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NUMERICAL INVARIANTS FOR MEASURABLE COCYCLES

Candidato: Filippo Sarti

Supervisore
Prof. Stefano Francaviglia
Cosupervisore
Dott. Alessio Savini

Coordinatrice Dottorato
Prof.ssa Valeria Simoncini

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Abstract

The theory of *numerical invariants* for representations can be generalized to measurable cocycles. This provides a natural notion of *maximality* for cocycles associated to complex hyperbolic lattices with values in groups of Hermitian type. Among maximal cocycles, the class of *Zariski dense* ones turns out to have a rigid behavior.

An alternative implementation of numerical invariants can be given by using equivariant maps at the level of boundaries and by exploiting the Burger–Monod approach to bounded cohomology. Through such *boundary maps* one can provide a useful characterization of maximal cocycles that is a fundamental ingredient to prove rigidity. Due to their crucial role in this theory, we prove existence results in two different contexts. Precisely, we construct boundary maps for non-elementary cocycles into the isometry group of CAT(0)-spaces of finite telescopic dimension and for Zariski dense cocycles into simple Lie groups.

Then we approach numerical invariants. Our first goal is to study cocycles from complex hyperbolic lattices $\Gamma < \mathrm{PU}(1,n)$ into the Hermitian group $\mathrm{SU}(p,q)$. Following the theory recently developed by Moraschini and Savini, we define the *Toledo invariant* by using the pullback along cocycles, also by involving boundary maps. For cocycles $\Gamma \times X \to \mathrm{SU}(p,q)$ with $1 \leq p \leq q < +\infty$, we prove that maximality and Zariski density imply superrigidity in the sense of Zimmer, namely such cocycles come from representations $\mathrm{PU}(1,n) \to \mathrm{SU}(p,q)$ of the ambient group. As a consequence, there is no Zariski dense such cocycle when $1 < p < q$.

Then we move to cocycles $\Gamma \times X \to \mathrm{PU}(p,\infty)$ where $\mathrm{PU}(p,\infty)$ is the infinite dimensional version of $\mathrm{SU}(p,q)$. Here we lose the algebraic structure of the target group, hence the algebraic hull is no more defined. However, we show that maximal cocycles are *reducible*, namely that, modulo cohomology, their image is contained in a finite dimensional algebraic subgroup of $\mathrm{PU}(p,\infty)$.

Finally, we classify Zariski dense measurable cocycles $\Gamma \times X \to G$ from finitely generated groups into Hermitian groups not of tube-type. Precisely, we show that the pullback of the Kähler class, called *parametrized Kähler class*, completely determines the cohomology class of such cocycles.

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Chapter 1

Introduction

1.1 Historical background

In the study of representations of lattices in semisimple Lie groups, the milestones can be identified in the pioneering works by Mostow [Mos68], Prasad [Pra73] and Margulis [Mar75]. Such phenomena are known as *Mostow–Prasad rigidity* and as *Margulis superrigidity*, where the meaning of the word *rigidity* in this context can be roughly explained as follows. A priori an isomorphism between two lattices in a topological group can be more general than the conjugation by an element of the ambient group. This is not the case for simple Lie groups not isomorphic to $\mathrm{SL}(2,\mathbb{R})$. Indeed Mostow [Mos68, Mos74] and Prasad [Pra73] proved that the isomorphism class of lattice boils down to its conjugacy class, that is two isomorphic lattices Γ_1 and Γ_2 must by conjugated.

For instance, if $G = \mathrm{PO}(n,1) = \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ is the group of positive isometries of the real hyperbolic space, any lattice $\Gamma < \mathrm{PO}(n,1)$ is the fundamental group of a complete hyperbolic n-manifold of finite volume. If $n \geq 3$, Mostow rigidity can be restated by saying that any π_1-isomorphism between two complete hyperbolic manifolds of finite volume is induced by an isometry. We observe that when $n = 2$ this is no more true. For instance, a lattice $\Gamma < \mathrm{SO}(2,1) \cong \mathrm{PSL}(2,\mathbb{R})$ corresponds to the fundamental group of a surface and such objects are never rigid, since the deformation space of the inclusion $\Gamma \hookrightarrow \mathrm{SO}(2,1)$, known as *Teichmüller space*, has dimension $6g - 6$ where g is the genus of the surface.

Mostow–Prasad theorem aroused the interest of many mathematicians in the last 50 years, and several efforts have been spent in the attempt of generalizing such phenomena. For instance, since it characterizes embeddings of lattices in the ambient group, one can go further by asking which representations from lattices into Lie groups have a similar rigid behavior. This is completely answered, in the higher rank case, by the celebrated Margulis superrigidity theorem [Mar75]. Given an irreducible lattices in a semisimple Lie group without compact factors of rank at least 2, Margulis proved that a Zariski dense representations of such a lattice into a simple algebraic group defined over a local field extends to a homomorphism of the ambient group.

The approach adopted by Margulis to prove his theorem is based on the construction of equivariant maps between *Furstenberg–Poisson boundaries* [Fur63]. Exploiting some peculiarities of higher rank Lie groups, he proved that any such map coincides almost everywhere with a rational one and, using this, he extended the starting homomorphism.

Since some of Margulis arguments rely on properties of higher rank groups, they cannot be straightforwardly adapted to the rank-one context, for instance to the case of lattices in $\mathrm{PU}(1,n) = \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{C}})$. In fact, superrigidity does not hold in general for such lattices, and explicit counterexamples have been exhibited, for instance in case of lattices in $\mathrm{PU}(1,2)$ [Mos80]. Motivated by this evidence and guided by the work of Goldman and Millson [GM87], Burger and Iozzi [BI07] studied systematically representations of lattices $\Gamma < \mathrm{PU}(1,n)$ into $\mathrm{PU}(1,q)$. They exploited the *Cartan angular invariant* of the complex hyperbolic space $\mathbb{H}^q_{\mathbb{C}}$ to define the *Kähler class*, denoted by k_q, which lies in the 2-bounded cohomology group $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(1,q);\mathbb{R})$. By applying the pullback and then the transfer map, they obtained a class in $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(1,n);\mathbb{R})$. Since the latter one is 1-dimensional, namely it is isomorphic to $\mathbb{R}$, such a class differs from the Kähler class k_n of $\mathrm{PU}(1,n)$ by a multiplicative constant, which in fact is a *numerical invariant* of the representation, called *Toledo invariant*. Such an invariant provides a way to select the class of representations with maximal Toledo invariant, called *maximal representations*. The main result in [BI07], proved independently by Koziarz and Maubon [KM08a] using techniques based on harmonic maps, is that maximal representations are rigid, in the sense that they admits an equivariant totally geodesic holomorphic embedding $\mathbb{H}^n_{\mathbb{C}} \to \mathbb{H}^q_{\mathbb{C}}$. Here the main idea is to study the behavior of equivariant maps $\partial\mathbb{H}^n_{\mathbb{C}} \to \partial\mathbb{H}^q_{\mathbb{C}}$ with respect to the incidence structure on the visual boundaries.

A generalized version of the Toledo invariant can be given for any representation of a complex hyperbolic lattice into groups of Hermitian type, and such an object has been thoroughly studied in the last years. For instance, homomorphisms from a surface group into a Hermitian Lie groups have been studied by Burger, Iozzi and Wienhard, who gave a complete characterization of such maximal representations [BIW10] and by Bradlow, García-Prada and Gothen [BGPG03, BGPG06] through Higgs bundles. On the other hand, it is conjectured that maximal representations $\Gamma \to G$, where $\Gamma < \mathrm{PU}(1,n)$ is a lattice and $n > 1$, are rigid.

Among the results about actions of lattices $\Gamma < \mathrm{PU}(1,n)$ with $n \geq 2$, we mention the work by Koziarz and Maubon [KM08b, KM17], in which case the target group is of rank 2, and the one by Pozzetti [Poz15]. In the latter paper, the author proves the above conjecture for Zariski dense representations into the group $\mathrm{SU}(p,q)$ for $1 < p \leq q$. Here a crucial result is that any Zariski dense equivariant map $\partial\mathbb{H}^n_{\mathbb{C}} \to \mathcal{S}_{p,q}$, where $\mathcal{S}_{p,q}$ denotes the Shilov boundary of $\mathrm{SU}(p,q)$, coincides almost everywhere with a rational one, and this is proved thanks to a deep study of the behavior of such maps with respect to the incidence structures of the boundaries.

A parallel but strictly related topic of research concerns the study of similar rigidity behaviors of a more general class of objects called *measurable cocycles*. A cocycle is a Borel measurable map $\Gamma \times X \to H$ satisfying the condition

$$\sigma(\gamma_1\gamma_2, x) = \sigma(\gamma_1, \gamma_2 x)\sigma(\gamma_2, x)\,,$$

where Γ and H are topological groups and (X, μ_X) is a standard Borel probability Γ-space. The interest in the study of cocycles has several motivations. On the one hand cocycles can be interpreted as a generalization of representations, since one can naturally embed the set of representations in the one of cocycles. Moreover, the comparison can be straightened by introducing the notion of *cohomology* between cocycles, which generalizes the one of conjugacy. On the other hand, cocycles play a role on their own in some fields of mathematics, for instance they describe the action by automorphisms on a principal bundle that has been measurably trivialized (see Example 2.4.3).

In the wider world of cocycles, Mostow rigidity can be restated by using *couplings* and the notion of *tautness*. Given two locally compact second countable groups G, H endowed with their Haar measurable structures, a coupling between them is the datum of a Lebesgue measure $G \times H$-space Ω, of two finite measure spaces X, Y and of G-equivariant (respectively H-equivariant) measurable isomorphism $G \times Y \to \Omega$ ($H \times X \to \Omega$). For a coupling to be *taut* means that it can be trivialized to the tautological one. Since to any (G, G)-coupling (X, μ_X) where G is a locally compact second countable group one can associate a *right measure equivalence cocycle* $G \times X \to G$, tautness can be expressed as follows. A coupling is taut if its associated right measure equivalence cocycle is cohomologous to the standard embedding. In this direction we mention the work by Monod and Shalom [MS04] about superrigidity and tautness of some classes of groups, and the one by Kida [Kid08, Kid10] about the tautness of certain mapping class groups. Concerning rank-one groups, Bader, Furman and Sauer [BFS13] studied the group $\mathrm{PO}(1,n)$ for $n \geq 3$, proving that under an integrability assumption any lattice is taut relatively to its standard embedding in $\mathrm{PO}(1,n)$.

The analogous of Margulis theorem in the context of cocycles is the famous

Zimmer superrigidity theorem [Zim80], which deals with lattices in higher rank groups. Roughly speaking, Zimmer showed that Zariski dense cocycles defined on higher rank groups acting ergodically on the probability space (X, μ_X) come from homomorphisms of the group [Zim80, Theorem 4.1]. Here Zariski dense means that the *algebraic hull* of the cocycle, namely the smaller subgroup of the target containing the image of a cohomologous cocycle, coincides with the ambient group (refer to Section 2.4.1 for the precise definition). More recently, an extension of Zimmer superrigidity has been proved by Fisher and Hitchman [FH06] using harmonic maps techniques. We also mention the beautiful proof of Zimmer's conjecture recently given by Brown, Fisher and Hurtado [BFH20b, BFH20a], where measurable cocycles are involved.

Coming back to the rank-one case, apart from the case of real hyperbolic lattices studied by Bader, Furman and Sauer, few things were known. In fact, as for representations, in this context Zimmer's approach fails. Motivated by these questions and inspired by the works by Bader, Furman and Sauer [BFS13] and by Burger and Iozzi [BI02], Moraschini and Savini have recently developed the theory of *numerical invariants* for cocycles [MS20, Sav20, Sav22, MS21]. An example can be found in [SS21c] and will be described in Chapter 4 in the case of cocycles $\Gamma \times X \to H$ from a lattice $\Gamma < \mathrm{PU}(1,n)$ into a group H of Hermitian type, where the authors define the *Toledo invariant* adapting the definition given for representations. Here, by integrating along X the *pullback* of the Kähler class of H and by composing with the *transfer map*, one gets a multiple of the Kähler class of G. This defines a multiplicative constant called Toledo invariant of the cocycle which is also bounded, and hence we can define *maximal cocycles* as those with maximal Toledo invariant. Such machinery can be alternatively implemented through *boundary maps* and this approach allows to study maximality by analyzing some properties of boundary maps, as done for instance in case of representations by Burger, Iozzi and Wienhard and by Pozzetti. Here we set the genesis of this thesis. In fact, on the one hand boundary maps are crucial to implement the pullback along cocycles, hence our first natural direction is to investigate their existence. Then we focus on the study of numerical invariants for some families of cocycles that are not covered by Zimmer's theorem. In particular, inspired by the results obtained for representations, we study cocycles from complex hyperbolic lattices into groups of Hermitian type.

1.2 Results of the thesis

As anticipated, our goal is to use numerical invariants to study rigidity behavior of measurable cocycles from lattices in rank-one groups. To this end, we first study *boundary maps*.

1.2.1 Boundary maps

In our context a boundary map for a cocycle $\sigma : \Gamma \times X \to H$ is a measurable equivariant functions $B \times X \to Y$ where B is a Γ-boundary in the sense of Bader and Furman [BF14b] and Y is a generic Lebesgue H-space. Such a notion of boundary generalizes the one of strong boundary in the sense of Burger and Monod [BM02], hence it can be used in general to compute bounded cohomology. Moreover, it also extend the notion of Furstenberg–Poisson boundary for semisimple Lie groups [Fur63]. In the particular case of a complex hyperbolic lattice in $\mathrm{PU}(1,n)$, it coincides with the visual boundary $\partial\mathbb{H}^n_{\mathbb{C}} \cong \mathbb{S}^{2n-1}$ and it can be used to implement the Toledo invariant.

