014, 2015, 2016

Direct estimates, i.e., the average economic security index by group of province, is computed using the following formula:

$$\hat{\mathcal{S}}_i^{(T)} = \frac{\sum_{j=1}^{n_i} w_j \quad \varphi_j \quad \mathcal{S}_j^{(T)}}{\sum_{j=1}^{n_i} w_j \quad \varphi_j} \tag{16}$$

with:

i	area index (province)	$i = 1, ..., M$
j	household index	j=1,...,n_i
$\mathcal{S}_j^{(T)}$	household economic security indicator	
φ_j	number of household members	
w_j	longitudinal household weights	
n_i	number of households in the area (province)	

Henceforth, for simplicity, we will call this indicator $\hat{\mathcal{S}}_i$.

So we have a complex indicator as well as the sample design, consequently the analytical expression of the variance will be complex. To overcome this problem we will employ a technique for estimating the variance of the direct estimator based on resampling.

6 Small Area Estimation

The term "small area (or domain)" covers a wide variety of meanings. The most intuitive is the one related to the geographical dimension (e.g. a region, a province, a municipality, a suburb, etc.). A small area can also be, for example, referred to, in the case of companies, a specific working area among many within the company itself. Hence, a concept linked to a sub-dimensionality with respect to a larger area, or rather it refers to subpopulations. This, in general terms, may lead to a misleading interpretation. In fact small area refers to size, but in more specific terms to the sample size from a survey that is not large enough to produce reliable direct estimates [6].

The main books dealing with small area estimation are those of Rao (2003, [6]) and its updated version by Rao & Molina (2015, [7]). In [7] the authors define small area estimation as follows:

> A domain (area) is regarded as large (or major) if the domain-specific sample is large enough to yield "direct estimates" of adequate precision. A domain is regarded as "small" if the domain-specific sample is not large enough to support direct estimates of adequate precision. Some other terms used to denote a domain with small sample size include "local area," "subdomain," "small subgroup," "subprovince," and "minor domain."

Other essential contributions in the field of small area estimation are those of Jiang & Lahiri (2006, [8]) and Pfeffermann (2013, [9]), which offer important reviews concerning the state of the subject at the time they were written. A further text that provides an overview of the various estimation methods for small areas, with a focus on the analysis of poverty indicators, is that of Pratesi (2016, [29]). The latter is a collection of papers dealing with small area estimation, but also with its study in different spatio-temporal frameworks.

Small area estimation has become increasingly used because it provides reliable estimates for subpopulations even if the sample data are not large enough. This is a widespread problem to overcome for many local authorities, such as regions, provinces, etc., but also for policy makers who need to make locally targeted policies. For example, they need to know how to allocate funds and resources in a more efficient way, how to use EU funding in a way that benefits different population groups, etc. Small area estimation statistics help to understand how best to implement these policies, which is why they are increasingly in demand. A further advantage of small area estimation is that even if resources are lacking for conducting large-scale surveys, reliable information can be derived for the portions of the population of interest, which inevitably leads to cost savings.

The underlying idea of small-area estimation techniques is to connected areas using a model and, throught that model take advantage from information available from administrative archives and census. In the literature, small area models are distinct into **Area Level** and **Unit Level** models.

6.1 Area Level models

The first area level model proposed in literature isthe Fay-Herriot model, Fay & Herriot (1979, [11]). This model represents the cornerstone of small area estimation:

$$y_d = \mu_d + e_d \tag{17}$$

with:

y_d	direct estimate of area d (economic security indicator)
$d = 1, ..., D$	area index
μ_d	characteristic of interest of area d
e_d	independent and normally distributed sampling errors where $e_d \vert \mu_d \sim \mathcal{N}(0, \sigma_d^2)$

Remark (I): σ_d^2 is the assumed to be known variance of direct estimator, in our case estimated using the bootstrap method and then smoothed using a Generalized Variance Function model

μ_d is structured as follows:

$$\mu_d = \boldsymbol{x_d'\beta} + u_d \tag{18}$$

with:

$\boldsymbol{x_d}$	p auxiliary variables taken from administrative or census sources (for which therefore there is no need to calculate a sampling error)
$\boldsymbol{\beta}$	p respective regression coefficients of auxiliary variables
u_d	model errors, I.I.D. that are distributed with $\mathcal{N}(0, \sigma_u^2)$
σ_u^2	independent from sampling errors e_d and unknown

So the basic Fay-Herriot model consists of two models, a **Sampling model** and a **Linking model**

For the sampling model, we therefore have the direct estimates from a sample survey and the corresponding sample variances, thus considered known. On the other hand, in linking model parameter of interest in the population are related to random effects specific to a given domain d.

To summarise, the Fay-Herriot model is an area-level model that is a two-stage one, where the first stage is the sampling model, used to represent the sampling error of the direct estimators, while the second stage is the linking model that is introduced to improve the reliability of the estimates by borrowing information (since this information is lacking and often inaccurate in small areas, in our case for provinces), thus:

- connect the small area through a model
- taking advantage of the relationship between the target result and some auxiliary information, available from the census or administrative data

Merging sampling and linking model we can obtain the extended form of the basic Fay-Herriot model at area level, which is:

$$y_d = \boldsymbol{x_d'}\boldsymbol{\beta} + u_d + e_d \tag{19}$$

Two sources of error are thus involved, one from sampling, e_d, and one from modelling, u_d.

6.2 Unit Level models

With regard to unit-level models, the main paper often referred to is that of Battese, Harter & Fuller (1988, [30]). Unit-level models are in general more difficult to develop, because it is more complicated to obtain auxiliary data from census or administrative sources concerning individuals. This problem arises from two factors: the first is that it costs much more to collect data on individuals and this often creates a major obstacle. The second is of course privacy concerns, with agencies holding this type of data being reluctant to provide it.

In general a basic unit level model is stuctured as follows

$$y_{dk} = \boldsymbol{x'_{dk}}\boldsymbol{\beta} + u_d + e_{dk} \tag{20}$$

with:

y_{dk}	direct estimate of area d for individual k
$d = 1, ..., D$	area index
$k = 1, ..., N_d$	individual index
$\boldsymbol{x_{dk}}$	p auxiliary variables taken from administrative or census sources available for each individuals and for each area
$\boldsymbol{\beta}$	p respective regression coefficients of auxiliary variables
u_d	area-specific random errors, I.I.D. that are distributed with $\mathcal{N}(0, \sigma_u^2)$
e_{dk}	independent and normally distributed sampling errors where $e_{dk} \sim \mathcal{N}(0, \sigma_e^2)$

where:

$$\boldsymbol{x_{dk}} = \begin{pmatrix} x_{1,d,k} \\ x_{2,d,k} \\ \vdots \\ x_{j,d,k} \\ \vdots \\ x_{p,d,k} \end{pmatrix}$$

It was already mentioned how difficult it is to find census or administrative source data at unit level and the problems associated with this. For this reason, an area-level model is used here, whose auxiliary data can be easily taken from various sources, mainly from ISTAT (which is the Italian national statistical institute) and MEF (i.e. data from tax declarations, available thanks to the Ministry of Economy and Finance).

After the analysis of the economic security indicator using a basic Fay-Herriot model, the focus will shift in order to see how this can be improved by borrowing information about time. In the following will be employed more elaborate Fay-Herriot models that deal with time-specific random effects by taking into account autoregressive processes of order 1 (AR1) and moving average of order 1 (MA1). But first it is necessary to discuss the estimation of the variance of the direct estimator and the methods used to estimate it.

7 Bootstrap variance estimation

The EU-SILC data are the result of a complex sampling plan, moreover the indicator itself is complex. These conditions make the estimation of the standard error particularly difficult to derive.

Several methods have been proposed in the literature for estimating the variance of direct estimators. The best-known are the balance repeated replication (BRR), the jackknife, the linearization and the bootstrap method (this in turn includes a multiplicity of methods). The first was first developed in Kish & Frankel (1970, [31]), the second has an excellent review in Miller (1974, [32]). Although all of these methods have been widely used, in the field of small area estimation the last two are the most applied in recent years. The reference paper with regard to linearisation is that of Deville (1999, [33]) based on the notion of influence function introduced by Hampel(1974, [34]), while with regard to boostrap one of the major contributions is that of Rao & Wu (1988, [35]), in which various types of resampling methods are studied, most of them based on the bootstrap methodology originally introduced by Efron (1979, [36]). Therefore, we had to choose one of these two methods to estimate the variance of the direct estimators. Although both have advantages and disadvantages, after a thorough review of the literature, the method that seems to fit our case best is the bootstrap method, as suggested in [35], in Kolenikov (2010, [37]) and in Valliant (2007, [38]). In fact we have a complex indicator and the EUSILC sample design is also complex, as we saw in Section 3. In addition, our indicator is constructed using sample weights, so to estimate the variance we need to recalibrate these weights, which with the bootstrap can be handled quite easily. Finally, a further reason for choosing bootstrapping over linearisation is that the latter requires much more computational effort.

For these reasons, a bootstrap resampling method was used that took into account the characteristics of the sampling design, although in a simplified version given the complexity of the design itself. The estimate of the bootstrap variance of the direct estimates is carried out in two steps, first recalibrating the weights and then estimating the bootstrap variance.

7.1 Weights recalibration

In order to proceed with the bootstrap it was necessary to develop an algorithm that allow to recalibrate the weights, in order to obtain for each bootstrap sample the appropriate weights. This is possible because the original weights were calibrated on known population totals. For the recalibration of the weights, the iterative proportional fitting method was chosen. This procedure starts with the definition of:

$$w_j$$

as the calibrated sample weight (of the survey), and

$$f_j^b$$

as the frequency of occurrence of observation j in bootstrap sample b, where $b = 1, ..., B$, so we can now define:

$$\widetilde{w}_j^b = f_j^b \cdot w_j$$

as the uncalibrated bootstrap weights. Now $\widetilde{w}_j^b$ may produce statistics which are different from those produced by the original weights w_j that are calculated from certain socio-demographic variables that refer to known population margins, this is because $\widetilde{w}_j^b$ are not calibrated while w_j have been calibrated. The calibration of original involve some socio-demographic variables that we can describe as follow:

$$n = n_P + n_H$$

with:

$$P = \{p_c, \quad c = 1, ..., n_P\}$$

$$H = \{h_c, \quad c = 1, ..., n_H\}$$

where n is the number of socio-demographic variables, that is the sum of personal, n_P, and household ones, n_H. Considering EUSILC data household variable can be for example country, region, province or household size, while personale ones can be age, employment status, income range or sex. Each variable in P and H can assume values:

$$P_c \quad \text{with population margin} \quad N_v^{p_c} \quad v = 1, ..., P_c$$

$$H_c \quad \text{with population margin} \quad N_v^{h_c} \quad v = 1, ..., H_c$$

The iterative proportional fitting starts initialising $k = 0$, and this procedure is applied to every $\widetilde{w}_j^b$ individually. The algorithm works in two step, involving first the personal variables, then the household ones, progressively adjusting and trimming them. An important aspect to underline is that if constraints regarding the populations margins are not met, then k will be raised by 1 and the procedure needs to start from the beginning.

The adjustment and trimming for personal variables work as follows:

$$\widetilde{w}_j^{[(n+1)k+c-1]}$$

is iteratively multiplied by a factor so that the projected distribution of the population correspond to the respective calibration specification:

$$N_{p_c} \quad c = 1, ..., n_P$$

and for each c the calculation of the calibrated weights against the population margin $N_v^{p_c}$ is:

$$\widetilde{w}_j^{[(n+1)k+c]} = \widetilde{w}_j^{[(n+1)k+c-1]} \cdot \frac{N_v^{p_c}}{\sum_l \widetilde{w}_l^{[(n+1)k+c-1]}}$$

where the sum in the denominator spreads over all observations that have the same value as observation j for the socio-demographic variable p_c. Now it is important to ensure that the weights are not too variable or at least not much more variable than the original weights, for this, following Potter (1990, [39]), it is necessary to define boundaries:

$$\left[\frac{w_j}{4}; 4w_j\right]$$

It is very likely that some weights will be outside this range, so the next step is as follows.If the calculated weights $\widetilde{w}_j^{[nk+c]}$ are lower than the lower boundary, the value of the lower boundary is assigned and, likewise, if the weights are higher than the upper boundary, the value of the upper boundary is assigned, in order to trim the weights.