We first focus on cocycles $\Gamma \times X \to H$ where Γ is a countable group, (X, μ_X) is an ergodic standard Borel probability Γ-space and the target group is the isometries of a CAT(0)-space of finite telescopic dimension. In the analogous context of actions on CAT(κ)-spaces, several efforts have been spent to investigate the existence of boundary maps. We mention the work by Burger–Mozes [BM96] and by Monod–Shalom [MS04] for CAT(-1)-spaces and the one by Duchesne, who first studied actions on the space $\mathcal{X}(p, \infty)$ [Duc12], and then, together with Bader and Lécureux [BDL16], on generic CAT(0)-spaces. In particular, for non-elementary actions by isometries of a locally compact second countable group Γ on a CAT(0)-space $\mathcal{X}$ of finite telescopic dimension, in [BDL16] the authors construct an equivariant map from a Γ-boundary into the visual boundary $\partial\mathcal{X}$. Our generalization of this result to cocycles is the following

Theorem 1 ([SS21a, Theorem 1])**.** *Let Γ be a discrete countable group, (X, μ_X) be an ergodic standard Borel probability Γ-space and B a Γ-boundary. For every non-elementary cocycle $\sigma : \Gamma \times X \to H$ into the isometry group of a CAT(0)-space $\mathcal{X}$ of finite telescopic dimension there exists a boundary map $\phi : B \times X \to \partial\mathcal{X}$.*

The techniques involved to prove Theorem 1 are essentially based on some geometric properties of CAT(0)-spaces and make use of *measurable fields*. In fact, the main result that we exploit to construct the desired map is a measurable version of the Adam–Ballmann theorem proved by Duchesne in [Duc12], and the proof of Theorem 1 follows some arguments adopted in [BDL16].Thanks to the Euclidean De Rham decomposition provided in [Duc12] and exploiting non-elementarity, first we reduce to study the case in which σ is minimal, which means that it does not fix a family of non-trivial convex subspaces of $\mathcal{X}$, and in which $\mathcal{X}$ has trivial Euclidean factor. Then the Adam–Ballmann dichotomy provides either a boundary map or a σ-equivariant family $\{E_{\xi,x}\}_{(\xi,x)\in B\times X}$ of Euclidean subspaces of $\mathcal{X}$. To rule out the second case we follow [BDL16], where the authors exploit relative metric ergodicity of the boundaries and the properties of convex closed subsets of CAT(0)-spaces of finite telescopic dimension due to Caprace and Lytchak [CL09]. In particular we use the fact that bounded subsets admits a circumcenter and the fact that filtering families of closed convex subsets intersects at $\mathcal{X}\cup\partial\mathcal{X}$. Those properties lead to construct either a boundary map, which contradicts the Adam–Ballmann dichotomy, or a family $\{F_x\}_{x\in X}$ of flats, that contradicts non-elementarity.

If $\Gamma < \mathrm{PU}(1,n)$ is a complex hyperbolic lattice and $\mathcal{X}=\mathcal{X}(p,q)$ is the Hermitian symmetric space associated to the group $\mathrm{PU}(p,q)$ with $1\leq p\leq q$ and $q\in\mathbb{N}\cup\{\infty\}$, Theorem 1 provides a boundary map $\partial\mathbb{H}^n_{\mathbb{C}}\times X\to\partial\mathcal{X}(p,q)$. Furthermore, ergodicity implies that the target is the set $\mathcal{I}_k(p,q)$ of isotropic k-subspace of $\mathbb{C}^{p,q}$ with respect to the Hermitian form of signature (p,q) for some $k\leq p$. In general we do not know whether $k=p$ or not. However, if $q<\infty$, the equality holds under the hypothesis of Zariski density. This is a consequence of a more general results on cocycles from locally compact groups into simple Lie groups of non-compact type that is our second result about boundary maps.

Theorem 2 ([SS21c, Theorem 1])**.** *Let Γ be a locally compact and second countable group and let H be a simple Lie group of non-compact type. Let (X,μ_X) be an ergodic standard Borel probability Γ-space and let $\sigma:\Gamma\times X\to H$ be a Zariski dense measurable cocycle. Then, for any Γ-boundary B there exists a σ-equivariant map $\phi:B\times X\to H/P$ where $P<H$ is a minimal parabolic subgroup.*

Here the arguments rely on the algebraic structure of the target group, and follow the line of [BF14b]. In fact, for Zariski dense representations into simple Lie groups of non-compact type Bader and Furman proved the existence of a boundary map into the Furstenberg boundary [BF14b, Theorem 3.4], and this result generalizes the one due to Burger and Iozzi [BI04] where the target is $\mathrm{PU}(p,q)$ (or any group of Hermitian type). The crucial point in [BF14b] is the following universal property. For any cocycle $\Gamma\times X\to H$ and for any Lebesgue Γ-space Y, there exists an algebraic subgroup $L<H$ and a Γ-equivariant universal map $\phi:Y\to V$ such that any other Γ-equivariant map $\psi:Y\to V$ into an algebraic H-space V factorizes through ϕ. Such a property, combined with relative metric ergodicity, allows us to construct a boundary map into a specific homogeneous H-space, identified with the *Furstenberg boundary*, which is the quotient H/Q by a minimal parabolic subgroup.

By composing the map provided by Theorem 2 with the natural projection on the *Shilov boundary*, which can be identified with the quotient of H by a specific maximal parabolic subgroup $Q < H$, we immediately get a boundary map that can be exploited in the next sections.

1.2.2 Toledo invariant and superrigidity

In the second part we apply Theorem 2 to study cocycles $\Gamma \times X \to \mathrm{SU}(p,q)$ when $\Gamma < \mathrm{PU}(1,n)$ is a complex hyperbolic lattice. In this setting, the machinery of numerical invariants applies, precisely we define the *Toledo invariant* of a cocycle and then *maximal cocycles*. Moreover, exploiting the algebraic structure of $\mathrm{SU}(p,q)$, one can give a natural notion of algebraic hull and of Zariski density. Under these two hypothesis, we are able to prove the following

Theorem 3 ([SS21c, Theorem 2])**.** *Fix integers $n \geq 2$ and $1 \leq p \leq q$. Let $\Gamma < \mathrm{PU}(1,n)$ be a torsion-free lattice and let (X, μ_X) be an ergodic standard Borel probability Γ-space. If $\sigma : \Gamma \times X \to \mathrm{SU}(p,q)$ is a maximal Zariski dense measurable cocycle, then it is cohomologous to the restriction of a cocycle associated to a representation $\rho : \mathrm{PU}(1,n) \to \mathrm{SU}(p,q)$.*

The strategy of the proof is the following one. We first study the slices of the boundary maps provided by Theorem 2, and we exploit the maximality of σ to show that such maps preserves the chain geometry of the boundaries. This fact, combined with Zariski density, allows to apply [Poz15, Theorem 4.1] to show that the slices coincide almost everywhere with a rational map. Here we can follow the line of [Zim80, Theorem 4.1], precisely by exploiting both Γ-ergodicity on X and the smoothness of the joint action of $\mathrm{PU}(1,n) \times \mathrm{SU}(p,q)$ on the space of rational functions from $\partial \mathbb{H}^n_{\mathbb{C}}$ to the Shilov boundary $\mathcal{S}_{p,q}$ in order to twist σ into a cocycle induced by a representation. Finally, we exploit again [Poz15] to extend the representation to the ambient group.

As an immediate consequence of Theorem 3 we rule the existence of such cocycle except when $p = q$, and this is the natural extension of [Poz15, Corollary 1.2].

Proposition 4 ([SS21c, Proposition 3])**.** *Consider $n \geq 2$. Let $\Gamma < \mathrm{PU}(1,n)$ be a torsion-free lattice and let (X, μ_X) be an ergodic standard Borel probability Γ-space. Assuming $1 < p < q$, there is no maximal Zariski dense measurable cocycle $\sigma : \Gamma \times X \to \mathrm{SU}(p,q)$.*

Pushing over the comparison with representations, it seems natural to investigate the behavior of maximal cocycles as in Theorem 3 that are not Zariski dense, since by Proposition 4 they are the only ones that can exist when $p \neq q$. Precisely, we focus on the structure of their *algebraic hull*, that is defined as the minimal algebraic subgroup of $\mathrm{SU}(p,q)$ containing the image of a cocycle cohomologous to the starting one. The following result characterize the factors of the algebraic hull of such a cocycle.

Proposition 5 ([SS21c, Proposition 4]). *Fix positive integers $n \geq 2$ and $1 < p < q$. Let $\Gamma < \mathrm{PU}(1,n)$ be a complex hyperbolic lattice, (X, μ_X) be an ergodic standard Borel probability Γ-space and consider a maximal cocycle $\sigma : \Gamma \times X \to \mathrm{PU}(p,q)$. Denoting by $\mathbf{H}$ the algebraic hull of σ and by $H = \mathbf{H}(\mathbb{R})^\circ$, then H splits as the product $K \times L_{nt} \times L_t$, where K is a compact subgroup of $\mathrm{SU}(p,q)$, L_t is a Hermitian Lie group of tube-type and L_{nt} is a Hermitian Lie group not of tube-type that splits again as a product of several copies of $\mathrm{SU}(1,q)$.*

The proof is essentially based on the arguments used in [Poz15, Theorem 1.3] and it indirectly exploits the notion of *tight cocycle*, introduced and studied by Savini [Sav20]. In fact [Sav20, Theorem 1] asserts that the algebraic hull of a maximal cocycle is reductive, and this is proved using the characterization of tight subgroups given in [BIW09] and the fact that maximality implies tightness. Now, since reductive groups split into a compact and a non-compact factor, by considering the composition of σ with the projections on the simple factors of the non-compact part, which are of the form $\mathrm{SU}(p_i, q_i)$, we obtain Zariski dense cocycles satisfying the hypothesis of Proposition 5 and we can conclude.

1.2.3 Finite reducibility

Since the space $\mathcal{X}(p,q)$ is Hermitian also for $q = \infty$, the definition of *Toledo invariant* can be adapted for cocycles $\Gamma \times X \to \mathrm{PU}(p,\infty)$. However, a key difficulty to overcome in the study of such objects is that $\mathrm{PU}(p,\infty)$ is no more algebraic, and this has remarkable consequences. In fact, since the notion of algebraic hull is based on the Noetherian property of algebraic groups, it cannot be defined in this context, hence we cannot to exploit Theorem 2 to get a boundary map in this context. Even if a notion of Zariski density can be given by using *standard algebraic groups*, as done for instance in [DLP21], under this assumptions we are not able to prove the existence of a boundary map with the desired target. However, we define *finite dimensional algebraic subgroups* of $\mathrm{PU}(p,\infty)$, that correspond to algebraic subgroups of the group of invertible linear operator of a finite dimensional Hilbert space. Then, by defining *finite reducible cocycles* as those that can be twisted so that the image is contained into a finite dimensional algebraic subgroup, we will prove the following

Theorem 6 ([SS21a, Theorem 2]). *Let $\Gamma < \mathrm{PU}(1,n)$ be a complex hyperbolic lattice with $n \geq 1$ and let (X, μ) be an ergodic standard Borel probability Γ-space.*

Consider a measurable cocycle $\sigma : \Gamma \times X \to \mathrm{PU}(p, \infty)$ with $p \geq 1$ and suppose there exists a boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{I}_p$ in the space of p-chains inside the visual boundary $\partial\mathcal{X}(p, \infty)$. If σ is maximal, then it is finitely reducible.

Although Theorem 6 is based on the analogous version for representations [DLP21, Theorem 1.6], we need a slight refinement of the arguments used by Duchesne, Lécureux and Pozzetti. Precisely, as in the finite dimensional case, the slices of the boundary map preserve the chain geometry of the boundaries. In this case we prove that there exists a family of minimal totally geodesic embedded subspaces of $\mathcal{X}(p, \infty)$ of the form $\mathcal{X}(p, q_x)$ with $p \leq q_x \leq np$. Since by ergodicity the dimension of such spaces is essentially constant, we have a σ-equivariant family of embedded copies of $\mathcal{X}(p, q)$ for some q and hence we can twist the cocycle in such a way that the image is contained into a copy of $\mathrm{PU}(p, q)$. Finally, the fact that such a group is finite dimensional algebraic implies that σ is reducible.

By combining Theorem 1 and Theorem 6 with results due to Moraschini and Savini [MS21], we can immediately prove the following infinite dimensional version of Mostow rigidity for cocycles.

Theorem 7 ([SS21a, Theorem 3])**.** *Let $\Gamma < \mathrm{PU}(1, n)$ be a complex hyperbolic lattice with $n \geq 1$ and let (X, μ_X) be an ergodic standard Borel probability Γ-space. Any maximal cocycle $\sigma : \Gamma \times X \to \mathrm{PU}(1, \infty)$ is cohomologous to a cocycle preserving a copy of $\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{H}^\infty_{\mathbb{C}}$ and acting on it via the standard lattice embedding.*

1.2.4 The parametrized Kähler class

In the last part of the thesis we study Zariski dense cocycles $\Gamma \times X \to G$ from a finitely generated group into a group G of Hermitian type whose symmetric spaces is irreducible and *not of tube-type*. Hermitian symmetric spaces can be classified in Hermitian spaces of *tube-type*, if they can biholomorphically realized as $V + i\Omega$ where V is a real vector space and $\Omega \subset V$ is a proper convex cone, or *not of tube-type* if not. As we will see, Burger Iozzi and Wienhard [BIW07] showed that such characterization can be also detected through the Kähler class, and it turns out that Zariski dense cocycles into the isometry group of Hermitian symmetric spaces not of tube-type are rigid. The natural notion of pullback along cocycles consists of a map $\mathrm{H}^2_{\mathrm{b}}(\sigma) : \mathrm{H}^2_{\mathrm{cb}}(G; \mathbb{R}) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathrm{L}^\infty(X; \mathbb{R}))$, where the latter denotes the bounded cohomology group of Γ with coefficients in the bounded measurable functions on X. The image under the pullback along σ of the bounded Kähler class k^b_H is called *parametrized Kähler class* of σ and it turns out to determine Zariski dense cocycles up to cohomology.

Theorem 8 ([SS21b, Theorem 1])**.** *Let Γ be a finitely generated group, let (X, μ_X) be an ergodic standard Borel probability Γ-space and consider a Zariski dense* *measurable cocycle $\sigma : \Gamma \times X \to G$ where $G = Isom(\mathcal{X})^\circ$ and $\mathcal{X}$ is an irreducible Hermitian symmetric space not of tube-type. Then the class $\mathrm{H}^2_{\mathrm{b}}(\sigma)(k^b_G)$ in $\mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathrm{L}^\infty(X; \mathbb{R}))$ is non-zero and it is a complete invariant of the cohomology class of σ.*

In the proof we adapt some arguments due to Burger, Iozzi and Wienhard in [BIW07]. Besides the definition of *Kähler class* given using the *Bergmann cocycle* that is already involved in our previous results, here we strongly use the metric structure inherited by an Hermitian symmetric space thanks to its *bounded domain realization*, that is a biholomorphism with a bounded domain $D_{\mathcal{X}} \subset \mathbb{C}^n$. The *Bergman metric* and its associated *Bergman kernel* allows to define the *Hermitian triple product*, that is a function on triples of points in the Shilov boundary $\mathcal{S}_{\mathcal{X}}$. The latter, already mentioned in the specific case of the group $\mathrm{SU}(p,q)$, can be defined in a purely analytic way as the set on which any holomorphic function on $\partial D_{\mathcal{X}}$ assumes its maximum. Thanks to the identification between the Shilov boundary with the quotient of the group by a specific maximal parabolic subgroup, Burger, Iozzi and Wienhard [BIW07] gave an extension of the Hermitian triple product, called *complex Hermitian triple product*, which can be used to characterize domains not of tube-types. With this tools, we deduce Theorem 8 as a consequence of the following more general result (see Theorem 6.1.4). The idea of the proof is to exploit such characterization to rule out a linear dependence between the parametrized Kähler classes of a finite family of Zariski dense independent cocycles $\{\sigma_i : \Gamma \times X \to G_i\}$ into Hermitian groups not of tube-type. Here the notion of independence generalizes the one of cohomology and it is given for cocycles with different targets.

Since cocycles $\Gamma \times X \to G$ can be interpreted as 1-cocycles in the cohomology of the orbital equivalence relation $\mathcal{R}_\Gamma \subset X \times X$ given by the Γ-action on X with values in G (see Feldmann–Moore [FM77] and Furman [Fur10]), we have a cohomological interpretation of their equivalence classes as the first cohomology group $\mathrm{H}^1(\Gamma \curvearrowright X; G)$. From this point of view, if we denote by $\mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X; G)$ the subgroup of Zariski dense cocycles, Theorem 8 provides an injection

$$\mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X; G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathrm{L}^\infty(X; \mathbb{R}))\,, \quad [\sigma] \mapsto \mathrm{H}^2_{\mathrm{b}}(\sigma)(k^b_G)$$

that can be combined with the injection $\mathrm{Rep}_{\mathrm{ZD}}(\Gamma; G) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathbb{R})$ proved in [BIW10] in order to get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_{\mathrm{ZD}}(\Gamma; G) \arrow[r] \arrow[d] & \mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathbb{R}) \arrow[d] \\
\mathrm{H}^1_{ZD}(\Gamma \curvearrowright X; G) \arrow[r] & \mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathrm{L}^\infty(X; \mathbb{R}))
\end{tikzcd} .$$

Here the vertical arrows are induced respectively by the inclusion of the set of representations into the one of cocycles and by the inclusion of constants $\mathbb{R} \hookrightarrow \mathrm{L}^\infty(X;\mathbb{R})$.

As a consequence of Theorem 8 we rule out the existence of Zariski dense cocycles $\Gamma \times X \to G$ if Γ is an irreducible lattice into a product of groups with trivial second bounded cohomology.

Proposition 9 ([SS21b, Proposition 4.1])**.** *Let $n \geq 2$. Consider an irreducible lattice $\Gamma < \prod_{i=1}^{n} H_i$ into a product of locally compact second countable groups with $\mathrm{H}^2_{\mathrm{cb}}(H_i;\mathbb{R}) = 0$ for $i = 1, \ldots, n$. Let (X, μ_X) be an irreducible standard Borel Γ-space and assume that the action is ergodic. Then there is no Zariski dense cocycle $\sigma : \Gamma \times X \to G$ where $G = Isom(\mathcal{X})^\circ$ and $\mathcal{X}$ is any irreducible Hermitian symmetric space not of tube-type.*

Here we exploit a result by Burger–Monod [BM02] that relates the cohomology of Γ with the cohomology of the factors H_i. Precisely, in the setting of Proposition 9, we have an isomorphism

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma;\mathbb{R}) \cong \bigoplus_{i=1}^{n} \mathrm{H}^2_{\mathrm{cb}}(H_i;\mathbb{R})$$

and, thanks to the injection $\mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X; G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathrm{L}^\infty(X;\mathbb{R}))$ provided by Theorem 8, we can conclude with a dimensional argument.

1.3 Structure of the thesis

The thesis is divided in five chapters. In Chapter 2 we recall the background material that we need in the following ones. More precisely, in Section 2.1 we give the definition of amenable, ergodic and smooth actions and of relative metric ergodicity. Section 2.2 is devoted to bounded cohomology, while in Section 2.3 we introduce Hermitian symmetric spaces, the notion of Shilov boundary and the Kähler class. Then we move to the main characters of this thesis, namely measurable cocycles, boundary maps and the machinery of pullback along cocycles (Section 2.4). Finally in Section 2.5, we give the basic notions about CAT(0)-spaces and about measurable fields.

In Chapter 3 we investigate the existence of boundary maps, providing two independent results: the first one if the target is the isometry group of a CAT(0)-space (Section 3.1) and the second one in the case of cocycles into algebraic groups (Section 3.2). We end with Section 3.3 by describing some useful properties of boundary maps.

In Chapter 4 we introduce the Toledo invariant and maximal cocycles (Section 4.1). Then we prove a superrigidity result for cocycles from complex hyperbolic lattices in the group $\mathrm{SU}(p,q)$ (Section 4.2).

Chapter 5 is spent analyzing cocycles from complex hyperbolic lattices into $\mathrm{PU}(p,\infty)$. Due to the absence of algebraicity, we need to introduce standard algebraic subgroups (Section 5.1) and this allows us to prove a result about reducibility (Section 5.2) and an infinite dimensional version of Mostow rigidity for cocycles (Section 5.3).

Finally, in Chapter 6 we classify Zariski dense cocycles from finitely generated groups into Hermitian groups not of tube-type via the pullback of the Kähler class.

Chapter 2

Preliminaries

2.1 Actions

In this section we are going to recall the definitions of smooth, ergodic, relatively metric ergodic and amenable actions, that will be ubiquitous throughout the dissertation. We point out that, although those concepts could be introduced independently, they are actually strongly related. For instance, one can characterize ergodicity through the notion of smoothness, as we will state in Proposition 2.1.7. Furthermore, in Section 2.4.2, we will introduce a notion of boundary for locally compact second countable groups, and in this context amenability and ergodicity will be involved. Our goal is to give the basic definitions and to suggest how those notions are related. We also take advantage of this section to show some examples that will appear in the next chapters. Since we do not want to give an exhaustive description for such tools, we refer the reader to Zimmer's book [Zim84] and to Bader–Furman's paper [BF14b] for more details, and we only discuss the definitions and the results that we will use in this thesis.

In the sequel G will be a locally compact second countable group. We fix once and for all the following basic concepts.

- A *Borel probability G-space* is a probability space (X, μ_X) equipped with a G-action that preserves μ_X.
- A *standard Borel probability G-space* is a Borel probability G-space whose Borel σ-algebra is the one of a separable and completely metrizable space.
- If a the measure μ_x on a standard Borel G-space (X, μ_X) is only quasi-invariant, namely $\mu_X(A) = 0$ if and only if $\mu_X(gA) = 0$ for every $g \in G$, we say that (X, μ_X) is a *Lebesgue G-space*.

We are ready to give the definition of ergodic actions.

Definition 2.1.1. Let G be a locally compact second countable group and let (X, μ_X) be a Lebesgue G-space. The action is *ergodic* if for every G-invariant Borel set A, we have either $\mu_X(A) = 0$ or $\mu_X(X \setminus A) = 0$.

The first trivial example of ergodicity are transitive actions. For instance, if $H < G$ is a closed subgroup, there exists a unique G-quasi-invariant probability measure on G/H and hence, since G/H is a G-homogeneous space, the action is ergodic.

A weaker notion of transitivity in the measurable framework is the one of *essential transitivity.* A G-action on a Borel probability space is essentially transitive if there exists a conull orbit. Essential transitivity implies ergodicity as well, since all orbits of an essentially transitive action are either null or conull. We call *properly ergodic* an ergodic action that is not essentially transitive, namely an action where any orbit has null measure (notice that orbits are actually measurable sets by [Zim84, Corollary 2.1.20]). We describe two examples of properly ergodic action.

Example 2.1.2 ([Zim84, Example 2.1.4])**.** Consider the action of $\mathbb{Z}$ on the circle $\mathbb{S}^1$ induced by the α-rotation

$$r_\alpha : \mathbb{S}^1 \to \mathbb{S}^1 \,, \quad r_\alpha(\xi) := e^{i\alpha}\xi \,,$$

where $\frac{\alpha}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$. Clearly, such an action preserves the Lebesgue measure and it is not essentially transitive, since every orbit is a dense set of measure zero. However, an invariant Borel set $A \subset \mathbb{S}^1$ must be either null or conull. In fact, taking the Fourier expansion of the characteristic function χ_A and exploiting invariance, we get

$$\sum_{n \geq 0} a_n \xi^n = \chi_A(\xi) = \chi_A(e^{i\alpha}\xi) = \sum_{n \geq 0} a_n e^{in\alpha} \xi^n$$

and hence $a_n = a_n e^{in\alpha}$ for every n. Since $\alpha/2\pi$ is irrational we must have $a_n = 0$ if $n \neq 0$, hence χ_A is forced to be constant. This shows that the $\mathbb{Z}$-action on the circle by rotation by an irrational multiple of π is properly ergodic.

Example 2.1.3. Since rational numbers are dense in the real line, by [Zim84, Lemma 2.2.13] the $\mathbb{Q}$-action on $\mathbb{R}$ is ergodic. As in the previous example, such an action is not essentially transitive, since $\mathbb{Q}$ has null Lebesgue measure. In general, any action of a dense subgroup $H < G$ on G is properly ergodic.

Properly ergodic actions generate complicated behavior of the orbits since, for example, any such continuous quasi-invariant G-action on a second countable space which is positive on open sets must have dense but null orbits [Zim84, Proposition 2.1.7]. To rule out such complications, we introduce the notions of *countably separated* Borel space and then of *smooth actions*. We show that, when smoothness and ergodicity coexist, then the situation described above cannot happen. Finally we provide an useful characterization of ergodicity in terms of Borel functions into countably separated spaces.

Definition 2.1.4. A Borel space $(X, \mathcal{B})$ is *countably separated* if there exists a countable family of Borel sets $\{B_j \in \mathcal{B}\}_{j \in \mathbb{N}}$ that separate points.

A relevant example of countably separated space is the quotient space of an algebraic variety defined over a local field of characteristic zero by an algebraic action of an algebraic group. This is a consequence of [Zim84, Theorem 2.1.14] together with [Zim84, Theorem 3.1.3].

Using the notion of countably separated space we are ready to define the concept of smooth action.

Definition 2.1.5. Let $(X, \mathcal{B})$ be a countably separated G-space. The action is called *smooth* if the quotient Borel structure on X/G is countably separated.

The $\mathbb{Q}$-action on $\mathbb{R}$ described in Example 2.1.3 is a first example of non-smooth action. In fact, the quotient $\mathbb{R}/\mathbb{Q}$ does not admit a countable family that separates points, hence it is not countably separated.

Proposition 2.1.6 ([Zim84, Proposition 2.1.10])**.** *Let G be a locally compact second countable group acting smoothly on (X, μ_X). If the action preserves the class of μ_X and is ergodic, then there exists a conull orbit.*

The importance of smoothness to (proper) ergodicity relies also in the following useful characterization.

Proposition 2.1.7 ([Zim84, Proposition 2.1.11])**.** *Let (X, μ) be an ergodic Borel probability G-space and let Y be a countably separated space. Then any G-invariant Borel function $f : X \to Y$ is essentially constant.*

Notice that the assumption on the space Y is actually necessary. Indeed, the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is $\mathbb{Q}$-invariant but clearly not constant, even if the $\mathbb{Q}$-action on $\mathbb{R}$ is ergodic (see Example 2.1.3).

As we will see in Chapter 4 and in the proof of Theorem 3, smooth actions are also crucial in the study of boundary theory. Indeed one of the key points of the proofs of both Margulis [Mar75] and Zimmer [Zim80] superrigidity results relies on the smoothness of the action of product groups on the set of rational functions between boundaries (see Definition 2.4.11 and [Zim84, Proposition 3.3.2]). To be more precise, we conclude this part by presenting an explicit example.

Example 2.1.8. Fix $n \geq 1$ and $1 \leq p \leq q$. Let $G := \mathrm{Isom}(\mathbb{H}^n_{\mathbb{C}})$ and $H := \mathrm{SU}(p,q) = \{h \in \mathrm{GL}(p+q,\mathbb{C})\, , \, hI_{p,q}h^* = I_{p,q}\}$, where $I_{p,q}$ is the diagonal matrix

$$I_{p,q} = \begin{pmatrix} \mathrm{Id}_p & \\ & -\mathrm{Id}_q \end{pmatrix}$$

and h^* denotes the adjoint matrix. Recall that G can be seen as the real points of its complexification $\mathbf{G} = \mathrm{PSL}(n+1,\mathbb{C})$ once we have suitably fixed a real structure on it, that is an antilinear involution $\mathbf{G} \to \mathbf{G}$. A similar thing holds for H and its complexification $\mathbf{H} = \mathrm{SL}(p+q,\mathbb{C})$. Recall that a *parabolic subgroup* of a connected real algebraic group $\mathbf{G}$ is a closed subgroup $\mathbf{P} < \mathbf{G}$ that contains a maximal connected solvable subgroup [Bor91]. In particular, minimal parabolic subgroups coincide with maximal solvable subgroups, and they play an important role in the context of semisimple groups (see the work of Furstenberg about *B-subgroups* and Poisson boundaries [Fur63] or Section 2.4.2).