So far the weights, $\widetilde{w}_j^{[nk+n_p]}$, are calculated based exclusively on personal variables, so these weights may currently be different for individuals in the same household, with the possibility of inconsistency between projected results with household and personal weights. To overcome this problem it is necessary to give each person in the household the average of the household weights, i.e:

$$\widetilde{w}_j^{[(n+1)k+n_p+1]} = \frac{\sum\limits_{l \in a} \widetilde{w}_l^{[(n+1)k+n_p]}}{h_a}$$

where j is the person, a is the household and h_a is the number of household member of household a. This can lead to loose the structure of population that we find above, so now it is necessary also to adjusting and trimming weights by household variables. The weights already computed by personal variables, $\widetilde{w}_j^{[nk+n_p+1]}$, must be adjusted by the household varible, setting a household constraint parameter ϵ_h, that is the allowed deviation from the population margin using the weights $\widetilde{w}_j^{[nk+n_p+1]}$, compared to $N_v^{h_c}$, $c = 1, ..., n_h$,$v = 1, ..., H_c$, and the new weights are:

$$\widetilde{w}_j^{[(n+1)k+n_p+c+1]} = \begin{cases} \widetilde{w}_j^{[(n+1)k+n_p+1]} & \cdot \dfrac{N_v^{h_c}}{\sum\limits_l \widetilde{w}_l^{[(n+1)k+n_p+1]}} \\ & \text{if} \sum\limits_l \widetilde{w}_l^{[(n+1)k+n_p+1]} \\ & \notin ((1-0.9\epsilon_h)N_v^{h_c}, (1+0.9\epsilon_h)N_v^{h_c}) \\ \widetilde{w}_j^{[(n+1)k+n_p+1]} & \text{otherwise} \end{cases}$$

where the sum in the denominator spreads over all households l that have the same value as observation j for the socio-demographic variable h_c. Also here the weights are trimmed using boundaries previously seen, i.e. $\left[\dfrac{w_j}{4}; 4w_j\right]$.

The next step in the algorithm is the convergence check, so for each update and refinement step, the factor:

$$\frac{N_v(\cdot)}{\sum\limits_l \widetilde{w}_l^{[(n+1)k+j]}}$$

with $j \in \{1, ..., n+1\} \backslash \{n_p+1\}$, $(\cdot)$ personal or househould variable, is checked against convergence constraints for households, ϵ_h, or personal variables, ϵ_p, so we want to verify personal and household constraints:

$$\frac{N_v^{p_c}}{\sum_l \widetilde{w}_l^{[(n+1)k+j]}} \in ((1-\epsilon_p)N_v^{p_c}, (1+\epsilon_p)N_v^{p_c})$$

$$\frac{N_v^{h_c}}{\sum_l \widetilde{w}_l^{[(n+1)k+j]}} \in ((1-\epsilon_h)N_v^{h_c}, (1+\epsilon_h)N_v^{h_c})$$

where the sum in the denominator spreads over all observations that have the same value for the socio-demographic variable p_c or h_c. In the case that constraints hold we reach the convergence, otherwise we have to increase k by 1 and repeat the algorithm.

In this work, we choose as household auxiliary characteristics for the calibration the region and the household size, while the personal ones are sex and age. This procedure implies many benefits with respect to weighting algorithms that involve only population totals, because we can use more variables to make recalibration more accurate. The population data with which comparisons are made come from ISTAT (that is the National Statistical Institute of Italy). For further details see Meraner *et al.* (2016, [40]) and Kolenikov (2014, [41]).

7.2 Bootstrap variance

The bootstrap resampling method works as follows, in the specific case of the economic security indicator. This technique consists of drawing from the original sample a number of samples with replacement (with the same size of the sample), say a number B of samples, and for each of these recalculate the economic security indicator, replacing the original weights with those calculated in the previous paragraph for each bootstrap sample. Instead, the generic bootstrap sample is denoted by b, where $b = 1, ..., B$ (in our work, we set $B = 1.000$):

$$\hat{\hat{\mathcal{S}}}_i^{(Boot)} = \frac{\sum_{b=1}^{B} \hat{\mathcal{S}}_i^{(b)}}{B} \tag{21}$$

with $i = 1, ..., M$ indicating the i-th area (thus within each bootstrap sample an estimate of economic insecurity is produced for each province). Once the bootstrap estimates are obtained, it is possible to proceed with the estimation of the variances:

$$\hat{V}_i^{(Boot)}(\hat{\hat{\mathcal{S}}}_i^{(Boot)}) = \frac{\sum_{b=1}^{B} (\hat{\mathcal{S}}_i^{(b)} - \hat{\hat{\mathcal{S}}}_i^{(Boot)})^2}{B} \tag{22}$$

8 Variance Smoothing

Further step often used in Small Area Estimation is variance smoothing.This procedure makes it possible to improve the estimation of variances by applying smoothing models. In particular, the methodologies used are those of the Generalized Variance Function (GVF), see Wolter (2007, [42]). These procedures are often necessary because the variances calculated directly from the survey tend to be inaccurate, given the scarcity of information present at the level of a small area and given that these usually have few or very few units within them.

An important aspect to emphasize (as Wolter himself argues) is that, although this methodology is widely used with regard to the estimation of household survey variances, it is primarily an empirical/practical methodology and, overall, a general theory supporting the use of one specific model over another has not yet been developed. On a practical level GVF allows to implement a model that enables to relate a characteristic of the population of interest to estimate, in our case the average of the Economic Security Indicator in the Italian provinces, to the estimate of its variance.

In order to obtain a variance that is more precise, we use as input the variance estimated through the bootstrap method, from which we obtained the standard deviation and coefficient of variation, so we can compare different types of models. Briefly, we recall that:

Standard Deviation:

$$\hat{\mathcal{SD}}_i^{(Boot)} = \sqrt{\hat{V}_i^{Boot}} \tag{23}$$

hence the square root of the bootstrap variance

Coefficient of Variation:

$$\hat{\mathcal{CV}}_i^{(Boot)} = \frac{\hat{\mathcal{SD}}_i^{(Boot)}}{|\hat{\mathcal{S}}_i|} \tag{24}$$

so the bootstrap standard error divided by the absolute value of direct estimator of economic security indicator.

Remark (I): For simplicity, below we will denote the direct estimation of economic security with Y, and the coefficient of variation explained above with CV.

After testing several models, the one that fits the data best will be chosen, based on AIC, BIC criteria and adjusted R^2.
The models compared are:

Model I:

$$log(CV^2) = \beta_0 + \beta_1 log(Y)$$

Model II:

$$CV^2 = \beta_0 + \frac{\beta_1}{Y}$$

Model III:

$$CV^2 = \beta_0 + \frac{\beta_1}{Y} + \frac{\beta_2}{Y^2}$$

Model IV:

$$SD = \beta_0 + \beta_1 Y + \beta_2 Y^2$$

Model V:

$$CV = \beta_0 + \frac{\beta_1}{Y} + \beta_2 Y$$

Model VI:

$$VAR = \beta_0 + \frac{\beta_1}{Y}$$

Model VII:

$$VAR = \beta_0 + \frac{\beta_1}{Y} + \frac{\beta_2}{Y^2}$$

Model VIII:

$$VAR = (\beta_0 + \beta_1 Y)^{-1}$$

Model IX:

$$VAR = (\beta_0 + \beta_1 Y + \beta_2 Y^2)^{-1}$$

Model X:

$$log(VAR) = \beta_0 - \beta_1 log(Y)$$

Model XI:

$$VAR = \beta_0 + \beta_1 Y + \beta_2 Y^2$$

Model XII:

$$log(VAR) = \beta_0 + \beta_1 Y$$

After testing those models the one who perform the best is **Model I**, that from now we will call **Model SMT**, that stands for smooth:

Model SMT:

$$log(CV^2) = \beta_0 + \beta_1 log(Y) \tag{25}$$

In order to compute this model, we used the ReGenesees package from R, which includes computational algorithms for GVF. Through this model we find the parameters we need to predict the estimated coefficient of variation that will allow us to obtain the smoothed variance, that is:

$$\widehat{CV}_{(SMT)(dt)} = \sqrt{\frac{\widehat{\sigma}^2_{(SMT)}}{2} \cdot e^{\hat{\beta}_0 + \hat{\beta}_1 log(Y_{dt})}} \tag{26}$$

where:

$\widehat{\sigma}^2_{(SMT)}$ variance of the model

From the CV thus obtained, we can simply find the smoothed variance with a procedure similar to that seen above.

9 The choice of covariates

Covariates must be available at population level, therefore were chosen taking into account only administrative sources, for example:

- **ISTAT (National Statistical Institute of Italy):**
  demographic variables such as number of people by age group and sex, Italians and foreign residents
- **MEF (Ministry of Economy and Finance):**
  economic variables such as sources of income, population by income bracket (frequency and amount)
- **BES-ISTAT (Fair and Sustainable Welfare):**
  several variables concerning economic well-being, for example, life expectancy at birth, municipal collection capacity, separate collection of urban waste, etc.

These auxiliary variables are available for the population, so they are not subject to sampling error. However, if we wanted to use covariates from a survey, we should estimate the sampling error. We choose the first path.

Normally, the model should be parsimonious, but in order to better explain the dependent variable, a large number of regressors had to be selected in this case. This is also due to the nature of our longitudinal indicator, which has a very high variability. The variables were chosen using stepwise regression, looking at AIC, BIC and adjusted R^2, and are the following:

- Average income from buildings:

$$\frac{\text{Income from buildings (amount)}}{\text{Income from buildings (frequency)}}$$

- Average retirement income:

$$\frac{\text{Retirement income (amount)}}{\text{Retirement income (frequency)}}$$

- Average income from 0 to 10.000 €:

$$\frac{\text{Income from 0 to 10.000 €(amount)}}{\text{Income from 0 to 10.000 €(frequency)}}$$

- Average income from 26.000 to 55.000 €:

$$\frac{\text{Income from 26.000 to 55.000 €(amount)}}{\text{Income from 26.000 to 55.000 €(frequency)}}$$

- Average income from 55.000 to 75.000 €:

$$\frac{\text{Income from 55.000 to 75.000 €(amount)}}{\text{Income from 55.000 to 75.000 €(frequency)}}$$

- Average income from 75.000 to 120.000 €:

$$\frac{\text{Income from 75.000 to 120.000 €(amount)}}{\text{Income from 75.000 to 120.000 €(frequency)}}$$

- Average income, more than 120.000 €:

$$\frac{\text{Income, more than 120.000 €(amount)}}{\text{Income, more than 120.000 €(frequency)}}$$

- Working-age population as a proportion of the total - Males:

$$\frac{\text{People between 15 and 64 years old-Males}}{\text{Total population-Males}}$$

- Youth dependency ratio-Females:

$$\frac{\text{People between 0 and 14 years old-Females}}{\text{People between 15 and 64 years old-Females}}$$

- Working-age population-Foreigners:

$$\frac{\text{People between 15 and 64 years old-Foreigners}}{\text{Total population-Foreigners}}$$

- Percentage of graduates in the province

10 The Small Area models considered

10.1 Fay-Herriot model

Now we have all the elements that allow us to pursue our objective, i.e. to produce small area estimates of the economic security of the Italian provinces between 2014 and 2016, and to see how these change the direct estimates of the indicator.

We recall again briefly how the Fay-Herriot model, the cornerstone of small area estimation, work. We have:

$$y_d = \mu_d + e_d \tag{27}$$

with:

y_d	direct estimate of area d (economic security indicator)
$d = 1, ..., D$	area index
μ_d	characteristic of interest of area d
e_d	independent and normally distributed sampling errors where $e_d \mid \mu_d \sim \mathcal{N}(0, \sigma_d^2)$

where σ_d^2 are the known variances of direct estimators, calculated using bootstrap and smoothing (GVF), and μ_d is structured as follows:

$$\mu_d = \boldsymbol{x_d'}\boldsymbol{\beta} + u_d \tag{28}$$

with:

$\boldsymbol{x_d}$	p auxiliary variables taken from administrative or census sources (for which therefore there is no need to calculate a sampling error)
$\boldsymbol{\beta}$	p respective regression coefficients of auxiliary variables
u_d	model errors, I.I.D. that are distributed with $\mathcal{N}(0, \sigma_u^2)$
σ_u^2	independent from sampling errors e_d and unknown

Combining sampling and linking model we obtain the extended form of the basic Fay-Herriot model at area level:

$$y_d = \boldsymbol{x'_d}\boldsymbol{\beta} + u_d + e_d \tag{29}$$

It is now necessary to obtain the EBLUP and the MSE, where EBLUP stands for Empirical Best Linear Unbiased Prediction, which is a weighted combination of a direct estimator of a particular domain and a synthetic regression estimator. See for more details Prasad & Rao (1990, [43]), Datta & Lahiri (2000, [44]):

In particular in [43] the authors use a moment estimator and in [44] REML to obtain EBLUP and MSE (claiming that this has several advantages over the ML method). The BLUP is:

$$\widetilde{y_d} = \boldsymbol{x'_d}\widetilde{\boldsymbol{\beta}} + \lambda_d \cdot (y_d - \boldsymbol{x'_d}\widetilde{\boldsymbol{\beta}}) \tag{30}$$

where:

$$\lambda_d = \frac{\sigma_u^2}{\sigma_d^2 + \sigma_u^2} \tag{31}$$

$$\widetilde{\boldsymbol{\beta}} = (\sum_{d=1}^{D} \frac{\boldsymbol{x_d} \cdot \boldsymbol{x'_d}}{\sigma_d^2 + \sigma_u^2})^{-1} \cdot (\sum_{d=1}^{D} \frac{\boldsymbol{x_d} \cdot \widehat{y_d}}{\sigma_d^2 + \sigma_u^2}) \tag{32}$$

The MSE is:

$$MSE(\widetilde{y_d}) = \lambda_d \cdot \sigma_d^2 + (1 - \lambda_d)^2 \cdot \boldsymbol{x'_d}(\sum_{d=1}^{D} \frac{\boldsymbol{x_d} \cdot \boldsymbol{x'_d}}{\sigma_d^2 + \sigma_u^2})^{-1} \boldsymbol{x_d} \tag{33}$$

where:

$$MSE(\widetilde{y_d}) = E(\widetilde{y_d} - y_d)^2 = g_{1d}(\sigma_u^2) + g_{2d}(\sigma_u^2) \tag{34}$$

with:

$$g_{1d}(\sigma_u^2) = \lambda_d \cdot \sigma_d^2 \tag{35}$$

$$g_{2d}(\sigma_u^2) = (1-\lambda_d)^2 \cdot \boldsymbol{x_d'} \Big(\sum_{d=1}^{D} \frac{\boldsymbol{x_d} \cdot \boldsymbol{x_d'}}{\sigma_d^2 + \sigma_u^2}\Big)^{-1} \boldsymbol{x_d} \tag{36}$$

The EBLUP is obtained simply by using the estimator $\widehat{\sigma}_u^2$ instead of σ_u^2, also in λ_d and in $\widetilde{\boldsymbol{\beta}}$, as follows:

$$\widehat{y_d} = \widehat{\lambda}_d \cdot y_d + (1-\widehat{\lambda}_d) \cdot \boldsymbol{x_d'} \widehat{\boldsymbol{\beta}} \tag{37}$$

10.2 AR1: temporal models that follow an autoregressive process of order 1

Over time, several extensions of the classical Fay-Herriot model have been developed. The idea behind these new theories is that models can be improved by borrowing information from space and time, in particular using simoultaneously time-varying effects and random effects. Therefore, to estimate the parameter for the last available year, previous information can be used through a cross-sectional or, in our case, a longitudinal model. The first models proposed were those including temporal correlation via an autoregressive process of order 1, or AR1, by Rao & Yu (1994, [45]). The model used here is recent Esteban *et al.* (2012, [12]), that will be described in this paragraph:

The model equation by equation is:

$$y_{dt} = \boldsymbol{x}_{dt}\boldsymbol{\beta} + u_{1,d} + u_{2,dt} + e_{dt} \tag{38}$$

where:

$d = 1, ..., D$	Areas index
$t = 1, ..., T$	Times index
y_{dt}	direct estimator of indicator for area d at time t
$\boldsymbol{x}_{dt}$	p auxiliary variables vector
$\boldsymbol{\beta}$	vector of p coefficients
$j = 1, ..., p$	Auxiliary variables index
$u_{1,d}$	area effects constant over time
$u_{2,dt}$	time varying effects following an AR(1)
e_{dt}	sampling errors

the same equation can be written in a more compact form as follows:

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Z_1 u_1} + \boldsymbol{Z_2 u_2} + \boldsymbol{e} \tag{39}$$

where:

$$\boldsymbol{y} = \underset{1\leq d\leq D}{col}(\boldsymbol{y}_d) \qquad \boldsymbol{e} = \underset{1\leq d\leq D}{col}(\boldsymbol{e}_d) \qquad \boldsymbol{\beta} = \underset{1\leq j\leq p}{col}(\beta_j)$$

$$\boldsymbol{y}_d = \underset{1\le t\le T}{col}(y_{dt}) \qquad \boldsymbol{e}_d = \underset{1\le t\le T}{col}(e_{dt}) \qquad \boldsymbol{Z}_1 = \underset{1\le d\le D}{diag}(\mathbf{1}_T)$$

$$\boldsymbol{u}_1 = \underset{1\le d\le D}{col}(u_{1,d}) \qquad \boldsymbol{X} = \underset{1\le d\le D}{col}(\boldsymbol{X}_d) \qquad \boldsymbol{Z}_2 = \boldsymbol{I}_{DT}$$

$$\boldsymbol{u}_2 = \underset{1\le d\le D}{col}(\boldsymbol{u}_{2,d}) \qquad \boldsymbol{X}_d = \underset{1\le t\le T}{col}(\boldsymbol{x}_{dt})$$

$$\boldsymbol{u}_{2,d} = \underset{1\le t\le T}{col}(u_{2,dt}) \qquad \boldsymbol{x}_{dt} = \underset{1\le j\le p}{col'}(x_{dtj})$$

where:

$$\begin{aligned}
\mathbf{u}_1 &\sim \mathcal{N}(\mathbf{0}, \mathbf{V}_{u_1})\\
\mathbf{u}_2 &\sim \mathcal{N}(\mathbf{0}, \mathbf{V}_{u_2})\\
\mathbf{e} &\sim \mathcal{N}(\mathbf{0}, \mathbf{V}_e)\\
\mathbf{V}_{u_1} &= \sigma_1^2 \boldsymbol{I}_D\\
\mathbf{V}_{u_2} &= \sigma_2^2 \Omega(\rho)\\
\Omega(\rho) &= \underset{1\le d\le D}{diag}(\Omega_d(\rho))\\
\mathbf{V}_e &= \underset{1\le d\le D}{diag}(\boldsymbol{V}_{ed})\\
\mathbf{V}_{ed} &= \underset{1\le t\le T}{diag}(\sigma_{dt}^2)\\
\sigma_{dt}^2 &= \text{sampling error variances}\\
\Omega_d(\rho) &= \text{AR(1) variance-covariance matrix}
\end{aligned}$$

and

$$\Omega_d = \Omega_d(\rho) = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & \rho & \cdots & \rho^{T-2} & \rho^{T-1} \\ \rho & 1 & \ddots & & \rho^{T-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \rho^{T-2} & & \ddots & 1 & \rho \\ \rho^{T-1} & \rho^{T-2} & \cdots & \rho & 1 \end{pmatrix} \tag{40}$$

BLU estimators are obtained as follows:

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} \\ \hat{\boldsymbol{u}} &= \boldsymbol{V_u}\boldsymbol{Z}'\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}) \end{aligned} \tag{41}$$

where:

$$\mathbf{V}_u = diag(\mathbf{V}_{u1}, \mathbf{V}_{u2})$$

and,

$$\begin{aligned} \mathbf{V} = var(\mathbf{y}) = \underset{1\le d\le D}{diag}(\mathbf{V}d) = \\ = \sigma_1^2\boldsymbol{Z}\boldsymbol{Z}' + \sigma_2^2 \underset{1\le d\le D}{diag}(\Omega_d(\rho)) + \boldsymbol{V}_e = \underset{1\le d\le D}{diag}(\sigma_1^2 \mathbf{1}_T\mathbf{1}_T{}' + \sigma_2^2\Omega_d(\rho) + \boldsymbol{V}_{ed}) \end{aligned}$$

The REML log-likelihood is given by:

$$\begin{aligned} l_{REML}(\sigma_1^2,\sigma_2^2,\rho) = -\frac{DT-p}{2}log2\pi + \frac{1}{2}log|\boldsymbol{X}\boldsymbol{X}'| - \frac{1}{2}log|\boldsymbol{V}| - \\ -\frac{1}{2}log|\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}| - \frac{1}{2}\boldsymbol{y}'\boldsymbol{P}\boldsymbol{y} \end{aligned} \tag{42}$$

with:

$$\boldsymbol{P} = \boldsymbol{V^{-1}} - \boldsymbol{V^{-1}X(X\,'V^{-1}X)^{-1}X\,'V^{-1}}$$

$$\boldsymbol{PVP} = \boldsymbol{P}$$

$$\boldsymbol{PX} = \boldsymbol{0}$$

$$\begin{aligned}
\boldsymbol{\theta} &= (\theta_1, \theta_2, \theta_3) = (\sigma_1^2, \sigma_2^2, \rho) \\
\boldsymbol{V_1} &= \frac{\delta \boldsymbol{V}}{\delta \sigma_1^2} = \underset{1 \le d \le D}{diag}(\boldsymbol{1}_T \boldsymbol{1}_T\,') \\
\boldsymbol{V_2} &= \frac{\delta \boldsymbol{V}}{\delta \sigma_2^2} = \underset{1 \le d \le D}{diag}(\Omega_d(\rho)) \\
\boldsymbol{V_3} &= \frac{\delta \boldsymbol{V}}{\delta \rho} = \sigma_2^2 \underset{1 \le d \le D}{diag}(\Omega_d(\rho))) \\
\boldsymbol{P}_a &= \frac{\delta \boldsymbol{P}}{\delta \theta_a} = -\boldsymbol{P}_a \frac{\delta \boldsymbol{V}}{\delta \theta_a} \boldsymbol{P} = -\boldsymbol{P V}_a \boldsymbol{P} \qquad a = 1, 2, 3
\end{aligned}$$

so the Score will be:

$$S_a = \frac{\delta l_{REML}}{\delta \theta_a} = -\frac{1}{2} Tr(\boldsymbol{P V}_a) + \frac{1}{2} \boldsymbol{y\,' P V}_a \boldsymbol{P y} \tag{43}$$

taking the partial derivatives with respect to θ_a and θ_b and performing this operation again we obtain the Fisher information matrix components:

$$F_{ab} = \frac{1}{2} Tr(\boldsymbol{P V}_a \boldsymbol{P V}_b) \qquad b = 1, 2, 3 \tag{44}$$

The REML estimator and its asymptotic distribution are:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\hat{\boldsymbol{V}}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{V}}^{-1}\boldsymbol{y} \tag{45}$$

$$\hat{\boldsymbol{\theta}} \sim \mathcal{N}_3(\boldsymbol{\theta}, (\boldsymbol{F}^{-1}(\boldsymbol{\theta}))) \tag{46}$$

$$\hat{\boldsymbol{\beta}} \sim \mathcal{N}_p(\boldsymbol{\beta}, (\boldsymbol{X}'\hat{\boldsymbol{V}}^{-1}\boldsymbol{X})^{-1}) \tag{47}$$

Further step, as done for the basic Fay-Herriot model, is to derive the EBLUP and MSE as follows:

$$\mu_{dt} = \boldsymbol{x}_{dt}\boldsymbol{\beta} + u_{1,d} + u_{2,dt} \tag{48}$$

with the EBLUP:

$$\widehat{\mu}_{dt} = \boldsymbol{x}_{dt}\widehat{\boldsymbol{\beta}} + \widehat{u}_{1,d} + \widehat{u}_{2,dt} \tag{49}$$

The aim is to predict:

$$y_{dt} = \boldsymbol{a}'\boldsymbol{y} \tag{50}$$

with:

$$\boldsymbol{a} = \underset{1\le l\le D}{col}\big(\underset{1\le k\le T_l}{col}(\delta_{dl}\delta_{tk})\big)$$
$$\boldsymbol{a} = \begin{cases} 1, & \text{in } t + \sum_{l=1}^{d-1} T_l \\ 0, & \text{otherwise} \end{cases} \tag{51}$$

By using the estimate of $\bar{Y}_{dt}$, i.e. $\widehat{\bar{Y}}_{dt} = \widehat{\mu}_{dt}$, the MSE is derived as follows:

$$MSE(\widehat{\bar{Y}}_{dt}) = g_1(\boldsymbol{\theta}) + g_2(\boldsymbol{\theta}) + g_3(\boldsymbol{\theta}) \tag{52}$$

where:

$$\begin{aligned} &g_1(\boldsymbol{\theta}) = \boldsymbol{a}'\boldsymbol{ZTZ}'\boldsymbol{a} \\ &g_2(\boldsymbol{\theta}) = \big[\boldsymbol{a}'\boldsymbol{X} - \boldsymbol{a}'\boldsymbol{ZTZ}'\boldsymbol{V_e}^{-1}\boldsymbol{X}\big]\boldsymbol{Q}\big[\boldsymbol{X}'\boldsymbol{a} - \boldsymbol{X}'\boldsymbol{V_e}^{-1}\boldsymbol{ZTZ}'\boldsymbol{a}\big] \\ &g_3(\boldsymbol{\theta}) \approx tr\Big\{(\nabla\boldsymbol{b}')\boldsymbol{V}(\nabla\boldsymbol{b}')'E\big[(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})'\big]\Big\} \end{aligned} \tag{53}$$

where:

$$\begin{aligned} \boldsymbol{Q} &= (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})' \\ \boldsymbol{T} &= \boldsymbol{V_u} - \boldsymbol{V_u}\boldsymbol{Z}'\boldsymbol{V}^{-1}\boldsymbol{Z}\boldsymbol{V_u} \\ \boldsymbol{b}' &= \boldsymbol{a}'\boldsymbol{Z}\boldsymbol{V_u}\boldsymbol{Z}'\boldsymbol{V}^{-1} \end{aligned} \tag{54}$$