We consider a minimal parabolic subgroup $\mathbf{P} < \mathbf{G}$ and a maximal parabolic subgroup $\mathbf{Q} < \mathbf{H}$ for which $\partial\mathbb{H}^n_{\mathbb{C}} = (\mathbf{G}/\mathbf{P})(\mathbb{R})$ and $(\mathbf{H}/\mathbf{Q})(\mathbb{R})$ is identified with the Shilov boundary $\mathcal{S}_{p,q}$ of $\mathrm{SU}(p,q)$ (see Definition 2.3.9). We say that a map between $\partial\mathbb{H}^n_{\mathbb{C}}$ and $\mathcal{S}_{p,q}$ is *rational* if it is the restriction of a rational map between $\mathbf{G}/\mathbf{P}$ and $\mathbf{H}/\mathbf{Q}$. Here a rational map between two algebraic varieties $\mathbf{U}$ and $\mathbf{V}$ is a function that is defined on charts by regular maps. More precisely, it is an equivalence class of pairs (U, f_U) where $U \subset \mathbf{U}$ is a non-empty Zariski open set, $f_U : U \to \mathbf{V}$ is a regular map and two such pairs (U,f) and (U',f') are identified if $U \cap U' \neq \emptyset$ and $f_{|U\cap U'} \equiv f'_{|U\cap U'}$. This enables us to speak about the set $\mathcal{Q} := \mathrm{Rat}(\partial\mathbb{H}^n_{\mathbb{C}}, \mathcal{S}_{p,q})$ of rational maps between $\partial\mathbb{H}^n_{\mathbb{C}}$ and $\mathcal{S}_{p,q}$. It is possible to define a joint action of G and H as follows

$$((g,h)\cdot f))(\xi) := h \cdot f(g^{-1}\xi) \ ,$$

for each $g \in G$, $h \in H$ and $f \in \mathcal{Q}$. Following [Zim84, Proposition 3.3.2] the actions of G, H and $G \times H$ on $\mathcal{Q}$ are all smooth.

We now move to a refinement of the notion of ergodicity, namely the one of *relative metric ergodicity*. The latter will be a necessary tool in Section 2.4.2 to introduce boundaries for locally compact second countable groups. Notice that this stronger version of ergodicity represents for us a mere tool to define boundaries and to give the results of Chapter 3 in whole generality. However, in the study of numerical invariants, we work with the Poisson boundaries (see Remark 2.4.12).

For this reason, we just list here the necessary definitions without comments or examples and we refer the reader to [BF14b] for further details. We start with the following

Definition 2.1.9. A Lebesgue G-space (X, μ_X) is *metrically ergodic* if for any isometric action $G \to \mathrm{Isom}(M, d)$ on a separable metric space (M, d), any G-equivariant measurable map $X \to M$ is essentially constant.

We note that a metrically ergodic action is actually ergodic, taking as M the space $\{0, 1\}$ with the trivial G-action. For our goal, we need a further refinement of metric ergodicity. Before doing that, we give the definition of fiber-wise isometric action.

Definition 2.1.10. A *metric* on a Borel function $p : M \to T$ between standard Borel probability spaces is a function $d : M \times_T M \to [0, \infty)$ whose restriction $d_{|p^{-1}(t)}$ on each p-fiber is a separable metric.

Given a metric on $p : M \to T$, an action of G on M is *fiber-wise isometric* if there exists a p-compatible G-action on T such that, for any $t \in T$, $x, y \in p^{-1}(t)$ and $g \in G$ we have

$$d_{|p^{-1}(gt)}(gx, gy) = d_{|p^{-1}(t)}(x, y).$$

The notion of fiberwise isometric action allows us to introduce the following

Definition 2.1.11. A map $q : X \to Y$ between Lebesgue G-spaces is *relatively metrically ergodic* if for any fiber-wise isometric G-action on $p : M \to T$ and measurable G-equivariant maps $f : X \to M$ and $g : Y \to T$ there exists a measurable G-equivariant map $\psi : Y \to M$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"] & M \arrow[d, "p"] \\ Y \arrow[r, "g"] \arrow[ur, dotted, "\psi"] & T. \end{tikzcd}$$

It is worth noticing that relative metric ergodicity boils down to metric ergodicity if we consider the trivial projection $q : X \to \{*\}$ on a point.

We devote the last part of this section to *amenability*. We notice that there are plenty of equivalent definitions for such a class of actions. For instance, one can rely upon through the notion of *mean* (see [SS21c, Definition 2.11]) or on a the fixed point property. The last approach is the one adopted by Zimmer [Zim78, Zim80, Zim84] and it turns out to be a better point of view for our purposes. We refer the reader to the book of Zimmer [Zim84, Chapter 4] for a detailed description of the theory.

Let G be a locally compact second countable group acting continuously on a compact metrizable space X. We denote by $\mathcal{M}^1(X) \subset C(X)^*$ the compact convex set of probability measures on X. Here $C(X)^*$ is endowed with the weak-* topology, namely the topology whose basis is given by the sets

$$U_{\varepsilon,f_1,\ldots,f_n}(\Psi) \coloneqq \{\Phi \in C(X)^* \mid |\Psi(f_i) - \Phi(f_i)| \leq \varepsilon\,,\ \forall\, 1,\ldots,n\}$$

where $\varepsilon > 0$, $n \in \mathbb{N}$, $f_i \in C(X)$ and $\Psi \in C(X)^*$. Then G acts continuously on $C(X)$ as follows

$$(gf)(x) \coloneqq f(g^{-1}x)$$

for every $g \in G$ and $f \in C(X)$ and this induces a natural action on $\mathcal{M}^1(X)$ defined as

$$g\mu(f) \coloneqq \mu(gf)$$

for every $\mu \in \mathcal{M}^1(X)$, $f \in C(X)$ and $g \in G$.

This is the setting to define amenable groups.

Definition 2.1.12. A locally compact second countable group G is *amenable* if for every continuous action on a compact metrizable space X, the induced G-action on $\mathcal{M}^1(X)$ admits a fixed point.

Examples of amenable groups are

- abelian groups [Zim84, Proposition 4.1.2],
- compact groups [Zim84, Proposition 4.1.5],
- solvable groups [Zim84, Corollary 4.1.7],
- compact extensions of solvable subgroups [Fur63, Theorem 1.7] (e.g. minimal parabolic subgroups).

Furthermore, in case of connected semisimple Lie groups one can shows that amenability is actually equivalent to be a compact extension of a solvable group [Fur63, Theorem 1.7].

An equivalent formulation of amenability of groups can be given in terms of the following class of actions on more general convex sets.

Definition 2.1.13. An *affine* G-action on a space A is the datum of a compact convex set $A \subset E_1^*$, where E_1^* is the unit ball in the dual of a separable Banach space E endowed with the weak-* topology, together with a G-action induced by the dual representation of some continuous isometric representation of G on E.

Such a class of actions gives rise to the following useful characterizations of amenable groups.

Proposition 2.1.14 ([Zim84, Proposition 4.1.4]). *A locally compact second countable group G is amenable if and only if every affine G-action has a fixed point.*

The previous characterization shows how actions of amenable groups inherits certain properties from the group. Conversely, one can find actions by non-amenable groups that satisfy a fixed point property. Our next step is to formalize this phenomena or, in other words, to generalize amenability of groups to actions. We start by defining a class of spaces that extends the one introduced in Definition 2.1.13. Consider a locally compact second countable group G, a measure G-space (S, μ) and a separable Banach space E. We suppose the existence of a measurable function $\sigma : G \times S \to \mathrm{Isom}(E)$ such that

$$\sigma(gh, s) = \sigma(g, hs)\sigma(h, s) \tag{2.1}$$

for every $g, h \in G$ and for almost every $s \in S$. We point out that the condition defined by Equation (2.1) actually corresponds to the cocycle condition defined by Equation (2.13) of Section 2.4.1. Nevertheless, we prefer not to introduce here the exact notion of measurable cocycle and to remind the reader to Section 2.4.1 where the theory is described more precisely.

The function σ induces a natural G-action on the space $\mathrm{L}^1(S; E)$ of μ-integrable measurable E-valued maps defined by

$$(g\varphi)(s) \coloneqq \sigma(g, s)\varphi(gs)$$

for every $\varphi \in \mathrm{L}^1(S; E)$, every $g \in G$ and almost every $s \in S$. The last action in turn induces, by duality, a G-action on $\mathrm{L}^\infty(S; E^*)$ as follows

$$(gf)(s) \coloneqq (\sigma(g, s)^{-1})^* f(gs)\,, \tag{2.2}$$

where $\mathrm{L}^\infty(S; E^*)$ is endowed with the weak-* topology and $(\sigma(g, s)^{-1})^*$ denotes the dual action of $\sigma(g, s)^{-1}$. We assume the existence of a family $\{A_s\}_{s \in S}$ of Borel subsets of the unit ball E^*_1 in E^* such that $\{(s, A_s)\} \subset S \times E^*_1$ is a Borel subset and we assume that $(\sigma(g, s)^{-1})^* A_s = A_{g^{-1}s}$ for every $g \in G$ and almost every $s \in S$ (such a family of Borel subsets is called a *G-equivariant family*). In this setting, we define the set

$$\mathcal{C}(S, \{A_s\}) \coloneqq \{f : S \to E^*_1 \,|\, f(s) \in A_s \text{ for almost every } s \in S\}\,,$$

that is a compact closed convex subset of the unit ball of $\mathrm{L}^\infty(S; E^*_1)$.

Definition 2.1.15. An affine G-space of the form $\mathcal{C}(S, \{A_s\})$ endowed with the action defined by Equation (2.2) is an *affine G-space over S*.

Affine G-spaces over a measure space S are the generalizations of affine spaces and allow us to give the following

Definition 2.1.16. A G-action on a measure space S is *amenable* if every affine G-space over S has a fixed point.

As anticipated, if G is amenable then every G-action is amenable. Conversely, the existence of an amenable G-action on some space X that preserves a finite measure ensures the amenability of G [Zim84, Proposition 4.3.3]. In particular, a group G is amenable if and only if the trivial G-action on a point is amenable. Amenable actions do not need to involve amenable groups. As a first example, the following results characterizes amenable actions of locally compact second countable groups on homogeneous spaces.

Proposition 2.1.17 ([Zim84, Proposition 4.3.2])**.** *Let $H < G$ be a closed subgroup of the locally compact second countable group. Then the G-action on G/H is amenable if and only if H is an amenable group.*

The previous characterization is exactly the context in which we exploit amenability. Precisely, if G is a connected semisimple Lie group of non-compact type, we can consider a minimal parabolic subgroup P. Since P is a compact extension of a solvable group, it is amenable. Hence the Furstenberg–Poisson boundary of G, which can be identified with the quotient G/P (see Remark 2.4.12 and Example 2.4.14), is an amenable G-space.

It is worth mentioning the existence of a different notion of amenable action that generalizes the characterization of amenable groups in terms of *means*. However, since we are not interested in this approach, we refer to [AEG94], where the relation between such definitions is discussed.

We conclude by recalling a property of amenable actions. This represent the crucial link between the theory of amenable actions and the one of boundary maps, that we are going to introduce in Section 2.4.2. In particular, it will be the starting point in the proof of Theorem 2.

Proposition 2.1.18 ([Zim84, Proposition 4.3.9])**.** *Let (S, μ) an amenable G-space and let Y be a compact metric G-space. Up to discarding a null measure subset of S, there exists a G-equivariant measurable map $S \to \mathcal{M}^1(Y)$ into the space of probability measures on Y.*

2.2 Bounded cohomology

This section is devoted to (continuous) bounded cohomology of locally compact second countable groups. In particular, after an introductory part about Banach modules, we give the classical definition using the complex of (continuous) bounded cochains. Then we provide a short but almost self-contained description of the Burger–Monod functorial characterization of continuous bounded cohomology. As a general reference for this section we refer the reader to Monod's book [Mon01].

2.2.1 Banach modules

In the sequel G will be a locally compact second countable group. Since we are going to describe a cohomology theory we start with the definition of a class of modules that will be the coefficients of our cohomology groups.

Definition 2.2.1. A *Banach G-module* E is a Banach space endowed with an isometric G-action $\pi : G \to \mathrm{Isom}(E)$. If the action is also continuous, E is called a *continuous Banach G-module.* A *G-morphism* is a continuous G-equivariant linear map $\alpha : E \to F$ between Banach G-modules that commutes with the actions.

We immediately recall some examples of Banach G-modules that are going to be ubiquitous along our dissertation.

Example 2.2.2. If E is a Banach G-module, the *maximal continuous submodule* is

$$\mathcal{C}E \coloneqq \{v \in E \,|\, g \mapsto gv \text{ is continuous } \} \ ,$$

and it coincides with the maximal submodule of E on which the G-action is continuous. As proved in [Mon01, Lemma 1.2.6], any G-morphism $E \to F$ of Banach space restricts to a G-morphism $\mathcal{C}E \to \mathcal{C}F$.

Example 2.2.3. For any topological space X on which G acts by homeomorphisms and for any Banach G-modules E, we consider the Banach space of *continuous bounded E-valued functions*

$$\mathrm{C_b}(X;E)$$

endowed with the supremum norm

$$||f||_\infty = \sup_{x\in X} ||f(x)||_E .$$

If $\pi : G \to \mathrm{Isom}(E)$ is the isometric G-action on E, we turn $\mathrm{C_b}(G;E)$ into a Banach G-module by defining the following G-action

$$(gf)(x) = \pi(g)f(g^{-1}x) \tag{2.3}$$

called *left regular representation.*

Example 2.2.4. Another example of Banach G-module is the Banach space of *essentially bounded measurable E-valued functions* on X

$$\mathrm{L}^\infty(X;E)$$

endowed with the supremum norm, where (X,μ_X) is any measure space and E is a Banach G-module. Here we identify measurable functions that coincide μ_X-almost everywhere. The space $\mathrm{L}^\infty(X;E)$ is a Banach G-module if endowed with the left regular representation defined by Equation (2.3).