Remark (I): we should remember from previous paragraph that:

$$\boldsymbol{\theta} = (\sigma_1^2, \sigma_2^2, \rho) \tag{55}$$

Finally, the estimator of MSE is:

$$MSE(\widehat{\bar{Y}}_{dt}) = g_1(\widehat{\boldsymbol{\theta}}) + g_2(\widehat{\boldsymbol{\theta}}) + 2g_3(\widehat{\boldsymbol{\theta}}) \tag{56}$$

10.3 MA1: temporal models that follow a moving average process of order 1

Another extension of the basic Fay-Herriot model at area level, more recent than the previous one, uses random effects at time level that follow a moving average of order 1 (MA1) process. In particular we use the approach of Esteban *et al.* (2016, [10]), in which the direct estimates are modelled as follows:

$$y_{dt} = \boldsymbol{x}_{dt}\boldsymbol{\beta} + u_{1,d} + u_{2,dt} + e_{dt} \tag{57}$$

in compact form:

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Z_1 u_1} + \boldsymbol{Z_2 u_2} + \boldsymbol{e} \tag{58}$$

we have now to remark that the main difference with the AR1 concerns the time varying random effects that here follow a MA1 process, so again:

$$\mathbf{u}_2 \sim \mathcal{N}(\mathbf{0}, \mathbf{V}_{u_2})$$

but with:

$$\begin{aligned}
\mathbf{V}_{u_2} &= \sigma_2^2 \Omega(\theta) \\
\Omega(\theta) &= \underset{1 \le d \le D}{diag}(\Omega_d(\theta)) \\
\Omega_d(\theta) &= \text{MA(1) variance-covariance matrix}
\end{aligned}$$

where:

$$\Omega_d = \Omega_d(\theta) = \begin{pmatrix} 1+\theta^2 & -\theta & \cdots & 0 & 0 \\ -\theta & 1+\theta^2 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 1+\theta^2 & -\theta \\ 0 & 0 & \cdots & -\theta & 1+\theta^2 \end{pmatrix} \tag{59}$$

The BLU estimator is:

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} \\ \hat{\boldsymbol{u}} &= \boldsymbol{V_u}\boldsymbol{Z}'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \end{aligned} \tag{60}$$

where:

$$\mathbf{V}_u = diag(\mathbf{V}_{u1}, \mathbf{V}_{u2})$$

and,

$$\begin{aligned} \mathbf{V} = var(\mathbf{y}) &= \underset{1\leq d\leq D}{diag}(\mathbf{V}d) = \\ &= \sigma_1^2\boldsymbol{Z}\boldsymbol{Z}' + \sigma_2^2 \underset{1\leq d\leq D}{diag}(\Omega_d(\theta)) + \boldsymbol{V}_e = \underset{1\leq d\leq D}{diag}(\sigma_1^2\mathbf{1}_T\mathbf{1}_T{}' + \sigma_2^2\Omega_d(\theta) + \boldsymbol{V}_{ed}) \end{aligned}$$

The REML log-likelihood is:

$$\begin{aligned} l_{REML}(\sigma_1^2, \sigma_2^2, \theta) = &-\frac{DT-p}{2}log2\pi + \frac{1}{2}log|\boldsymbol{X}\boldsymbol{X}'| - \frac{1}{2}log|\boldsymbol{V}| - \\ &-\frac{1}{2}log|\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}| - \frac{1}{2}\boldsymbol{y}'\boldsymbol{P}\boldsymbol{y} \end{aligned} \tag{61}$$

where:

$$\boldsymbol{P} = \boldsymbol{V^{-1}} - \boldsymbol{V^{-1}X}(\boldsymbol{X}\,'\boldsymbol{V^{-1}X})^{-1}\boldsymbol{X}\,'\boldsymbol{V^{-1}}$$

$$\boldsymbol{PVP} = \boldsymbol{P}$$

$$\boldsymbol{PX} = \boldsymbol{0}$$

$$\begin{aligned}
\boldsymbol{\theta} &= (\theta_1, \theta_2, \theta_3) = (\sigma_1^2, \sigma_2^2, \theta) \\
\boldsymbol{V_1} &= \frac{\delta \boldsymbol{V}}{\delta \sigma_1^2} = \underset{1\leq d\leq D}{diag}(\boldsymbol{1}_T \boldsymbol{1}_T\,') \\
\boldsymbol{V_2} &= \frac{\delta \boldsymbol{V}}{\delta \sigma_2^2} = \underset{1\leq d\leq D}{diag}(\Omega_d(\theta)) \\
\boldsymbol{V_3} &= \frac{\delta \boldsymbol{V}}{\delta \theta} = \sigma_2^2 \underset{1\leq d\leq D}{diag}(\Omega_d(\theta))) \\
\boldsymbol{P}_a &= \frac{\delta \boldsymbol{P}}{\delta \theta_a} = -\boldsymbol{P}_a \frac{\delta \boldsymbol{V}}{\delta \theta_a} \boldsymbol{P} = -\boldsymbol{P}\boldsymbol{V}_a\boldsymbol{P} \qquad a = 1,2,3
\end{aligned}$$

with the Score:

$$S_a = \frac{\delta l_{REML}}{\delta \theta_a} = -\frac{1}{2} Tr(\boldsymbol{P}\boldsymbol{V}_a) + \frac{1}{2} \boldsymbol{y}\,'\boldsymbol{P}\boldsymbol{V}_a\boldsymbol{P}\boldsymbol{y} \tag{62}$$

Taking the partial derivatives with respect to θ_a and θ_b and performing this operation again, we obtain the Fisher information matrix components:

$$F_{ab} = \frac{1}{2} Tr(\boldsymbol{P}\boldsymbol{V}_a\boldsymbol{P}\boldsymbol{V}_b) \qquad b = 1,2,3 \tag{63}$$

The REML estimator and its asymptotic distribution are:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\hat{\boldsymbol{V}}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\boldsymbol{V}}^{-1}\boldsymbol{y} \tag{64}$$

$$\hat{\boldsymbol{\theta}} \sim \mathcal{N}_3(\boldsymbol{\theta}, (\boldsymbol{F}^{-1}(\boldsymbol{\theta}))) \tag{65}$$

$$\hat{\boldsymbol{\beta}} \sim \mathcal{N}_p(\boldsymbol{\beta}, (\boldsymbol{X}'\hat{\boldsymbol{V}}^{-1}\boldsymbol{X})^{-1}) \tag{66}$$

Prediction of the EBLUP and MSE are given by:

$$\mu_{dt} = \boldsymbol{x}_{dt}\boldsymbol{\beta} + u_{1,d} + u_{2,dt} \tag{67}$$

with the EBLUP:

$$\widehat{\mu}_{dt} = \boldsymbol{x}_{dt}\widehat{\boldsymbol{\beta}} + \widehat{u}_{1,d} + \widehat{u}_{2,dt} \tag{68}$$

This is basically the same we did with AR1 process, but we have to remember that now we have time varying random effects that follow a MA1, sothe aim is to predict:

$$y_{dt} = \boldsymbol{a}'\boldsymbol{y} \tag{69}$$

with:

$$\begin{aligned} \boldsymbol{a} &= \underset{1\leq l\leq D}{col}\big(\underset{1\leq k\leq T_l}{col}(\delta_{dl}\delta_{tk})\big) \\ \boldsymbol{a} &= \begin{cases} 1, & \text{in } t+\sum\limits_{l=1}^{d-1} T_l \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{70}$$

By using the estimate of $\bar{Y}_{dt}$, i.e. $\widehat{\bar{Y}}_{dt} = \widehat{\mu}_{dt}$, the MSE follows:

$$MSE(\widehat{\bar{Y}}_{dt}) = g_1(\boldsymbol{\theta}) + g_2(\boldsymbol{\theta}) + g_3(\boldsymbol{\theta}) \tag{71}$$

where:

$$\begin{aligned} &g_1(\boldsymbol{\theta}) = \boldsymbol{a}'\boldsymbol{ZTZ}'\boldsymbol{a} \\ &g_2(\boldsymbol{\theta}) = \big[\boldsymbol{a}'\boldsymbol{X} - \boldsymbol{a}'\boldsymbol{ZTZ}'\boldsymbol{V_e^{-1}X}\big]\boldsymbol{Q}\big[\boldsymbol{X}'\boldsymbol{a} - \boldsymbol{X}'\boldsymbol{V_e^{-1}ZTZ}'\boldsymbol{a}\big] \\ &g_3(\boldsymbol{\theta}) \approx tr\Big\{(\nabla\boldsymbol{b}')\boldsymbol{V}(\nabla\boldsymbol{b}')'E\big[(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta})(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta})'\big]\Big\} \end{aligned} \tag{72}$$

where:

$$\begin{aligned} \boldsymbol{Q} &= (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})' \\ \boldsymbol{T} &= \boldsymbol{V_u} - \boldsymbol{V_u Z}'\boldsymbol{V}^{-1}\boldsymbol{ZV_u} \\ \boldsymbol{b}' &= \boldsymbol{a}'\boldsymbol{ZV_uZ}'\boldsymbol{V}^{-1} \end{aligned} \tag{73}$$

Remark (I): we should remember again that now we have:

$$\boldsymbol{\theta} = (\sigma_1^2, \sigma_2^2, \theta) \tag{74}$$

Finally, the estimator of MSE is:

$$MSE(\widehat{\widehat{Y}}_{dt}) = g_1(\widehat{\boldsymbol{\theta}}) + g_2(\widehat{\boldsymbol{\theta}}) + 2g_3(\widehat{\boldsymbol{\theta}}) \tag{75}$$

11 Application to EU-SILC data

The economic security indicator, as anticipated, is calculated considering three waves, and we compute the index for three years, from 2014 to 2016 (therefore the first index is calculated considering 2012, 2013 and 2014; the second one considering 2013, 2014 and 2015; the third one considering 2014, 2015 and 2016). The data are longitudinal data from the EUSILC survey and the indicator that has been constructed is based on the equivalent disposable income and provides a level of security (in case it is positive) or insecurity (in case it is negative).

The following table shows some summary statistics calculated for the direct estimates, the estimates of their variances, obtained by using bootstrap and smoothing as described in Section 7 and 8, and the coefficients of variation.

Stat	Dir. Est.	Var.	CV
Min.	-1.865	54.702	0,15
1st Qu.	-185	138.558	0,46
Median	276	165.286	0,82
3rd Qu.	871	194.082	1,54
Max.	3.526	268.490	41,59
Mean	376	165.047	1,83

Table 2: Direct estimates, variances and coefficient of variation: 2014-2016.

Table 2 shows that the indicator is highly variable and ranges from about -2.000 up to 3.500, with a mean of 376, so the mean seems to suggest that on average, the Italian groups of provinces are more secure than economically insecure. The same information comes from position measures (i.e. median, 3rd quartile), which show that economic security prevails over insecurity. However, looking at the 1st quartile it can be seen that for 25% of the areas economic insecurity is recorded. The range between the minimum and maximum is very wide, in fact the indicator is highly variable, as can be seen from the coefficients of variation. In particular CVs calculated for all these summary statistics assume high values, indicating a high dispersion around the mean for areas, so this led us to use small area estimation. It would be naïve to claim that the Italian groups of provinces are more secure than economically insecure by looking only direct estimator of the economic security indicator. This is because direct estimation is inefficient so we need to

consider small area models in order to improve the estimates by borrowing information from connected related areas and auxiliary variables available from administrative archives files.
To this purpose we consider Fay-Herriot model (FH), and its extensions to the temporal level, i.e. the AR1 and the MA1, that may be particularly suitable for our longitudinal indicator.

Table 3 results obtained from the models considered. We can note that all model estimates are in general lower than direct estimates for positive values, higher for negative values. The highest minimum is that of MA1, with both AR1 and FH very close to each other, lower by about 100, and the direct estimates lower by about 850. The highest 1st quartile is again that of the MA1 model, but this time closer to that of the FH (-6) than the AR1's (-14) one, while again that of the direct estimates is the lowest. Above the first quartile, all estimators move from values displaying insecurity to ones showing economic security. In the case of the median, FH and AR1 have identical values, with MA1 being slightly higher. The situation change again when looking at the 3rd quartile, with MA1 having the lowest values, followed by AR1 and FH. This trend is confirmed for the maximum and for the mean, with the difference that FH and AR1 have almost the same values.

Stat	Direct Estimate	Eblup FH	Eblup AR1	Eblup MA1
Min.	-1.865	-1.134	-1.120	-1.004
1st Qu.	-185	-61	-69	-55
Median	276	254	254	263
3rd Qu.	871	706	699	680
Max.	3.526	2.443	2.388	2.270
Mean	376	355	355	352

Table 3: Direct estimates, Eblup FH, AR1, MA1 estimates: 2014-2016

Table 4 variances obtained for direct estimates and EBLUPs as explained in Section 7, 8, 10. It is possible to note that all model MSE estimates are in general lower than the variance of direct estimates. The lowest minimum is that of the AR1, followed by the MA1 which is very close to that of the Fay-Herriot, while for the 1st quartile the trend is reversed with the MA1 having the lowest value and the FH and AR1 having very similar values. The same applies to the median, the 3rd quartile, the maximum and the mean, with MA1 having the lowest values followed by FH and AR1 with almost identical values, but still much lower than the variances of the direct estimates.

Stat	DIR	FH	AR1	MA1
Min.	54.701	48.490	40.362	47.142
1st Qu.	138.557	105.157	105.664	97.122
Median	165.286	121.423	122.380	111.281
3rd Qu.	194.081	136.010	137.461	122.737
Max.	268.490	178.073	183.526	167.195
Mean	165.047	120.026	120.754	109.723

Table 4: Direct estimates variance and MSE of FH, AR1, MA1 estimates: 2014-2016

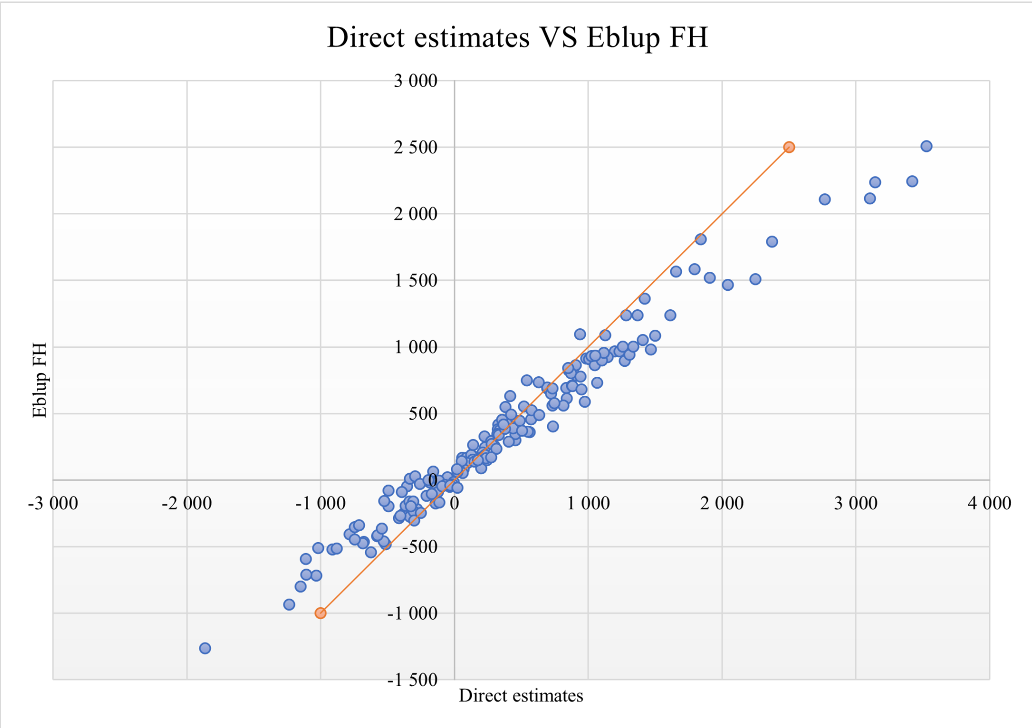

Figure 1: Comparison between Direct estimates and Eblup FH estimates: 2014-2016

In Figures 1, 2 and 3 direct estimates are compared with EBLUPs obtained respectively from FH, AR1 and MA1 models. Figure 1 highlights that the Eblup of the FH model tends to lower the absolute value of the economic security level of the direct estimator. In other words, model based estimates tend to shrink toward 0. This phenomenon, called “shrinkage” is well known in small area literature, see for example Rao and Molina (2015, [7]), Yoshimory and Lahiri (2014, [46]), Datta and Ghosh (2012, [47]). In fact, in some cases models have a propensity to overestimate or underestimate small area estimators, this is because EBLUP tends to reduce the sample mean of the small area towards a certain quantity which results from pooling all the data. In Table 4 we can see also that the estimates of the MSE of the Eblup of FH are always lower than those of the variances of the direct estimator, and this tells us that the precision of the direct estimates can be improved, just as we expected. Fay-Herriot models extended to the case of area-level random effects following an autoregressive process of order 1 (AR1), first, and a moving average process of order 1 (MA1), later, work as follows:

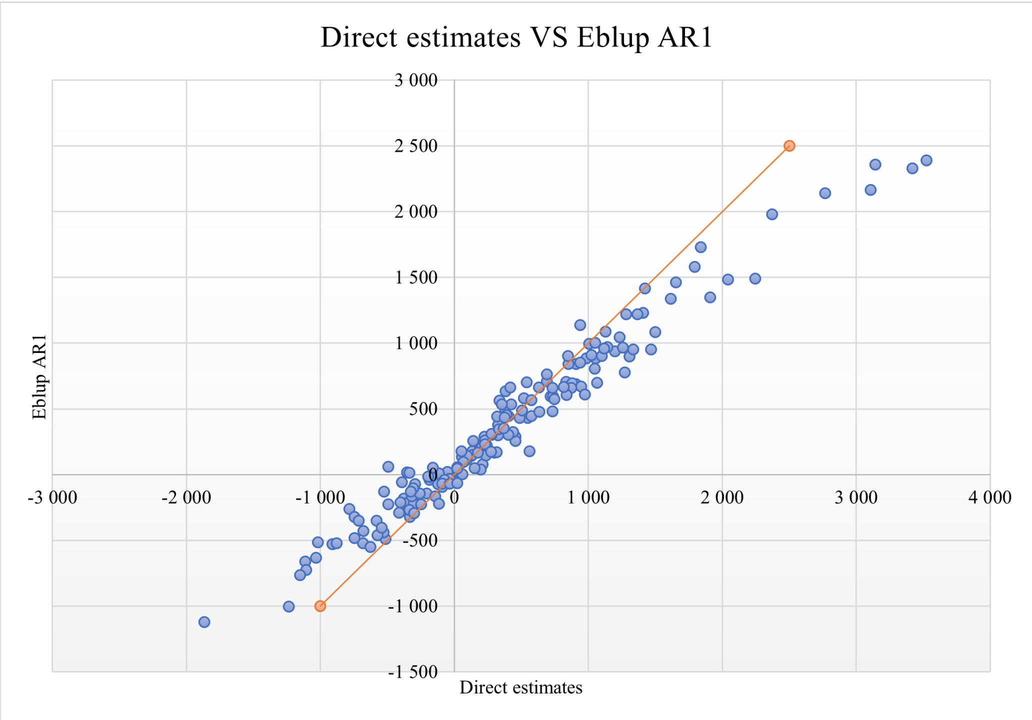

Figure 2: Comparison between Direct estimates and Eblup AR1 estimates: 2014-2016

Figure 2 displays that also in this case positive values of the direct estimation correspond to lower values of the Eblup of AR1 model, while negative values are higher, as recorded also by the values of Eblup FH estimates. The model parameter $\rho = -0,36$ is significant and we have a medium-low time autocorrelation. In addition, Table 4 shows that also the MSE of the AR1 is always lower when compared with the variance of the direct estimator. MA1 model performs better.

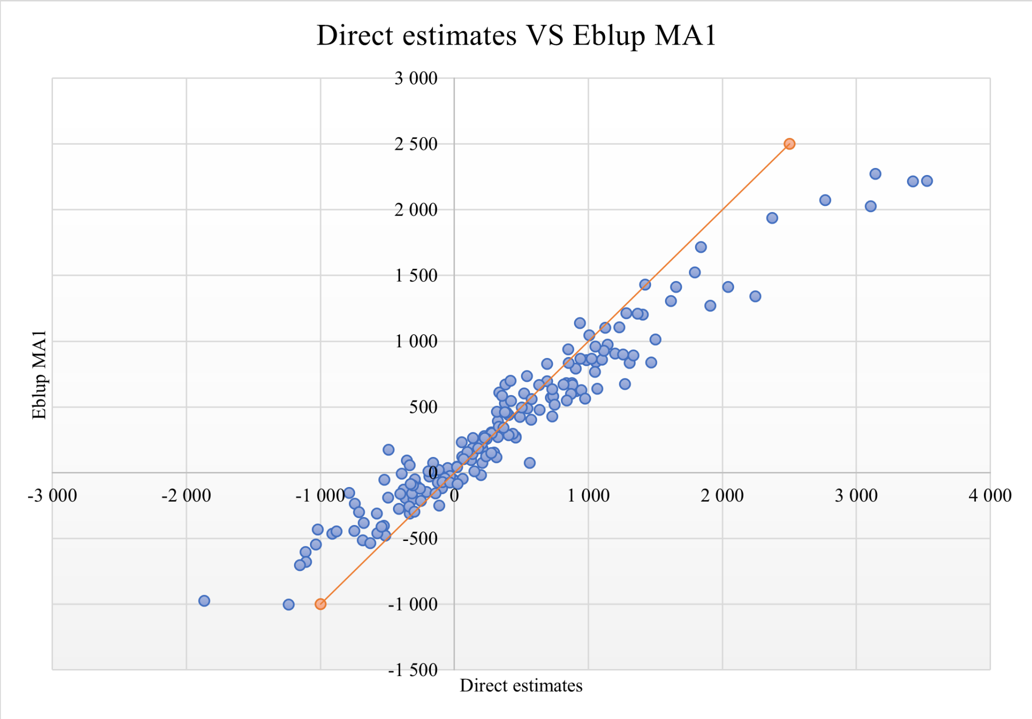

Figure 3: Comparison between Direct estimates and Eblup MA1 estimates: 2014-2016

In fact Figure 3 shows that the Eblup of the MA1 model tends to lower the values of the economic security level of the direct estimator, even if it seems to be less concentrated around zero, when compared to the FH and AR1 models. The Eblup estimates are even more compressed than in the FH model, with lower and lower extreme positive values and higher extreme negative Eblup values. The model parameter $\theta = 0,71$ is significant and indicates that a medium-high temporal autocorrelation is involved. In addition, in this case the roots of the MSE estimates of the MA1 model are lower than the SDs of the direct estimates, but also than the MSEs of the other models..