We point out that the notion of continuous bounded cohomology for locally compact second countable groups could be given here, since it only requires the concept of Banach G-modules. Nevertheless, the powerful machinery developed by Burger and Monod that we are going to describe in Section 2.2.3 needs some additional definitions. We first consider a particular class of Banach G-modules.

Definition 2.2.5. A Banach G-module E is a *coefficient module* if it is the dual of some separable continuous Banach G-module F and, denoting by $\pi^\flat : G \to \mathrm{Isom}(F)$ the isometric G-action on F, the action $\pi : G \to \mathrm{Isom}(E)$ satisfies

$$\pi(g)(\phi)(v) = \phi(\pi^\flat(g)^{-1}(v)) \tag{2.4}$$

for every $g \in G$, $\phi \in E$ and $v \in F$.

Since we are interested in the space of measurable bounded functions on some measure space, we show how the Banach module introduced in Example 2.2.4 satisfies the conditions of Definition 2.2.5.

Example 2.2.6. Let G be a locally compact second countable group and let (X,μ_X) be a measure G-space. We consider the Banach G-module

$$\mathrm{L}^\infty(X) = \mathrm{L}^\infty(X;\mathbb{R})$$

of bounded measurable functions on X and the Banach space

$$\mathrm{L}^1(X)$$

of μ_X-integrable functions endowed with the G-action defined as

$$\pi^\flat : G \to \mathrm{L}^1(X)\,, \quad (\pi^\flat(g)(f))(x) := f(gx)\,.$$

It is a classical fact the existence of an isometric isomorphism between $\mathrm{L}^\infty(X)^*$ and $\mathrm{L}^1(X)$, and one can also check that the isometric G-actions satisfies the equality in Equation (2.4). This implies that the space $\mathrm{L}^\infty(X)$ is a coefficient G-module (see [Mon01] for more details).

A second class of objects that we need are described below.

Definition 2.2.7. A *regular G-space* is a standard Borel space S endowed with a measure class preserving G-action such that, if μ is a probability measure that represents the invariant class, then

$$G \to \mathrm{Isom}(\mathrm{L}^1(S)), \quad g \mapsto \lambda^\flat(g)\varphi$$

is continuous where $\lambda^\flat(g)\varphi \coloneqq \varphi(g^{-1}s)\frac{dg^{-1}\mu}{d\mu}(s)$.

Although the Definition 2.2.7 may appears quite mysterious, we recall two well-known examples of regular Banach G-modules.

- Any locally compact second countable group G is a regular G-space itself if endowed with the class of its Haar measure. More generally, for any closed subgroup $H < G$, the homogeneous space G/H admits a G-quasi-invariant measure; hence G preserves the class of this measure and this turns G/H into a regular space. As a consequence, the Furstenberg–Poisson boundary of a semisimple Lie group, that can be identified with the quotient of the group by any minimal parabolic subgroup, is a regular space (see also Example 2.4.14).

- For any probability measure μ on a locally compact second countable group G, there exist probability G-spaces called *Poisson boundaries* that are regular G-spaces for the class of corresponding stationary measure (see the work of Furstenberg [Fur63, Fur67, Fur73] or Section 2.4.2). In case of semisimple Lie groups the wide family of Poisson boundaries includes the Furstenberg–Poisson ones.

Regular spaces give us the chance to present a class of Banach G-modules that will be crucial in the functorial characterization.

Example 2.2.8. Let S be a regular G-space and let E be a coefficient G-module with the isometric representation $\pi : G \to \mathrm{Isom}(E)$. We consider the space

$$\mathcal{B}^\infty_{\mathrm{w}^*}(S; E)$$

of *bounded weak-* measurable E-valued functions* on S, which becomes a Banach G-module if it is equipped with the supremum norm and with the left regular representation defined by Equation (2.3).

As we will see, instead of bounded measurable functions, sometimes it is convenient to work with the space

$$\mathrm{L}^\infty_{\mathrm{w}^*}(S; E)$$

of *essentially bounded weak-* measurable E-valued functions on S*, namely the quotient of $\mathcal{B}^\infty_{\mathrm{w}^*}(S;E)$ by the equivalence relation that identifies two measurable functions that coincide almost everywhere. It is also a Banach G-module if considered with the essential supremum norm and with the same G-action as on $\mathcal{B}^\infty_{\mathrm{w}^*}(S;E)$ and, as we will see, its role is crucial in the Burger–Monod theory.

Finally, the subspace of *alternating* functions on the product $S^{\bullet+1}$ is denoted by

$$\mathcal{B}^\infty_{\mathrm{w}^*,\mathrm{alt}}(S^{\bullet+1};E) \quad \text{or} \quad \mathrm{L}^\infty_{\mathrm{w}^*,\mathrm{alt}}(S^{\bullet+1};E)$$

and consists of functions $f \in \mathcal{B}^\infty_{\mathrm{w}^*}(S^{\bullet+1};E)$ (respectively $\mathrm{L}^\infty_{\mathrm{w}^*}(S^{\bullet+1};E)$) such that

$$f(s_0,\dots,s_\bullet) = \mathrm{sgn}(\sigma) f(s_{\sigma(0)},\dots,s_{\sigma(\bullet)})$$

for every $s_0,\dots,s_\bullet \in S^{\bullet+1}$ and $\sigma \in \mathfrak{S}^{\bullet+1}$.

Remark 2.2.9. We notice that an element in $\mathrm{L}^\infty_{\mathrm{w}^*}(S;E)$ is an equivalence class of functions that coincide almost everywhere. However, to simplify the exposition, we slightly abuse the notation and we write $f \in \mathrm{L}^\infty_{\mathrm{w}^*}(S;E)$ to consider a representative of some class.

We conclude by introducing an extension property for Banach G-modules, that is the last technical ingredient for the Burger–Monod approach to bounded cohomology.

Definition 2.2.10. A G-morphism $\eta : A \to B$ is said to be *admissible* if there exists a morphism $\sigma : B \to A$ with $||\sigma||_\infty \leq 1$ and such that $\eta\sigma\eta = \eta$.

In particular, if η is injective, admissibility is equivalent to the existence of a left inverse for η of norm one. Injective admissible morphisms are the object of the next

Definition 2.2.11. A Banach G-module E is *relatively injective* if for any injective admissible G-morphism between continuous G-modules $i : A \to B$ and for any G-morphism $\alpha : A \to E$ there exists a G-morphism $\beta : B \to E$ with $||\beta||_\infty \leq 1$ and such that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "\alpha"'] & & B \arrow[dl, dotted, "\beta"] \\ & E. & \end{tikzcd}$$

The extension property that holds for a relatively injective G-module E is strictly related to the G-module $\mathcal{C}E$, as established by the following

Lemma 2.2.12 ([Mon01, Lemma 4.1.5]). *A Banach G-module E is relatively injective if and only if $\mathcal{C}E$ is.*

The relevance of relatively injective G-modules in bounded cohomology, which will become completely clear in Section 2.2.3, is also explained by the next result.

Theorem 2.2.13 ([Mon01, Theorem 4.5.2]). *Let G be a locally compact group and let E be a Banach G-module. Then the space $\mathrm{C_b}(G;E)$ is relatively injective.*

2.2.2 The standard definition of bounded cohomology

We now introduce the theory of continuous and continuous bounded cohomology of locally compact second countable groups. We will not be exhaustive and we recall only the basic definitions and tools that we need. Standard references are Monod's book [Mon01] and Burger–Monod [BM02]. We notice that most of the notions that we are going to introduce below can be given in a more general context. Nevertheless we prefer to restrict to the specific setting needed for our purposes.

In the sequel G will be a locally compact second countable group and E will be a Banach G-module. We denote the set of *E-valued continuous functions* on G as

$$\mathrm{C}^\bullet_\mathrm{c}(G;E) \coloneqq \{f : G^{\bullet+1} \to E \,|\, f \text{ is continuous }\}$$

and we define the *standard homogeneous coboundary operator* by

$$\delta^\bullet : \mathrm{C}^\bullet_\mathrm{c}(G;E) \to \mathrm{C}^{\bullet+1}_\mathrm{c}(G;E)\,,$$

$$(\delta^\bullet f)(g_0,\dots,g_{\bullet+1}) \coloneqq \sum_{i=0}^{\bullet+1}(-1)^i f(g_0,\dots,\widehat{g_i},\dots,g_{\bullet+1})\,,$$

that clearly preserves continuity. Similarly, we consider the subset of *E-valued continuous bounded function* on G

$$\mathrm{C}^\bullet_\mathrm{cb}(G;E) \coloneqq \{f \in \mathrm{C}^\bullet_\mathrm{c}(G;E) \,|\, ||f||_\infty < +\infty\}$$

where $||\cdot||_\infty$ denotes the norm

$$||f||_\infty \coloneqq \sup_{g_0,\dots,g_\bullet} ||f(g_0,\dots,g_\bullet)||_E. \tag{2.5}$$

The G-action on $\mathrm{C}^\bullet_\mathrm{c}(G;E)$ corresponds to the one introduced in Equation (2.3) by taking the diagonal G-action on $G^{\bullet+1}$. The set of *G-invariant* E-valued continuous (bounded) functions on G is

$$\mathrm{C}^\bullet_{\mathrm{c(b)}}(G;E)^G \coloneqq \{f \in \mathrm{C}^\bullet_{\mathrm{c(b)}}(G;E) \,|\, gf = f\,,\, \forall g \in G\}\,.$$

Since the coboundary operator $\delta^\bullet$ preserves both boundedness and G-invariance, the collection

$$(\mathrm{C}^\bullet_{\mathrm{c(b)}}(G;E)^G, \delta^\bullet)$$

forms a cochain complex.

Definition 2.2.14. The *continuous (bounded) cohomology* of a locally compact group G with coefficient into a Banach G-module E is the cohomology of the complex $(\mathrm{C}^\bullet_{\mathrm{c}}(G;E)^G, \delta^\bullet)$ (respectively $(\mathrm{C}^\bullet_{\mathrm{cb}}(G;E)^G, \delta^\bullet)$) and it is denoted by $\mathrm{H}^\bullet_{\mathrm{c}}(G;E)$ (respectively $\mathrm{H}^\bullet_{\mathrm{cb}}(G;E)$).

We notice that the supremum norm on each $\mathrm{C}^\bullet_{\mathrm{cb}}(G;E)$ defined by Equation (2.5) induces a seminorm on $\mathrm{H}^\bullet_{\mathrm{cb}}(G;E)$ given by

$$||\alpha|| := \inf\{||c||_\infty \,|\, [c] = \alpha\} .$$

Two seminormed spaces are said to be *isometric* if there exists a linear isomorphism between them that preserves the seminormed structures.

Before passing to the functorial approach, we recall two maps that play an important role in the whole theory of bounded cohomology. First, the inclusion $\mathrm{C}^\bullet_{\mathrm{cb}}(G;E) \hookrightarrow \mathrm{C}^\bullet_{\mathrm{c}}(G;E)$ induces a map at the cohomological level

$$\mathrm{comp}^\bullet : \mathrm{H}^\bullet_{\mathrm{cb}}(G;E) \to \mathrm{H}^\bullet_{\mathrm{c}}(G;E) \tag{2.6}$$

that is called *comparison map*.

Similarly, any G-morphism $\alpha : E \to F$ between Banach G-modules naturally induces a cochain map $\mathrm{C}^\bullet_{\mathrm{c(b)}}(G;E) \to \mathrm{C}^\bullet_{\mathrm{c(b)}}(G;F)$ and hence a map between cohomology groups

$$\mathrm{H}^\bullet_{\mathrm{c(b)}}(G;E) \to \mathrm{H}^\bullet_{\mathrm{c(b)}}(G;F). \tag{2.7}$$

Both the comparison map and the map induced by a G-morphism have aroused the interest of many mathematicians so far. For instance, Dupont [Dup79] investigated the surjectivity of the comparison map in degree two and Hartnick and Ott [HO11] generalized the result in the Hermitian case. On the other hand, several useful results involving change of coefficients are due to Burger and Monod [BM02]. In particular, we will exploit the boundedness of the Kähler class in the context of semisimple Lie groups of Hermitian type, that follows from the surjectivity of the comparison map in degree two [Dup79] (see Section 2.3), and the isomorphism $\mathrm{H}^2_{\mathrm{cb}}(G;\mathbb{R}) \cong \mathrm{H}^2_{\mathrm{cb}}(G;\mathrm{L}^\infty(X))$ induced by the inclusion $G \hookrightarrow \mathrm{L}^\infty(X)$, if G is a product acting irreducibly on a measure space (X, μ_X) (see [BM02, Corollary 15] and Chapter 6).

2.2.3 The functorial characterization

Despite its crystalline definition, continuous bounded cohomology turns out to be almost inaccessible for direct computations. A powerful solution to this problem has been provided by Burger and Monod [BM02], who obtained a nice functorial characterization of continuous bounded cohomology. In the sequel we recall a list of definitions and results that we will use for our purposes. For an exhaustive description of the theory we refer to Monod's book [Mon01] and to Burger–Iozzi appendix [BI02] of [BM02].

A *complex of Banach G-modules* is a cochain complex $(E^\bullet, d^\bullet)$ where $d^\bullet$ are G-morphisms. In this context, we can naturally define the *n-th cohomology* of a G-complex $(E^\bullet, d^\bullet)$ as the space

$$\mathrm{H}^n(E^\bullet) \coloneqq \mathrm{Ker}(d^\bullet)/\mathrm{Im}(d^{\bullet-1})$$

endowed with the semi-norm induced by the norm of $E^\bullet$.

A *G-morphism of complexes* is a sequence of G-morphisms $\alpha^\bullet : E^\bullet \to F^\bullet$ intertwining the coboundary operators $d_E^\bullet$ and $d_F^\bullet$. A G-morphism $\alpha^\bullet : E^\bullet \to F^\bullet$ naturally induces continuous linear maps at the cohomological level

$$\mathrm{H}^n(\alpha^\bullet) : \mathrm{H}^n(E^\bullet) \to \mathrm{H}^n(F^\bullet)$$

in any degree.

Given G-morphisms $\alpha^\bullet, \beta^\bullet : E^\bullet \to F^\bullet$, a *$G$-homotopy* from $\alpha^\bullet$ to $\beta^\bullet$ is a family of G-morphisms $h^\bullet : E^\bullet \to F^{\bullet-1}$ such that

$$h^{\bullet+1} d_E^\bullet - d_F^{\bullet-1} h^\bullet = \alpha^\bullet - \beta^\bullet ,$$

namely such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & E^{\bullet-1} \arrow[r, "d_E^{\bullet-1}"] \arrow[d, "\alpha^{\bullet-1}"] & E^\bullet \arrow[r, "d_E^\bullet"] \arrow[d, "\alpha^\bullet"] \arrow[dl, "h^\bullet"] & E^{\bullet+1} \arrow[r] \arrow[d, "\alpha^{\bullet+1}"] \arrow[dl, "h^{\bullet+1}"] & \cdots \\
\cdots \arrow[r] & F^{\bullet-1} \arrow[r, "d_F^{\bullet-1}"] & F^\bullet \arrow[r, "d_F^\bullet"] & F^{\bullet+1} \arrow[r] & \cdots
\end{tikzcd}$$

A *contracting homotopy* of a G-complex $(E^\bullet, d^\bullet)$ is a homotopy $h^\bullet$ between the identity and the null map with $||h^n|| \le 1$ for every $n \in \mathbb{Z}$.
The complexes we are interested in are called strong complexes. A G-complex $(E^\bullet, d^\bullet)$ is *strong* if the maximal continuous subcomplex $(\mathcal{C}E^\bullet, d^\bullet_|)$ admits a contracting homotopy.

A *resolution* of a Banach G-module E is an exact complex with $E^0 = E$ and $E^n = 0$ for $n \leq -1$. Finally, a resolution is *strong* if it is realized by a strong complex.

Example 2.2.15. Given a locally compact second countable group and a Banach G-module E, the complex

$$\cdots \longrightarrow 0 \longrightarrow E \xrightarrow{\epsilon} \mathrm{C}_{\mathrm{b}}^0(G;E) \xrightarrow{\delta^0} \mathrm{C}_{\mathrm{b}}^1(G;E) \xrightarrow{\delta^1} \cdots$$

where $\epsilon : E \to \mathrm{C}_{\mathrm{b}}^0(G;E)$ is the inclusion of coefficients, is a strong resolution of E by relatively injective modules (see Definition 2.2.11).

Strong resolutions by relatively injective modules give a characterization of the bounded cohomology of a group G with coefficients into a Banach G-module E in terms of the G-invariants. More precisely, we have the following

Theorem 2.2.16 ([Mon01, Theorem 7.2.1])**.** *Let G be a locally compact second countable group and let E be a Banach G-module. Then for every strong resolution $(E^\bullet, d^\bullet)$ by relatively injective Banach G-modules, the cohomology $\mathrm{H}^n((E^\bullet)^G)$ is canonically isomorphic, as a topological vector space, to the continuous bounded cohomology $\mathrm{H}_{\mathrm{cb}}^n(G;E)$ of G for every $n \geq 0$.*

A priori, the isomorphism provided by Theorem 2.2.16 is not isometric. The natural step that immediately follows Theorem 2.2.16 is the search for strong resolutions by relatively injective modules that realize the isometry at the cohomological level. This is the content of the next result.

Theorem 2.2.17 ([Mon01, Theorem 7.5.3])**.** *Let G be a locally compact second countable group, let E be a coefficient G-module and consider an amenable regular G-space S. If $\epsilon : E \to \mathrm{L}_{\mathrm{w}^*}^\infty(S;E)$ denotes the inclusion of coefficients, the complex*

$$\cdots \longrightarrow 0 \longrightarrow E \xrightarrow{\epsilon} \mathrm{L}_{\mathrm{w}^*}^\infty(S;E) \xrightarrow{\delta^0} \mathrm{L}_{\mathrm{w}^*}^\infty(S^2;E) \xrightarrow{\delta^1} \cdots$$

is a strong resolution by relatively injective modules for E. Moreover, the isomorphism

$$\mathrm{H}^k(\mathrm{L}_{\mathrm{w}^*}^\infty((S)^{\bullet+1};E)^G) \cong \mathrm{H}_{\mathrm{cb}}^k(G;E) \tag{2.8}$$

is isometric for every $k \geq 0$.

The same holds for the complex $(\mathrm{L}_{\mathrm{w}^*,\mathrm{alt}}^\infty((S)^{\bullet+1};E)), \delta^\bullet)$ of alternating essentially bounded measurable functions on S.

Sometimes, instead of essentially bounded functions, it is convenient to consider the complex of bounded weak-* measurable functions $(\mathcal{B}^\infty_{\mathrm{w}^*}((S)^{\bullet+1};E),\delta^\bullet))$ on a (not necessarily amenable) regular G-space, that actually is a strong resolution for E [BI02, Proposition 2.1]. The drawback is the loss of relative injectivity and hence the fact that the resolution

$$\cdots \longrightarrow 0 \longrightarrow E^G \xrightarrow{\ \epsilon\ } \mathcal{B}^\infty_{\mathrm{w}^*}(S;E)^G \xrightarrow{\ \delta^1\ } \mathcal{B}^\infty_{\mathrm{w}^*}(S^2;E)^G \xrightarrow{\ \delta^2\ } \cdots$$

does not compute the bounded cohomology of G. Nevertheless, thanks to the work done by Burger–Iozzi [BI02] which is based on [BM02, Proposition 1.5.2], we know the existence of canonical maps

$$\mathfrak{c}^n : \mathrm{H}^n(\mathcal{B}^\infty_{\mathrm{w}^*}((S)^{\bullet+1};E)^G) \to \mathrm{H}^n(\mathrm{L}^\infty_{\mathrm{w}^*}((S)^{\bullet+1};E)^G) \tag{2.9}$$

for every $n \geq 0$. The same holds for the complex $(\mathcal{B}^\infty_{\mathrm{w}^*,\mathrm{alt}}((S)^{\bullet+1};E),\delta^\bullet)$ of alternating measurable bounded functions on S, as observed in [BI02, Remark 2.8].

2.3 Hermitian symmetric spaces

The goal of this section is to recall the basics about Hermitian symmetric spaces and then to introduce some related notions. In particular, we will first introduce the definition, recalling both the classification in tube-type and non-tube-type spaces as well as their bounded domain realization. The latter will allow us to present the Shilov boundary and its basic properties. Exploiting the natural Kähler structure on Hermitian symmetric spaces we will define the Bergman cocycle. This will correspond, under the canonical map of Equation (2.9), to an element in the second bounded cohomology of the isometry group of the Hermitian symmetric space called bounded Kähler class. Finally, we will introduce the Hermitian triple product and we will show its relation with the Bergman cocycle.

We start with the definition of Hermitian symmetric space.

Definition 2.3.1. A symmetric space $\mathcal{X}$ with $G = \mathrm{Isom}(\mathcal{X})^\circ$ is of *Hermitian type* if it admits a G-invariant complex structure. Given a semisimple real algebraic group $\mathbf{G}$, we say that $G = \mathbf{G}(\mathbb{R})$ is of *Hermitian type* if its associated symmetric space is. The *rank* of symmetric space of Hermitian type is the maximum dimension of a totally geodesic embedded Euclidean space.

From now on, G will be a group of Hermitian type and $\mathcal{X}$ will be the associated symmetric space. It is well-known that any Hermitian symmetric space $\mathcal{X}$ admits a *bounded domain realization* [FKK$^+$00, Theorem III. 2.6], that is a biholomorphism between $\mathcal{X}$ and a bounded domain $\mathcal{D}_{\mathcal{X}} \subset \mathbb{C}^n$ for some n, on which G acts via biholomorphisms. Furthermore, one can distinguish Hermitian symmetric spaces among *tube-type* and *not tube-type*. More precisely, we have the following

Definition 2.3.2. A Hermitian symmetric space is of *tube-type* if it is biholomorphic to a domain of the form

$$V + i\Omega$$

where V is a real vector space and $\Omega \subset V$ is a proper open convex cone.

To clarify the meaning of the above classification, we describe three examples of Hermitian symmetric spaces. We warn the reader that our last example is actually the generalization of the second one, as well as the first one is a special case of the second one. However, because of their different roles in our results, we prefer to distinguish them. For a complete description of these examples refer to Pozzetti's thesis [Poz14] and to [DLP21, SS21a].

Example 2.3.3. On the complex space $\mathbb{C}^{n+1}$ we consider the Hermitian form $Q_{1,n}$ of signature $(1,n)$ defined as

$$Q_{1,n}(x) \coloneqq x_0\overline{x_0} - \sum_{i=1}^{n} x_i\overline{x_i}$$

for every $x = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$. Since the positivity of $Q_{1,n}$ is invariant under scalar multiplication, it makes sense to consider the set of Q-positive lines, that actually coincides with the *complex hyperbolic space*

$$\mathbb{H}^n_{\mathbb{C}} \coloneqq \{x = [x_0, \ldots, x_n] \in \mathbb{P}(\mathbb{C}^{n+1}) \,|\, Q_{1,n}(x) > 0\} .$$

Since this is contained in the affine chart $U_0 = \{[1, x_1, \ldots, x_n] \in \mathbb{P}(\mathbb{C}^{n+1})\}$, there is a natural identification with the unit ball

$$\mathcal{D}_{\mathcal{X}} \coloneqq \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, \sum_{i=1}^{n} |z_i|^2 < 1 \right\} \subset \mathbb{C}^n .$$

Here the notation $\mathcal{D}_{\mathcal{X}}$ refers to the bounded domain realization of $\mathcal{X}$. Notice that $\mathbb{H}^n_{\mathbb{C}}$ can be identified with the rank-one Hermitian symmetric space $\mathcal{X}$ associated to the group $\mathrm{SU}(1,n)$ of matrices in $\mathrm{SL}(n+1,\mathbb{C})$ that preserve the Hermitian form $Q_{1,n}$, that is

$$\mathrm{SU}(1,n) \coloneqq \{h \in \mathrm{SL}(n+1,\mathbb{C}) \,|\, Q_{1,n}(hx,hy) = Q_{1,n}(x,y) \,,\ \forall x,y \in \mathbb{C}^{n+1}\} .$$

Moreover, there exists a biholomorphism between $\mathbb{H}^n_{\mathbb{C}}$ and the *Siegel domain* [Gol99, Section 4.1.1]

$$\mathbf{h}^n \coloneqq \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, 2\mathrm{Re}(z_1) - \sum_{i=2}^{n} z_i \overline{z_i} > 0\}.$$

This shows that $\mathbb{H}^n_{\mathbb{C}}$ is of tube-type if and only if $n = 1$, since in this case $\mathbf{h}^1 = \{w \in \mathbb{C} \,|\, \mathrm{Re} w > 0\}$ is the upper half-plane and it satisfies the condition of Definition 2.3.2.

For further details on the complex hyperbolic space we refer to Goldman's book [Gol99].

Example 2.3.4. In view of Example 2.3.3, we fix two positive integers p and q and we define the Hermitian form of signature (p, q) as

$$Q_{p,q}(x) \coloneqq \sum_{i=1}^{p} x_i \overline{x_i} - \sum_{i=p+1}^{q} x_i \overline{x_i}$$

for every $x = (x_1, \ldots, x_{p+q}) \in \mathbb{C}^{p+q}$. The set of matrices in $\mathrm{SL}(p+q, \mathbb{C})$ preserving $Q_{p,q}$, namely the subgroup

$$\mathrm{SU}(p, q) \coloneqq \{h \in \mathrm{SL}(p+q, \mathbb{C}) \,|\, Q_{p,q}(hx, hy) = Q_{p,q}(x, y)\,,\ \forall\, x, y \in \mathbb{C}^{p+q}\}\,,$$

is a group of Hermitian type. In particular, denoting by $\mathrm{Gr}_p(\mathbb{C}^{p+q})$ the Grassmannian of p-dimensional subspaces of $\mathbb{C}^{p+q}$, one can identify the associated symmetric space with the set

$$\mathcal{X}(p, q) \coloneqq \{V \in \mathrm{Gr}_p(\mathbb{C}^{p+q}) \,|\, Q_{|_V} > 0\}.$$

Indeed, the latter is a homogeneous $\mathrm{SU}(p, q)$-space and the stabilizer of the point $V_0 = \langle e_0, \ldots, e_p\rangle$ is the subgroup $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$. We notice that $\mathcal{X}(p, q)$ generalizes the complex hyperbolic space introduced in Example 2.3.3, precisely $\mathcal{X}(1, n) = \mathbb{H}^n_{\mathbb{C}}$. Moreover, the analogous of the bounded realization of $\mathbb{H}^n_{\mathbb{C}}$ is the space of matrix

$$X_{p,q} = \{X \in M(p \times q, \mathbb{C}) \,|\, X^*X - \mathrm{Id} < 0\} \subset \mathbb{C}^{pq}\,,$$

where $X^*X - \mathrm{Id} < 0$ means that the Hermitian matrix $\mathrm{Id} - X^*X$ is positive definite (see also [Poz14]). Finally, it is well known that the rank of $\mathcal{X}(p, q)$ is the minimum between p and q, and $\mathcal{X}(p, q)$ is of tube-type if and only if $p = q$. This observations are the coherent extension of those ones discussed in Example 2.3.3.

Example 2.3.5. Pushing further Example 2.3.4, we fix $p \in \mathbb{N}$ and we consider an infinite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$ together with an Hilbert basis $\{e_i\}_{i\in\mathbb{N}}$. We denote by $L(\mathcal{H})$ the set of $\mathbb{C}$-linear bounded operators with respect to the operator norm and by $\mathrm{GL}(\mathcal{H})$ the group of bounded invertible $\mathbb{C}$-linear operators of $\mathcal{H}$ with bounded inverse.