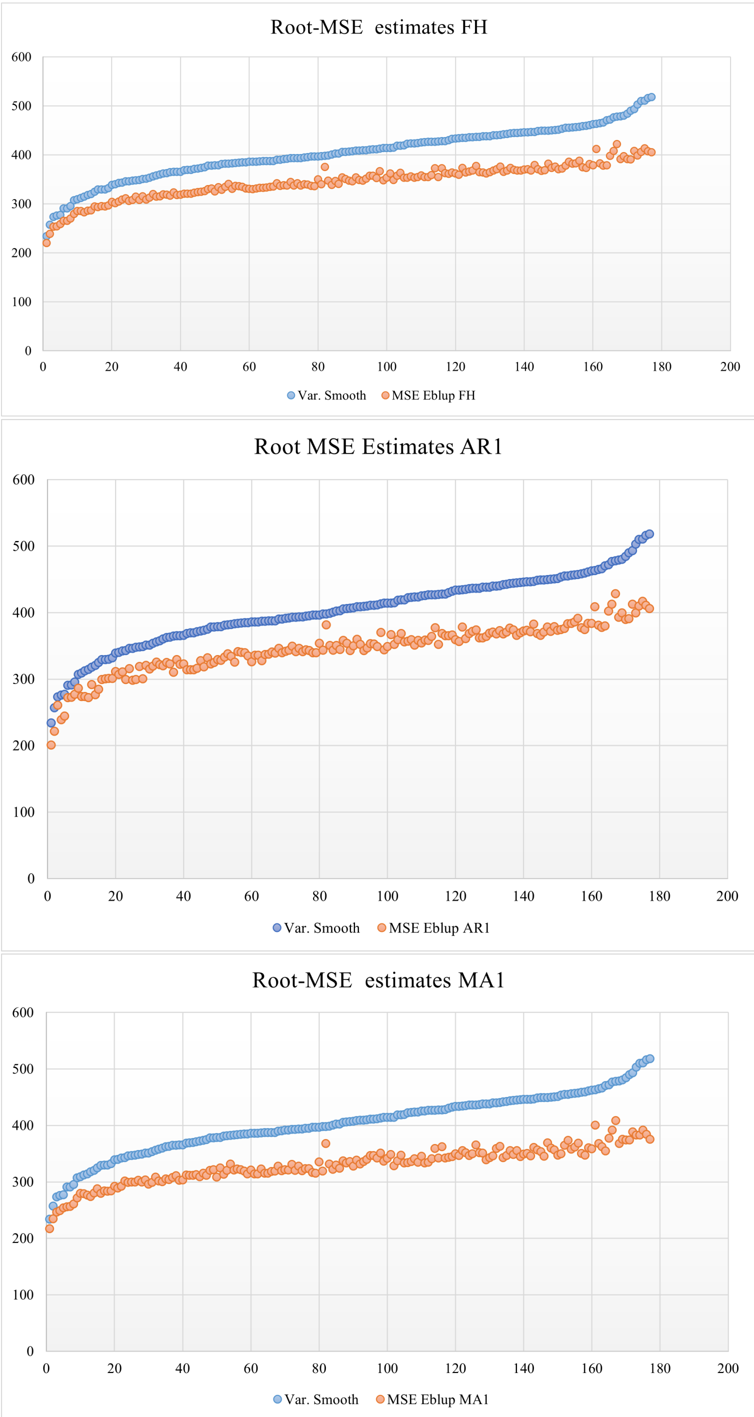

Figure 4: Standard Deviation of direct estimates, Root-MSE estimates of FH, AR1, MA1: 2014-2016

Figure 4 highlights even more than Table 4, that MSE of the Eblup of all the models are always lower than those of the variances of the direct estimator. However, all estimates suffer from shrinkage towards zero, which is why a benchmarking procedure will be carried out later. This technique allows to limit the effect of shrinkage (which is a very common problem in small area estimation), in order to ensure that the estimates of the FH, AR1, MA1 models will be more consistent with the direct ones.

Stat.	Gain Eff. FH	Gain Eff. AR1	Gain Eff. MA1
10th Perc.	-18,9	-9,4	24,5
25th Perc.	13,2	8,3	28,7
Median	28,2	27,7	32,8
75th Perc.	40,6	40,3	36,7
90th Perc.	49,7	50,7	39,9
Average	22,1	21,3	32,0

Table 5: Gain in efficiency of MSE FH, AR1, MA1 with respect to direct estimator variances: 2014-2016.

Table 5 shows the gain in efficiency provided by the small area estimates, calculated as calculated as the percentage of the difference between one and the ratio of the MSE of the model estimates and the variance of the direct estimates. Each of the models used provides gains in efficiency with respect to direct estimates. In particular the average efficiency gain is always positive, with the FH and AR1 models showing similar values of just over 20%, while the MA1 model with an efficiency gain of 32% turns out to be the best on average. This also shows better values than the other two for values up to the median, above which the models that seem to perform better in terms of MSE efficiency are the FH and AR1. The only model that also shows positive values for the 10th percentile is the MA1 model, which also displays better values for the 25th percentile and the median. This changes when looking at the 75th and 90th percentiles where the FH and AR1 show a greater efficiency gain. AR1 gives results that are fully comparable with FH; this result is unexpected and highlights the inadequacy of this model for estimating our indicator. Both, compared to MA1, provide rather variable results, i.e. they perform very well in some areas, but very badly in others (see negative 10th percentile). On the other hand, MA1 produces results that are always more reliable than direct, although with a somewhat more limited range for efficiency gain, but still superior to all on average. This leads the MA1 to appear as the desirable model, although this is something that needs to be investigated further through a simulation study.

To overcome the shrinkage problem outlined above, a benchmarking procedure was conducted in order to limit the effect of shrinkage and to obtain estimates that are more consistent with the original ones. This procedure generally consists of calibrating estimates for small areas to direct estimates obtained for larger areas, available within the same survey (see Rao and Molina, 2015 [7]). In this way we ensure that the estimates resulting from the models are consistent with a design-based estimator, which unlike the one calculated for small areas, is calculated on an aggregate that has a sufficiently large sample size to produce reliable estimates (see Pfeffermann, 2013 [9]; Bell, Datta and Ghosh, 2013 [48]), in our case for the economic security indicator. This procedure was carried out as follows. The goal is to calibrate the Eblup estimates of the various models with survey estimates, taking advantage of population proportions. In our case, the calibration of the population proportions of groups of provinces (small areas) is done by year considering as larger areas the territorial repartitions (Nuts1), so that these sum to 1 within a given repartition for each specific year. Applying this ensures consistent results by Nuts1.

$$\widehat{\mu}_d^{St}(bench) = \widehat{\mu}_d^{St} + \widehat{p}_d(\widehat{\mu}^{N1} - \widehat{\mu}^{St}) \tag{76}$$

where $\widehat{\mu}^{N1}$ are the survey estimates for Nuts1, $\widehat{\mu}^{St}$ is the sum of model estimates weighted by area proportions (i.e., model estimates weighted by the proportion of small areas taking the population of Nuts1 as total, for each year) and:

$$\widehat{p}_{dt} = \frac{W_d \cdot MSE(\widehat{\mu}_{dt}^{St})}{\sum\limits_{d=1}^{D} W_d^2 \cdot MSE(\widehat{\mu}_{dt}^{St})} \tag{77}$$

and W_d is the proportion of population belonging to the $d-th$ area:

$$\sum_{d=1}^{D} W_d \widehat{\mu}_d^{St}(bench) = \widehat{\mu}^{N1} \tag{78}$$

Stat	Direct Estimate	Bench. FH	Bench. AR1	Bench. MA1
Min.	-1.865	-1205	-1172	-1038
1st Qu.	-185	-101	-104	-106
Median	276	283	293	305
3rd Qu.	871	785	782	788
Max.	3.526	2745	2727	2601
Mean	376	382	383	386

Table 6: Direct estimates, Benchmark FH, AR1, MA1 estimates: 2014-2016

Table 6 shows that benchmarking results in estimates that are in general still lower than direct estimates for positive values and also for negative ones, showing less concentration of model estimates around zero. In particular, the comparison with Table 3 highlights the following results: model estimates for areas with the highest level of economic insecurity are even more insecure. The same is true for 25% of domains, where insecurity has almost doubled compared to the Eblup estimates calculated before the benchmarking procedure. In contrast, economic security is again recorded when half of the groups of provinces are taken into account, with economic security being more pronounced on the median for the benchmarked MA1 model, and always slighlty higher than direct estimates one. The same applies when looking at the average values of the economic security indicator for models and direct estimates. In the 3rd quartile, the economic security of the benchmarked model estimators is about 90 lower than that of the direct estimates, but about 90 higher on average than that of the Eblup estimates, with model MA1 again scoring the biggest difference, with the highest value of economic security. Finally, this trend is confirmed when looking at the maximum, for which on average the three benchmarked models gain an economic security value of +340 compared to the models' Eblup, although they do not reach the maximum value of the direct estimates, compared to which they are still lower.

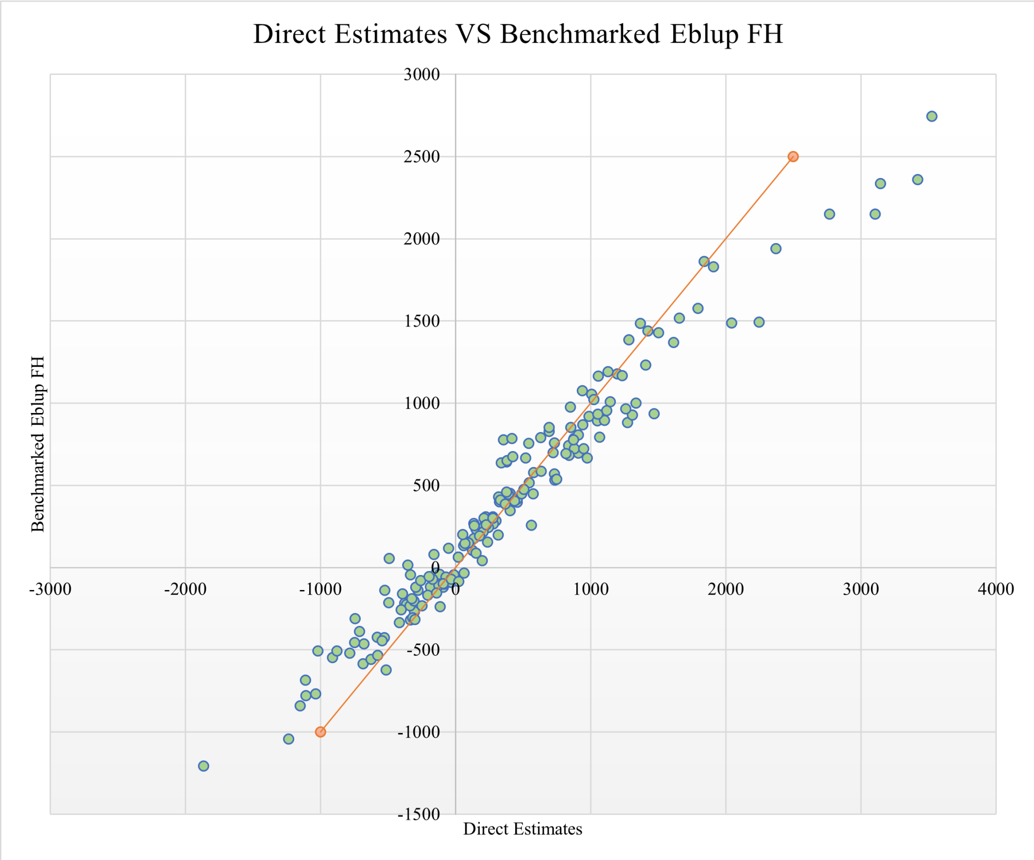

Figure 5: Comparison between Direct estimates and Benchmarked Eblup FH estimates: 2014-2016

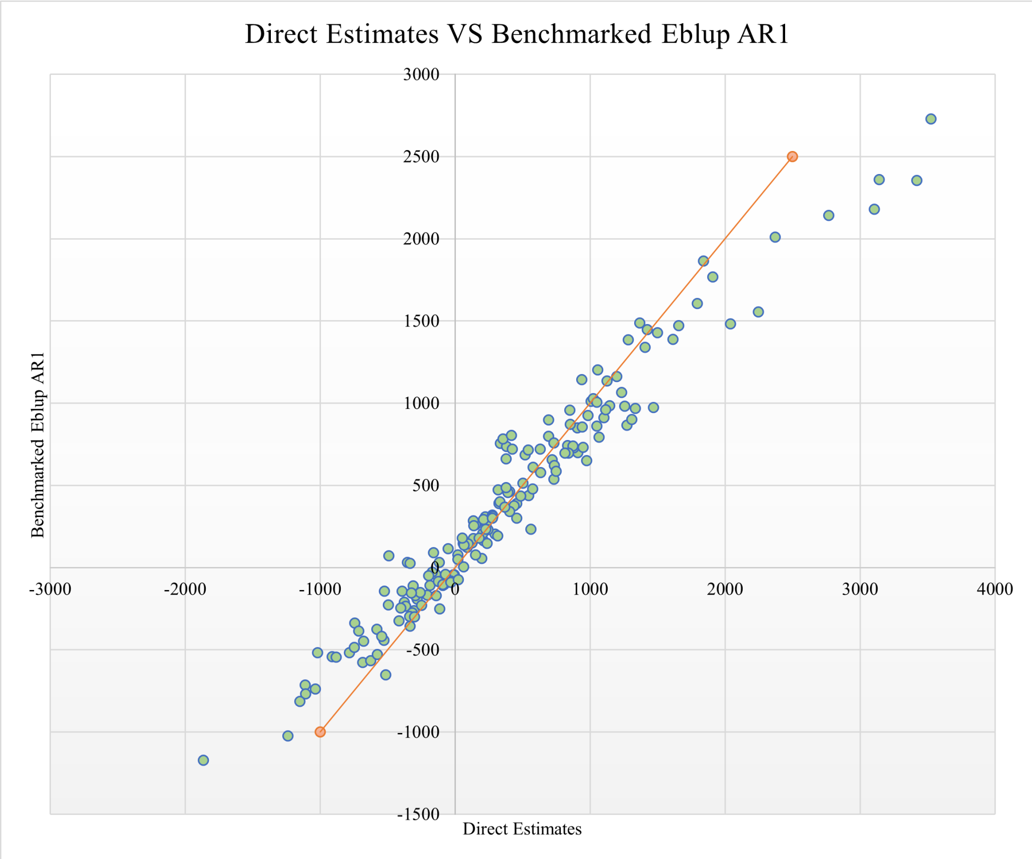

Figure 6: Comparison between Direct estimates and Benchmarked Eblup AR1 estimates: 2014-2016

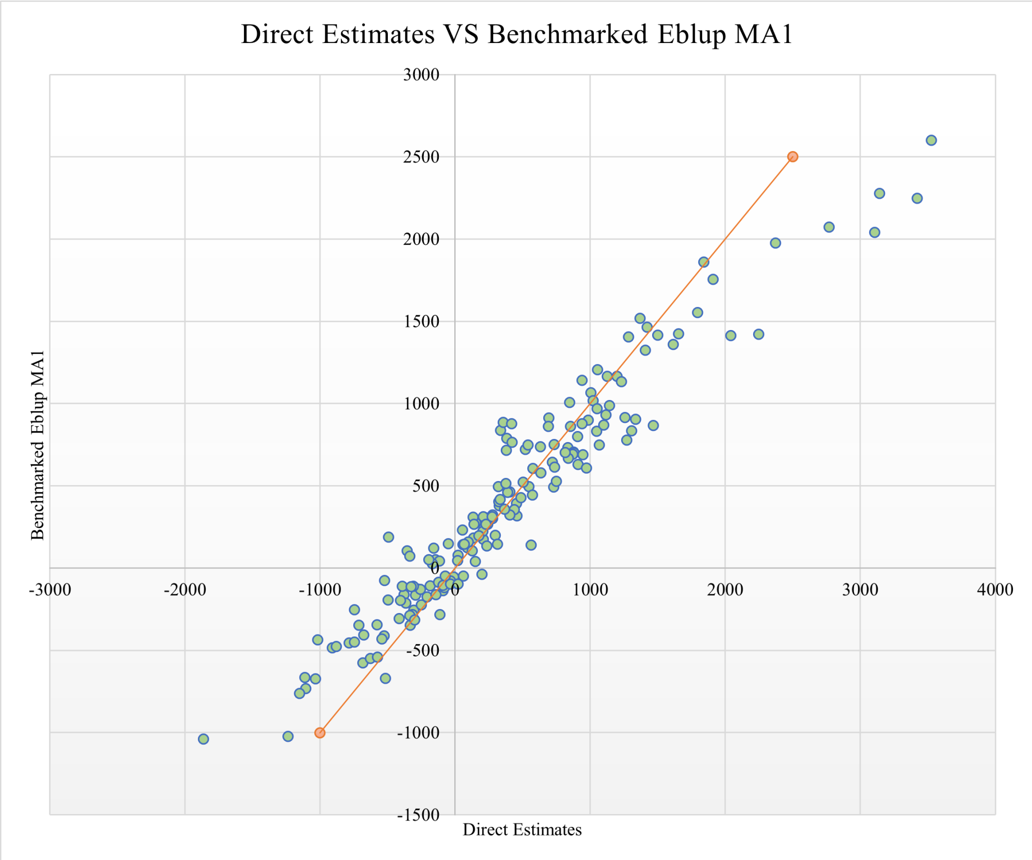

Figure 7: Comparison between Direct estimates and Benchmarked Eblup MA1 estimates: 2014-2016

This shift away from zero due to the benchmarking procedure used is even more evident when looking at Figures 5, 6, 7 and comparing them with Figures 1, 2, 3. Specifically, a widening of the highest and lowest estimates can be seen, which are closer to the values of the original estimates. In addition, a large number of observations, which for the comparison between Eblup and direct estimates were almost right at zero, now yields observations that are more spreaded out and closer to those of the direct estimates. However, even in the case of the benchmarked estimates, as Table 6 also shows, there is still a tendency for the estimates to be lower in absolute value than the original ones, which we may now consider acceptable.

12 Simulation

A simulation was performed to understand the properties of the small area estimators better. To this purpose, a design-based simulation is carried out. The advantage of design-based simulation is that the bias, and other estimator properties are evaluated under the randomised distribution, i.e. the distribution over all possible samples that could be selected from the population of interest under the sampling design. In contrast, model-based methods have a tendency to affect the selected sample, it follows that inference is made with respect to the underlying models, that are always approximations. On the other hand, design-based allows the robustness of model-based estimation methods to be assessed against misspecification by repeatedly sampling from a realistic population.The advantages between choosing a design-based or model-based simulation has been widely discussed, see for example Salvati *et al.*(2010, [49]), Pfeffermann (2013, [9]) and Warnholz and Schmid (2016, [50]).

In this simulation study regions (Nuts2) were chosen as areas, which, having a larger sample size than groups of provinces, allow for a better assessment of the bias and MSE of our estimators. This simulation as mentioned above is sample-based and the following steps were followed: within the original sample, for each area in each year, 1,000 simple random samples were drawn without replacement considering a 15 % sampling rate. We consider simple random samples because the number of units available in the original longitudinal samples does not easily allow to carry out the two stage sampling selection used in EU-SILC survey.

	Ni	ni (sr 15 %)
Min.	122	18
Average	545	81
Max.	1745	261

Table 7: Number of units in the area (Nuts2), total and with 15 % sampling rate

Table 7 shows the total number of “population” units per area (Nuts 2) and in the samples, considering a 15% sampling rate. The smallest domain has 122 units, so 18 units will be extracted at each replication, while the largest has 1745 units, so 261 units will be extracted, and on average the areas have 545 units with a sampling rate that allows 81 units to be extracted.

In more detail, the steps to run the simulation involve first deriving the area numerosity for each year and each region. We consider a 15% sampling rate, we select simple random samples of households without replacement from the region for each year and calculate the average of the security values obtained for the individuals in the households selected. This procedure is repeated 1,000 times.

The averages obtained from each sample represent the direct estimate. Then the Fay-Herriot, AR1 and MA1 EBLUPs were recalculated using the methodologies seen in Section 10. The following performance measures are calculated in order to compare the performance of small area estimators proposed:

$$ARB = \frac{1}{D}\sum_{d=1}^{D}\Big|\frac{1}{1000}\sum_{s=1}^{1000}\Big(\frac{\widehat{Y}_{ds}}{Y_d} - 1\Big)\Big| \tag{79}$$

where ARB means average absolute relative bias and it is a measure of the bias of an estimator (see Rao , 2003 [6]). In this formula $d = 1, ..., D$ denotes the area, while $s = 1, ..., 1000$ denotes the sample, $\widehat{Y}_{ds}$ is the estimate for the d-th area and the s-th sample (Direct, Fay-Herriot, AR1 or MA1) and Y_d is the parameter in population for the d-th domain. Then we measure the accuracy of estimates considering:

$$AMSE = \frac{1}{D}\sum_{d=1}^{D}\frac{1}{1000}\sum_{s=1}^{1000}(\widehat{Y}_{ds} - Y_d)^2 \tag{80}$$

where $AMSE$ is the average mean-squared error of an estimator (Direct, Fay-Herriot, AR1 or MA1). Finally:

$$AEFF(St) = \sqrt{\frac{AMSE(Dir)}{AMSE(St)}} \tag{81}$$

where St stands for estimate (FH, AR1, MA1), measures the gain in efficiency provided on average by the small area estimator. AMSE and AEFF provide us a measure of accuracy of an estimator in terms of its mean square error ([7]).

	Direct	FH	AR1	MA1
ARB (%)	0,00	99,39	112,33	43,97
AMSE	5,81	2,77	2,53	1,09
AEFF (%)	-	144,94	151,51	230,62

Table 8: Average relative bias, average mean-squared errors, and average relative efficiency

Table 8 shows that the MA1 model performs better than both FH and AR1 models in terms of bias. The ARB of the direct estimator is really close to zero, as expected from theory. On the other hand, the ARB of the Fay-Herriot model is slightly lower than that of the AR1, but definitely higher than that of the MA1, so when choosing between the various models, the one preferred in terms of bias is the MA1.
All small area models provide significantly lower value for AMSE than the direct estimator. The most efficient estimates are produced by MA1 model, followed by AR1 and then by FH model. Therefore, even though AR1 model provides more biased estimates than FH model, it overall provides more efficient estimates than FH model. The simulation study shows more clearly the gain in efficiency provided by the small area estimators. The simulation still highlights that the best model in our case is MA1, but AR1 also appears to provide an overall efficiency gain compared to the FH, although it is more biased.

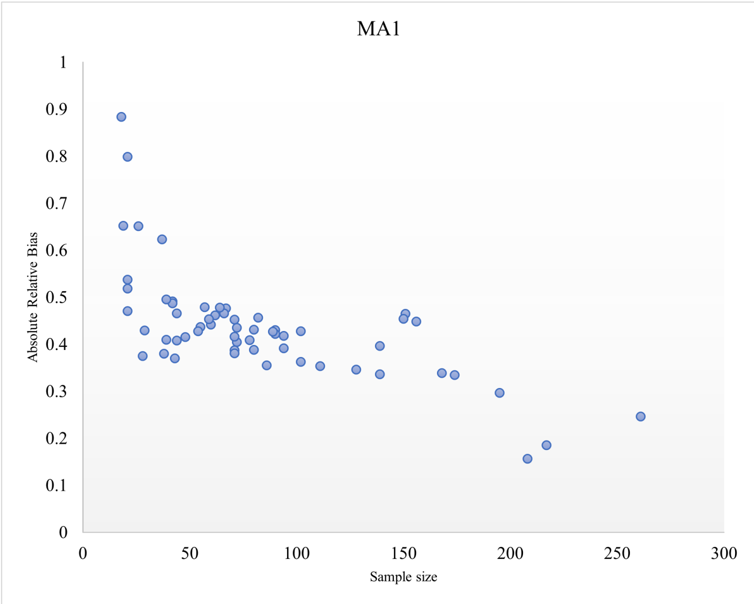

Figure 8: Absolute relative bias of MA1 estimates plotted against the domain sample size

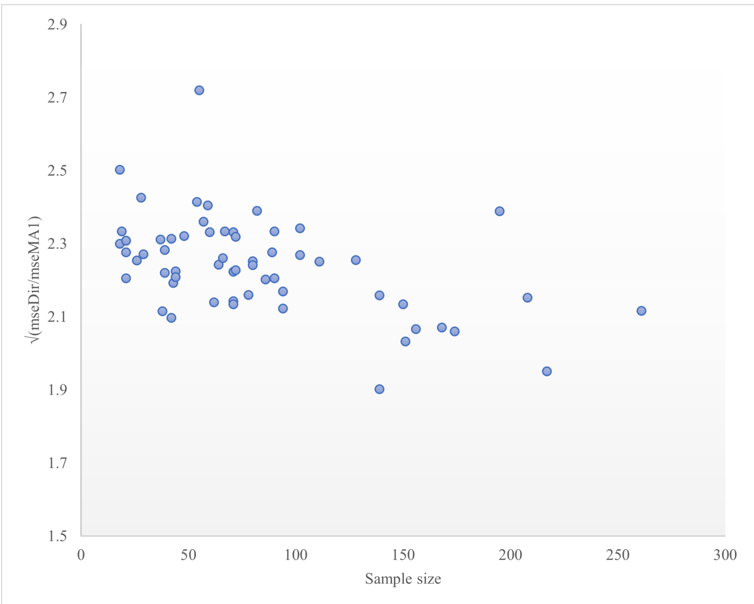

Figure 9: Comparison between $\sqrt{(}MSE(Dir)/MSE(MA1))$ versus domain sample size

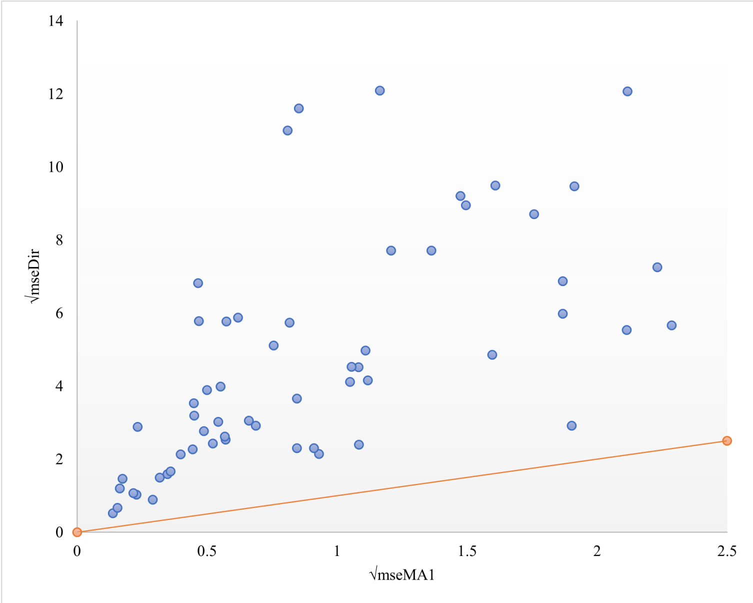

Figure 10: Comparison between $\sqrt{(}MSE(MA1))$ versus $\sqrt{(}MSE(Dir))$

To understand better the properties of results obtained from the best performing small area model, MA1, we carry out a graphical analysis. Figure 8 highlights that as the sample size increases, the absolute relative bias decreases. In fact, it can be seen that the highest values for bias are recorded in the areas with the lowest sample size, and then decrease as the sample size increases. This is evidence that the small area estimator based on the MA1 model tends to be asymptotically design-unbiased and consistent. In Figure 9 we plot the square root of the ratio of $MSE(Dir)$ and $MSE(MA1)$ against the sample size, so that we can see the efficiency gain of the estimator and compare it with the sample size. This gain seems to be particularly pronounced for those areas with a low value of n_i, although, in general, we can notice that, even in the largest areas, MA1 model provides a noticeable, although slightly less pronounced, gain in efficiency. Figure 10 shows that all observations are above the diagonal, so the MSE of the direct is greater than the MSE of the MA1 in all areas.