Following the previous examples, we define the Hermitian form $Q_{p,\infty}$ of signature (p,∞) as follows

$$Q_{p,\infty}(x) = \sum_{i=1}^{p} x_i\overline{x}_i - \sum_{i\geq p+1} x_i\overline{x}_i$$

where $x = \sum_{i\geq 1} x_i e_i$ for every $x \in \mathcal{H}$. We denote by $\mathrm{U}(p,\infty)$ the subgroup of $\mathrm{GL}(\mathcal{H})$ of isometries with respect to $Q_{p,\infty}$, namely

$$\mathrm{U}(p,\infty) \coloneqq \{h \in \mathrm{GL}(\mathcal{H}) \,|\, Q_{p,\infty}(hx,hy) = Q_{p,\infty}(x,y)\,,\ \forall\, x,y \in \mathcal{H}\}.$$

If we define the space

$$\mathcal{X}(p,\infty) \coloneqq \left\{V < \mathcal{H} \mid \dim V = p\ ,\ Q_{|_V} > 0\right\},$$

by Witt's theorem the group $\mathrm{U}(p,\infty)$ acts transitively on it (see for instance [Art11, Theorem 3.9]). Moreover, the stabilizer of $V_0 = \langle e_1,\ldots,e_p\rangle$ is the product $\mathrm{U}(p)\times\mathrm{U}(\infty)$, where $\mathrm{U}(m)$ is the orthogonal group of the Hilbert space of dimension m, for any $m \in \mathbb{N}\cup\{\infty\}$. Hence we can identify $\mathcal{X}(p,\infty)$ with the quotient

$$\mathrm{U}(p,\infty)/\mathrm{U}(p)\times\mathrm{U}(\infty)$$

and one can show that it has a structure of simply connected non-positively curved Riemannian symmetric space (see [Duc12]). Finally, the rank of $\mathcal{X}(p,\infty)$ is p.

Homotheties act trivially on $\mathcal{X}(p,\infty)$, hence the quotient

$$\mathrm{PU}(p,\infty) \coloneqq \mathrm{U}(p,\infty)/\{\lambda\mathrm{Id}\ ,\ |\lambda| = 1\}$$

acts by isometries on $\mathcal{X}(p,\infty)$.

Beyond the structure of Riemannian symmetric space, in [DLP21] the authors describe also a complex structure on $\mathcal{X}(p,\infty)$ that makes it a Hermitian symmetric space. Finally, the space $\mathcal{X}(p,\infty)$ is not of tube-type [DLP21, Lemma 2.11].

As suggested by the notation, the space $\mathcal{X}(p,\infty)$ is the natural extension of $\mathcal{X}(p,q)$ to the infinite dimensional case. However, it loses all the algebraic properties inherited by $\mathcal{X}(p,q)$ thanks to its embedding into the Grasmannian. The lack of algebraicity will be deeply discussed in Chapter 5, where we will consider actions of complex hyperbolic lattices on the space $\mathcal{X}(p,\infty)$.

We conclude this part by formalizing the notion of embedding between the symmetric spaces introduced in Examples 2.3.4 ans 2.3.5. Fix positive integers $1 \leq p < q_1$ and a second integer $q_1 \leq q_2 \leq \infty$. We denote by $\{e_i\}_{i=1}^{p+q_1}$ and by $\{f_i\}_{i=1}^{p+q_2}$ two basis respectively of $\mathbb{C}^{p+q_1}$ and of $\mathbb{C}^{p+q_2}$, where eventually $\mathbb{C}^{p+\infty}$ denotes the infinite dimensional Hilbert space $\mathcal{H}$ of Example 2.3.5.

Definition 2.3.6. An *embedding* of $\mathcal{X}(p, q_1)$ into $\mathcal{X}(p, q_2)$ is an isometric linear map $\iota : \mathbb{C}^{p+q_1} \to \mathbb{C}^{p+q_2}$ that preserves the Hermitian forms Q_{p,q_1} and Q_{p,q_2}, namely such that

$$Q_{p,q_2}(\iota(x), \iota(y)) = Q_{p,q_1}(x, y)$$

for every $x, y \in \mathbb{C}^{p+q_1}$. Moreover, the group $\mathrm{U}(p, q_1)$ of linear bounded transformations preserving Q_{p,q_1} embeds in $\mathrm{U}(p, q_2)$ in the following way: the action on $\iota(\mathbb{C}^{p+q_1})$ is that of $\mathrm{U}(p, q_1)$ and is trivial on the orthogonal of $\iota(\mathbb{C}^{p+q_1})$.

Among all embeddings of $\mathcal{X}(p, q_1)$ in $\mathcal{X}(p, q_2)$, we consider the *standard embedding* defined by the map $\iota_0 : \mathbb{C}^{p+q_1} \to \mathbb{C}^{p+q_2}$ where $\iota_0(e_i) = f_i$ for $i = 1, \ldots, p+q_1$. In this special case, the space $\mathcal{X}(p, q_1)$ inside $\mathcal{X}(p, q_2)$ can be identified with the set

$$\mathcal{V}_0 = \{V < \langle e_1, \ldots, e_{p+q_1} \rangle \mid \mathrm{dim} V = p \ , \ Q_{p,q_{2|V}} > 0\}$$

and the group $\mathrm{U}(p, q_1)$ is identified with elements g in $\mathrm{U}(p, q_2)$ such that

$$g(e_i) = \sum_{j=1}^{q_2} a_{ij} e_j$$

where, for either i or j greater than $p + q_1$, then $a_{ij} = \delta_{ij}$, and the matrix $A = (a_{ij})_{i,j=1}^{p+q_1}$ represents an element in $\mathrm{U}(p, q_1)$. In other words it satisfies

$$A^* \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_{q_1} \end{pmatrix} A = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_{q_1} \end{pmatrix} .$$

The role of the standard embedding is clarified by the following

Proposition 2.3.7 ([SS21a, Proposition 2.3])**.** *Any embedding $\mathcal{X}(p, q_1) \hookrightarrow \mathcal{X}(p, q_2)$ can be obtained as the composition of an element $g \in \mathrm{U}(p, q_2)$ with the standard embedding.*

Proof. Let $\iota : \mathbb{C}^{p+q_1} \to \mathbb{C}^{p+q_2}$ be an isometric linear map. For each e_i we set $u_i := \iota(e_i)$ and

$$U_\iota := \mathrm{Span}\{u_1, \ldots, u_{p+q_1}\}.$$

There is a natural identification of $\mathcal{X}(p, q_1)$ with the subspace of $\mathcal{X}(p, q_2)$ defined by

$$\mathcal{V}_\iota = \{V < U_\iota \mid \mathrm{dim} V = p \ , \ Q_{p,q_{2|V}} > 0\}.$$

If we denote by U_0 the subspace of $\mathbb{C}^{p+q_2}$ spanned by the first $p+q_1$ vectors of the basis $\{f_i\}$, we can define an isometric linear map $h: U_0 \to U_\iota$ on the basis as follows

$$h(f_i) = u_i$$

and then extend it by linearity. Since h preserves the Hermitian form Q_{p,q_2}, by Witt's theorem it extends to an isometry of $\mathbb{C}^{p+q_2}$ with respect to Q_{p,q_2}, namely to an element $g \in \mathrm{U}(p,q_2)$. The assertion follows by noticing that the isometric linear map $g \circ \iota$ actually realizes the standard embedding. □

Remark 2.3.8. As a subspace of the Grassmannian $\mathrm{Gr}(p+q,\mathcal{H})$, the set of embedding of $\mathcal{X}(p,q)$ inside $\mathcal{X}(p,\infty)$ naturally inherits the topology induced by principal angles, that in this case coincide with the Wisjman topology (see [DLP21]). Since by Lemma 2.3.7 the group $\mathrm{U}(p,\infty)$ acts transitively on the set of all such embeddings, we have an identification with the $\mathrm{PU}(p,\infty)$-orbit of the standard embedding in $\mathrm{Gr}(p,\mathcal{H})$. Moreover, if V_0 is the image of the standard embedding, such an orbit can be identified with the quotient $\mathrm{PU}(p,\infty)/\mathrm{Stab}_{\mathrm{PU}(p,\infty)}V_0$.

We finally notice that any Hermitian symmetric space of the form $\mathcal{X}(p,q)$ with $p<q$ contains maximal tube-type subdomains: these have the same rank of the ambient space and hence are embedded copies of $\mathcal{X}(p,p)$. Moreover, such tube-type subdomains are all conjugated under the G-action. This property will be crucial in the next section, where we will focus on the restriction of such embeddings to the Shilov boundary.

2.3.1 Shilov boundary

We mentioned in the previous section the existence, for any Hermitian symmetric space $\mathcal{X}$, of a bounded domain realization, that is a biholomorphism between $\mathcal{X}$ and a bounded domain $\mathcal{D}_\mathcal{X} \subset \mathbb{C}^n$ for some n. Our next goal is to characterize a subset of the topological boundary of $\mathcal{D}_\mathcal{X}$, that is called *Shilov boundary of G*. As we will see, such a set involves both the analytic, the geometric and the algebraic structure of G and $\mathcal{X}$ and hence it has several equivalent characterizations. We start with the analytic one, that we give for a generic bounded domain.

Definition 2.3.9. The *Shilov boundary* of a bounded domain $\mathcal{D} \subset \mathbb{C}^n$ is the unique closed subset $\mathcal{S}_\mathcal{D}$ of $\partial\mathcal{D}$ such that, for any continuous function f on $\overline{\mathcal{D}}$ which is holomorphic on $\mathcal{D}$ it holds

$$|f(z)| \leq \max_{y \in \mathcal{S}} |f(y)|$$

for every $z \in \mathcal{D}$.

Coming back to domain realizations of symmetric spaces, we also recalled in the previous section that the group $G = \mathrm{Isom}(\mathcal{X})^\circ$ acts on $\mathcal{D}_\mathcal{X}$ by biholomorphisms. Such an action can be extended continuously on the topological boundary of $\mathcal{D}_\mathcal{X}$. The latter, if the rank of G is greater than one, is not a homogeneous G-space. In any case, there exists a unique closed G-orbit that actually coincides with the Shilov boundary of $\mathcal{D}_\mathcal{X}$, that we denote by $\mathcal{S}_G$. More precisely, if $\mathcal{X}$ is irreducible and $\mathbf{G}$ denotes the algebraic group associated to the complexified Lie algebra of G, the stabilizer of any point in $\mathcal{S}_G$ is a maximal parabolic subgroup, and it turns out that $\mathcal{S}_G$ can be identified with $\mathbf{G}/\mathbf{Q}(\mathbb{R})$ where $\mathbf{Q} < \mathbf{G}$ is one of such stabilizers (see Burger, Iozzi and Wienhard for a description of this identification [BIW07, Section 2.3.2]). If $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is a product of irreducible factors $\mathcal{X}_i$, then $\mathcal{S}_G = \mathcal{S}_{G_1} \times \cdots \times \mathcal{S}_{G_n}$ where $G_i = \mathrm{Isom}(\mathcal{X}_i)^\circ$ and the previous argument works for any irreducible factors.

We now consider again Examples 2.3.3 and 2.3.4 in order to describe explicitly the Shilov boundaries in this cases.

Example 2.3.10. In view of Example 2.3.4, we fix two positive integers $1 \leq p < q$ and we consider the group $\mathrm{SU}(p,q)$ together with its symmetric space $\mathcal{X}(p,q)$. The unique closed orbit of the $\mathrm{SU}(p,q)$-action on the boundary of the domain realization $X(p,q)$ is the space

$$S_{p,q} = \{X \in M(p \times q, \mathbb{C}) \,|\, XX^* - \mathrm{Id} = 0\} \,,$$

that corresponds to the set of isotropic subspaces

$$\mathcal{S}_{p,q} = \{V \in \mathrm{Gr}_p(\mathbb{C}^{p+q}) \,|\, Q_{p,q|_V} = 0\} \,,$$

(see for instance [Poz14]). In particular, we have that

$$\mathcal{S}_{1,n} = \left\{ [x_0, \ldots, x_n] \in \mathbb{P}(\mathbb{C}^{n+1}) \,|\, |x_0|^2 - \sum_{i=1}^n |x_i|^2 = 0 \right\}$$

which corresponds, in the affine chart U_0 of Example 2.3.3, to the set

$$\left\{ (x_1, \ldots, x_n) \in \mathbb{C}^n \,|\, \sum_{i=1}^n |x_i|^2 = 1 \right\} \cong \mathbb{S}^{2n-1} \,.$$

Moreover, $\mathcal{S}_{1,n}$ contains embedded copies of $\mathcal{S}_{1,1} \cong \mathbb{S}^1$ that are boundaries of embedded copies of $\mathcal{X}(1,1)$ in $\mathcal{X}(1,n)$ (see Figure 2.1).

In view of the phenomena observed in the previous example, we give the following

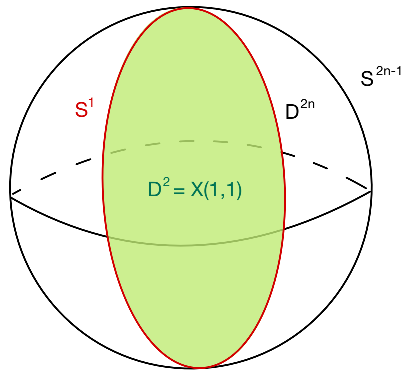

Figure 2.1: An embedded copy of $\mathcal{S}_{1,1} \cong \mathbb{S}^1$ (red) inside $\mathcal{S}_{1,n} \cong \mathbb{S}^{2n-1}$.

Definition 2.3.11. Fix $1 \leq p < q$ with $q \in \mathbb{N} \cup \{\infty\}$. A *p-chain* is the boundary of an embedded copy of $\mathcal{X}(p,p)$ inside $\mathcal{X}(p,q)$. The set of p-chains of $\mathcal{X}(p,q)$ is denoted by $\mathcal{I}_p(p,q)$ or simply $\mathcal{I}_p$.