13 Conclusions

In this work the small area estimation of economic security is considered. To this purpose an indicator obtained as the sum absolute changes of the differences between levels of equivalent disposable income in consecutive years, which includes a loss function, a win function and a discount factor, and constructed using an axiomatic approach, is calculated using data taken from EU-SILC sample survey carried out for Italy from 2014 and 2016. The target parameter is the average of the individual economic security for groups of Italian provinces. The lack of sufficient sample size to obtain reliable direct estimates for these domains led us to the use of small area models. In particular we focus on models specified at area level. This indicator has been used here for the first time in a small area estimation context, whereas poverty and inequality indicators have usually been considered in the literature so far. In addition to the basic Fay-Herriot model, given the nature of the indicator, we proposed to consider some longitudinal extensions of the Fay-Herriot model, specifically two models including temporal random effects. In the first longitudinal model temporal random effects follow an autoregressive process of order 1 (AR1), in the second one they follow a moving average process of order 1 (MA1). Thus, we try to improve the reliability of estimates by borrowing information, not only from auxiliary variables available from administrative archives, but also from temporal correlation. The variances of small area direct estimates, to be included in the small area models, are estimated by using a bootstrap methodology, and then smoothing using the GVF method.

All the models used show a significant efficiency gain in terms of the MSE with respect to the variance of the direct estimates. Especially MA1 model shows the highest efficiency gain on average and median, while AR1's performance appears very similar to the FH's one. This result can be due to the small number of available waves. Moreover, we notice a shrinkage effect towards zero. In order to limit this effect and to obtain model estimates consistent with direct estimates calculated for larger areas, the territorial repartions, a benchmarking procedure is carried out. The benchmarking procedure enables us to obtain small area estimates that are more coherent with direct estimates, although some shrinkage remains. Finally, a simulation study was carried out to understand the properties of the small area estimators better. Results obtained from the simulation study highlight that the best performing small area model, both in terms of bias and overall efficiency, is the MA1 model. Moreover, the simulation study provides further evidence of some properties of the estimators: indeed, all models perform better than direct estimates in terms of mean square error. Furthermore, the

graphical analysis shows that the small area estimators based on best performing model, MA1 model, tends to be asymptotically design-unbiased and consistent. This also emphasises the efficiency gain, which is considerable for all areas.
However, this work still suffers from some limitations. Direct estimates are highly variable. This extreme variability is only partially reduced by the use of models for small areas. The available auxiliary variables are not higly correlated with the target variable. Although this is a problem in general for poverty and inequality indicators, the variability of the economic security indicator makes it even more difficult to find suitable covariates to explain our dependent variable. Finally, a further limitation is the low number of waves available. In fact, with a larger number of waves available, it would have been possible to obtain more efficient and precise estimates for the economic security indicator, estimated with models for small areas, and with less variability.

References

[1] C. D'Ambrosio and N. Rohde, "The distribution of economic insecurity: I taly and the us over the great recession," *Review of Income and Wealth*, vol. 60, pp. S33–S52, 2014.

[2] E. Dvoryadkina, K. Guseynly, and A. Sobyanin, "Economic security of the region's periphery in conditions of digitalization, urbanization and covid-19," in *SHS Web of Conferences*, vol. 93. EDP Sciences, 2021.

[3] W. Bossert, A. E. Clark, C. d'Ambrosio, and A. Lepinteur, "Economic insecurity and the rise of the right," 2019.

[4] W. Bossert and C. D'Ambrosio, "Measuring economic insecurity," *International Economic Review*, vol. 54, no. 3, pp. 1017–1030, 2013.

[5] L. Osberg and A. Sharpe, "An index of economic well–being for selected oecd countries," *Review of Income and Wealth*, vol. 48, no. 3, pp. 291–316, 2002.

[6] J. N. Rao, *Small area estimation.* John Wiley & Sons, 2003.

[7] J. N. Rao and I. Molina, *Small area estimation.* John Wiley & Sons, 2015.

[8] J. Jiang and P. Lahiri, "Mixed model prediction and small area estimation," *Test*, vol. 15, no. 1, pp. 1–96, 2006.

[9] D. Pfeffermann, "New important developments in small area estimation," *Statistical Science*, vol. 28, no. 1, pp. 40–68, 2013.

[10] M. D. Esteban, D. Morales, and A. Pérez, "Area-level spatio-temporal small area estimation models," *Analysis of Poverty Data by Small Area Estimation*, pp. 205–226, 2016.

[11] R. E. Fay III and R. A. Herriot, "Estimates of income for small places: an application of james-stein procedures to census data," *Journal of the American Statistical Association*, vol. 74, no. 366a, pp. 269–277, 1979.

[12] M. D. Esteban, D. Morales, A. Pérez, and L. Santamaría, "Small area estimation of poverty proportions under area-level time models," *Computational Statistics & Data Analysis*, vol. 56, no. 10, pp. 2840–2855, 2012.

[13] K. Dynan, "Household economic security and public policy," *Business Economics*, vol. 51, no. 2, pp. 83–89, 2016.

[14] J. S. Hacker, G. A. Huber, A. Nichols, P. Rehm, M. Schlesinger, R. Valletta, and S. Craig, "The economic security index: A new measure for research and policy analysis," *Review of Income and Wealth*, vol. 60, pp. S5–S32, 2014.

[15] T. K. Lin, R. Law, J. Beaman, and D. G. Foster, "The impact of the covid-19 pandemic on economic security and pregnancy intentions among people at risk of pregnancy," *Contraception*, vol. 103, no. 6, pp. 380–385, 2021.

[16] J. E. Edgerton, "Principles of economic security," *The ANNALS of the American Academy of Political and Social Science*, vol. 154, no. 1, pp. 73–77, 1931.

[17] J. Sperling and E. Kirchner, "Economic security and the problem of cooperation in post-cold war europe," *Review of International Studies*, vol. 24, no. 2, pp. 221–237, 1998.

[18] H. E. Nesadurai, "Introduction: economic security, globalization and governance," *The Pacific Review*, vol. 17, no. 4, pp. 459–484, 2004.

[19] J. Pinder, "European economic security: How can we master the modern economy?" *International Journal*, vol. 40, no. 1, pp. 128–144, 1985.

[20] J. S. Nye, "Collective economic security," *International Affairs (Royal Institute of International Affairs 1944-)*, vol. 50, no. 4, pp. 584–598, 1974.

[21] M. Kahler, "Economic security in an era of globalization: definition and provision," *The Pacific Review*, vol. 17, no. 4, pp. 485–502, 2004.

[22] W. Zhengyi, "Conceptualizing economic security and governance: China confronts globalization," *The Pacific Review*, vol. 17, no. 4, pp. 523–545, 2004.

[23] P. Murias, S. Novello, and F. Martinez, "The regions of economic well-being in italy and spain," *Regional Studies*, vol. 46, no. 6, pp. 793–816, 2012.

[24] I. Svetkina, "Smart city: ensuring economic security of administrative center of the russian entity," in *Current Achievements, Challenges and*

Digital Chances of Knowledge Based Economy. Springer, 2021, pp. 539–547.

[25] L. Osberg and A. Sharpe, "Measuring economic insecurity in rich and poor nations," *Review of Income and Wealth*, vol. 60, pp. S53–S76, 2014.

[26] C. T. Whelan, B. Nolan, and B. Maitre, "Measuring material deprivation in the enlarged eu," 2008.

[27] L. Osberg and A. Sharpe, "An index of economic well–being for selected oecd countries," *Review of Income and Wealth*, vol. 48, no. 3, pp. 291–316, 2002.

[28] J. S. Hacker, G. A. Huber, P. Rehm, M. Schlesinger, and R. Valletta, "Economic security at risk," *The Rockefeller Foundation*, 2010.

[29] M. Pratesi, "Analysis of poverty data by small area estimation," 2016.

[30] G. E. Battese, R. M. Harter, and W. A. Fuller, "An error-components model for prediction of county crop areas using survey and satellite data," *Journal of the American Statistical Association*, vol. 83, no. 401, pp. 28–36, 1988.

[31] L. Kish and M. R. Frankel, "Balanced repeated replications for standard errors," *Journal of the American Statistical Association*, vol. 65, no. 331, pp. 1071–1094, 1970.

[32] R. G. Miller, "The jackknife-a review," *Biometrika*, vol. 61, no. 1, pp. 1–15, 1974.

[33] J.-C. Deville, "Variance estimation for complex statistics and estimators: linearization and residual techniques," *Survey methodology*, vol. 25, no. 2, pp. 193–204, 1999.

[34] F. R. Hampel, "The influence curve and its role in robust estimation," *Journal of the american statistical association*, vol. 69, no. 346, pp. 383–393, 1974.

[35] J. N. Rao and C. Wu, "Resampling inference with complex survey data," *Journal of the american statistical association*, vol. 83, no. 401, pp. 231–241, 1988.

[36] B. Efron, "Bootstrap methods: Another look at the jackknife," *The Annals of Statistics*, vol. 7, no. 1, pp. 1–26, 1979.

[37] S. Kolenikov, "Resampling variance estimation for complex survey data," *The Stata Journal*, vol. 10, no. 2, pp. 165–199, 2010.

[38] R. Valliant, "An overview of the pros and cons of linearization versus replication in establishment surveys," in *Proceedings of the Third International Conference on Establishment Surveys*, 2007, pp. 929–940.

[39] F. J. Potter, "A study of procedures to identify and trim extreme sampling weights," in *Proceedings of the American Statistical Association, Section on Survey Research Methods*, vol. 225230. American Statistical Association Washington, DC, 1990.

[40] A. Meraner, D. Gumprecht, and A. Kowarik, "Weighting procedure of the austrian microcensus using administrative data," *Austrian Journal of Statistics*, vol. 45, no. 3, pp. 3–14, 2016.

[41] S. Kolenikov, "Calibrating survey data using iterative proportional fitting (raking)," *The Stata Journal*, vol. 14, no. 1, pp. 22–59, 2014.

[42] K. M. Wolter, *Introduction to variance estimation.* Springer, 2007, vol. 53.

[43] N. N. Prasad and J. N. Rao, "The estimation of the mean squared error of small-area estimators," *Journal of the American statistical association*, vol. 85, no. 409, pp. 163–171, 1990.

[44] G. S. Datta and P. Lahiri, "A unified measure of uncertainty of estimated best linear unbiased predictors in small area estimation problems," *Statistica Sinica*, pp. 613–627, 2000.

[45] J. N. Rao and M. Yu, "Small-area estimation by combining time-series and cross-sectional data," *Canadian Journal of Statistics*, vol. 22, no. 4, pp. 511–528, 1994.

[46] M. Yoshimori and P. Lahiri, "A new adjusted maximum likelihood method for the fay–herriot small area model," *Journal of Multivariate Analysis*, vol. 124, pp. 281–294, 2014.

[47] G. Datta and M. Ghosh, "Small area shrinkage estimation," *Statistical Science*, vol. 27, no. 1, pp. 95–114, 2012.

[48] W. R. Bell, G. S. Datta, and M. Ghosh, "Benchmarking small area estimators," *Biometrika*, vol. 100, no. 1, pp. 189–202, 2013.

[49] N. Salvati, H. Chandra, M. G. Ranalli, and R. Chambers, “Small area estimation using a nonparametric model-based direct estimator,” *Computational Statistics & Data Analysis*, vol. 54, no. 9, pp. 2159–2171, 2010.

[50] S. Warnholz and T. Schmid, “Simulation tools for small area estimation: Introducing the r-package saesim,” *Austrian Journal of Statistics*, vol. 45, no. 1, pp. 55–69, 2016.