We conclude this section by noticing that the diagonal action of G on pairs of points $(s_1, s_2) \in \mathcal{S}_G^2$ has a unique open orbits, which corresponds to pairs of opposite parabolic subgroups. Two points belonging to this orbits are said *opposite* and the set of opposite pairs is denoted by $\mathcal{S}_G^{(2)}$. We first notice that the set of points that are opposite to a fixed point is a Zariski closed subset of $\mathcal{S}_G$. Furthermore, whenever a pair of opposite points is fixed, there exists a unique maximal tube-type subdomain $\mathcal{Y} \subset \mathcal{X}$ such that the two points belong to its Shilov boundary. In the case of $\mathcal{X}(p,q)$ with $1 \leq p < q$, this fact can be reformulated in a more familiar way as follows: given two points in $\mathcal{S}_{p,q}$ that correspond to opposite $Q_{p,q}$-isotropic points V_1 and V_2 of the Grasmannian $\mathrm{Gr}_p(\mathbb{C}^{p+q})$, the direct sum $V_1 \oplus V_2$ is an $Q_{p,p}$-isotropic point in $\mathrm{Gr}_p(\mathbb{C}^{2p})$.

To understand the relevance of chains in our theory we need to introduce cocycles and boundary maps. The first intuition of the potential of this notion is due to Cartan [Car32] and then it was developed by Burger and Iozzi [BI02, BI04] and by Pozzetti [Poz14, Poz15]. In particular, Pozzetti exploited the incidence structure on the Shilov boundary defined by opposition to reach a characterization of Zariski dense rational maps between $\mathcal{S}_{1,n}$ and $\mathcal{S}_{p,q}$ in terms of their behavior on chains [Poz15, Theorem 4.1]. We postpone to Chapter 4 a deeper discussion about this result.

2.3.2 Kähler structure and Bergman cocycle

Let $\mathcal{X}$ be a Hermitian symmetric space with $G = \mathrm{Isom}(\mathcal{X})^\circ$. Denoting by $J : \mathcal{X} \to \mathcal{X}$ the G-invariant complex structure on $\mathcal{X}$ and by $\langle \cdot, \cdot \rangle$ the G-invariant Riemannian metric of minimal holomorphic curvature -1, we can define a 2-differential form as

$$\omega_{\mathcal{X}}(X, Y) \coloneqq \langle X, JY \rangle$$

where X, Y vary over all the vector fields on $\mathcal{X}$. By definition $\omega_{\mathcal{X}}$ is G-invariant, hence an element of $\Omega(\mathcal{X}; \mathbb{R})^G$. It was observed by Cartan that any such differential form must be closed [Hel01, VII.4], so that $\omega_{\mathcal{X}}$ is a *Kähler form* and $\mathcal{X}$ is a *Kähler manifold*. In this setting, if we define the function

$$\beta : \mathcal{X}^{(3)} \to \mathbb{R}, \quad \beta(x_0, x_1, x_2) \coloneqq \frac{1}{\pi} \int\limits_{\Delta(x_0,x_1,x_2)} \omega \tag{2.10}$$

where $\Delta(x_0, x_1, x_2)$ denotes any triangle with vertices x_0, x_1, x_2 and geodesic sides, we obtain a well-defined G-invariant bounded 2-cocycle. Clerc and Ørsted [CØ03] provided a measurable extension of β to triples of points on the Shilov boundary $\mathcal{S}_G$ which is bounded, that allows to give the following

Definition 2.3.12. The measurable extension

$$\beta_G : \mathcal{S}_G^3 \to \mathbb{R}$$

of β is the *Bergman cocycle* of G.

We immediately recall some properties of the Bergman cocycle that are the content of [BIW07, Theorem 1] and [Poz15, Proposition 2.1].

Proposition 2.3.13. *Let $\beta_G : \mathcal{S}_G^3 \to \mathbb{R}$ the Bergman cocycle defined above. Then it is a strict alternating G-invariant bounded cocycle taking values in the interval $[-\mathrm{rk}\mathcal{X}, \mathrm{rk}\mathcal{X}]$. Moreover, if $|\beta_G(s_0, s_1, s_2)| = \mathrm{rk}\mathcal{X}$, then the triple (s_0, s_1, s_2) is contained in the Shilov boundary of a tube-type subdomain. If $\mathcal{X}$ is also irreducible, then the following are equivalent:*

- *(i) $\mathcal{X}$ is not of tube-type;*
- *(ii) the set of triples of distinct points on $\mathcal{S}_G$, denoted by $\mathcal{S}_G^{(3)}$, is connected;*
- *(iii) the Bergman cocycle attains all values in $[-\mathrm{rk}\mathcal{X}, \mathrm{rk}\mathcal{X}]$.*

The three conditions of Proposition 2.3.13 will be completed in the next section with another equivalent condition, that requires the notion Hermitian triple product.

Thanks to Proposition 2.3.13 we can interpret the Bergman cocycle as an element in $\mathcal{B}^{\infty}_{\mathrm{w}^*,\mathrm{alt}}(\mathcal{S}^3_G;\mathbb{R})$. Hence it corresponds, under the canonical map of Equation (2.9), to an element $k^b_G \in \mathrm{H}^2_{\mathrm{cb}}(G;\mathbb{R})$ that we call *bounded Kähler class* of G (here we are using alternating bounded functions). Furthermore, k^b_G corresponds under the comparison map

$$\mathrm{comp}^2 : \mathrm{H}^2_{\mathrm{cb}}(G;\mathbb{R}) \to \mathrm{H}^2_{\mathrm{c}}(G;\mathbb{R})$$

to a class $k_G \in \mathrm{H}^2_{\mathrm{c}}(G;\mathbb{R})$ that we call *Kähler class* of G.

Remark 2.3.14. When $G = \mathrm{SU}(1,n)$, the Kähler class $k^b_{\mathrm{SU}(1,n)}$ coincides with the *Cartan class*, that is the element of $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{SU}(1,n),\mathbb{R})$ defined by the Cartan angular invariant of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$.

2.3.3 Hermitian triple product

We conclude with the definition of the *Hermitian triple product* and by showing its link with the Bergman cocycle.

Before introducing such object we need to recall the *Bergman kernel*. The latter is the reproducing kernel associated to the *Bergman metric* defined thanks to the Harish-Chandra embedding, that is a bounded domain realization $\mathcal{D}_{\mathcal{X}}$ of a Hermitian symmetric space of non-compact type $\mathcal{X}$ (see [BI04, BIW07] for more details). Precisely, the Bergman kernel is a function

$$k_{\mathcal{D}_{\mathcal{X}}} : \mathcal{D}_{\mathcal{X}} \times \mathcal{D}_{\mathcal{X}} \to \mathbb{C}$$

that can be extended continuously to the set $\overline{\partial\mathcal{D}_{\mathcal{X}}}^{(2)}$ of pairs of distinct points on the boundary and that is nowhere zero [Sat80].

In this setting, we consider the function

$$\begin{gathered}\langle \cdot,\cdot,\cdot\rangle : \overline{\mathcal{D}_{\mathcal{X}}}^{(3)} \to \mathbb{C}^* \,, \\ \langle x_0,x_1,x_2\rangle := k_{\mathcal{D}_{\mathcal{X}}}(x_0,x_1)k_{\mathcal{D}_{\mathcal{X}}}(x_1,x_2)k_{\mathcal{D}_{\mathcal{X}}}(x_2,x_0)\end{gathered}$$

that is continuous and, by [BIW07, Theorem 3.7], it satisfies the relation

$$\langle x_0,x_1,x_2\rangle \equiv e^{i\beta(x_0,x_1,x_2)} \mod \mathbb{R}^* \tag{2.11}$$

for any $(x_0,x_1,x_2) \in \overline{\mathcal{D}_{\mathcal{X}}}^{(3)}$. Hence we have the following

Definition 2.3.15. The *Hermitian triple product* is the map

$$\langle\langle \cdot,\cdot,\cdot \rangle\rangle : \mathcal{S}_G^{(3)} \to \mathbb{R}^*\backslash\mathbb{C}^*$$

obtained by restricting $\langle \cdot,\cdot,\cdot \rangle$ to $\mathcal{S}_G^{(3)}$ and by composing with the projection $\mathbb{C}^* \to \mathbb{R}^*\backslash\mathbb{C}^*$, where we quotient $\mathbb{C}^*$ by the $\mathbb{R}^*$-action by dilations.

Starting from the identification between $\mathcal{S}_G$ and the quotient $\mathbf{G}/\mathbf{Q}(\mathbb{R})$ that we mentioned in Section 2.3.1, Burger, Iozzi and Wienhard reached an extension of the Hermitian triple product to a *complex Hermitian triple product.* More precisely, denoting by A^* the group $\mathbb{C}^* \times \mathbb{C}^*$ endowed with the real structure $(\lambda, \mu) \mapsto (\overline{\mu}, \overline{\lambda})$ and by Δ^* the image in $\mathbb{C}^* \times \mathbb{C}^*$ of the diagonal embedding of $\mathbb{C}^*$, they were able to define a map

$$\langle\langle \cdot,\cdot,\cdot \rangle\rangle_{\mathbb{C}} : (\mathbf{G}/\mathbf{Q})^3 \to \Delta^*\backslash A^*$$

that makes commutative the following diagram

$$\begin{tikzcd} \mathcal{S}_G^{(3)} \arrow[r, "\langle\langle \cdot,\cdot,\cdot \rangle\rangle"] \arrow[d, "(i)^3"] & \mathbb{R}^*\backslash\mathbb{C}^* \arrow[d, "\Delta"] \\ (\mathbf{G}/\mathbf{Q})^3 \arrow[r, "\langle\langle \cdot,\cdot,\cdot \rangle\rangle_{\mathbb{C}}"] & \Delta^*\backslash A^*. \end{tikzcd} \tag{2.12}$$

Here $i : \mathcal{S}_G \to \mathbf{G}/\mathbf{Q}$ refers to the G-equivariant identification between $\mathcal{S}_G$ and $\mathbf{G}/\mathbf{Q}(\mathbb{R})$, and Δ stands for the map induced by the diagonal embedding.

We conclude with a result characterizing Hermitian symmetric spaces not of tube-type via the complex Hermitian triple product that we will use in the proof of Theorem 8. We first provide an equivalent condition to the ones of Proposition 2.3.13 that implies the above characterization.

Proposition 2.3.16 ([BIW07, Theorem 1])**.** *Let $\mathcal{X}$ an irreducible Hermitian symmetric space. Then $\mathcal{X}$ is not of tube-type if and only if the Hermitian triple product (equivalently the complex Hermitian triple product) is not constant.*

For any pair of transverse points $(s_0, s_1) \in \mathcal{S}_G^{(2)}$ we denote by $\mathcal{O}_{s_0,s_1} \subset \mathbf{G}/\mathbf{Q}(\mathbb{R})$ the Zariski open subset such that the map

$$P_{s_0,s_1} : \mathcal{O}_{s_0,s_1} \to \Delta^*\backslash A^* \ , \quad \eta \mapsto \langle\langle s_0, s_1, s \rangle\rangle_{\mathbb{C}}$$

is defined. Hence the following is a direct consequence of Proposition 2.3.16.

Lemma 2.3.17 ([BIW07, Lemma 5.1])**.** *If $\mathcal{X}$ is not of tube-type, then for any $m \in \mathbb{Z}$ the map*

$$\mathcal{O}_{s_0,s_1} \to \Delta^*\backslash A^* \ , \quad s \mapsto P_{s_0,s_1}(\eta)^m$$

is not constant.

2.4 Measurable cocycles and boundary maps

This section is the core of our preliminary chapter, since we are going to introduce the main objects of the dissertation. After the definition, we will list several examples, with the purpose to make the reader more familiar with those objects. We will make a parallel with the world of representations, for instance by introducing an equivalence relation between cocycles which extend the one of conjugacy and the notion of algebraic hull. Pushing further the comparison, we will introduce boundaries of locally compact groups and then boundary maps for cocycles. Finally, we describe the pullback of cohomology classes along cocycles, that is the fundamental ingredient in the theory of numerical invariants.

Regarding cocycles and boundary maps references are the classic work by Furstenberg [Fur81], Zimmer's book [Zim84] or the more recent project by Moraschini and Savini [MS20, Sav20, Sav22, MS21, SS21c]. For boundaries we remind to Bader-Furman's paper [BF14b].

2.4.1 Measurable cocycles

Throughout the section G and H will denote two locally compact second countable groups, both endowed with their Haar measurable structures. We also fix a standard Borel probability G-space (X, μ_X) and we assume that μ_X is atom-free. As noticed by Moraschini and Savini [MS21, Remark 1], every essentially-free actions on a measure space (X, μ_X) guarantees the absence of atoms for μ_X. Even if essentially-free actions form a wide family of standard Borel spaces and sometimes it is convenient to work with this assumption, we prefer to drop this condition and to tacitly assume absence of atoms for μ_X.

Given two measure spaces (X, μ_X) and (Y, μ_Y) and a distance d_Y on Y which is compatible with its measurable structure, the set of equivalence classes of *measurable functions from X to Y* that coincides almost everywhere is denoted by $\mathrm{Meas}(X, Y)$ and it is a topological space if endowed with the topology induced by convergence in measure. We recall that a base for this topology is given by the sets

$$U_{\varepsilon,f} := \{g \in \mathrm{Meas}(X, Y) \,|\, d(f, g) < \varepsilon\}$$

where $\varepsilon > 0$, $f \in \mathrm{Meas}(X, Y)$ and the distance between two functions f and g is

$$d(f, g) := \inf_{\delta>0} \mu_X \left(\{x \in X \,|\, d_Y(f(x), g(x)) > \delta\}\right) + \delta.$$

We are now ready to introduce the main object of our dissertation

Definition 2.4.1. A *measurable cocycle* is a measurable function $\sigma : G \times X \to H$ which satisfies the cocycle condition

$$\sigma(g_1 g_2, x) = \sigma(g_1, g_2 x)\sigma(g_2, x) \tag{2.13}$$

for almost every $g_1, g_2 \in G$ and for almost every $x \in X$.

At a first sight, Equation (2.13) may appear quite mysterious. For this reason, we are going to provide some examples of cocycles, with the aim of motivating for the interest in the study of those objects.

Example 2.4.2. Our first example may appear trivial, but it justifies the interpretation of measurable cocycles as a generalization of representations. Indeed, given a continuous homomorphism $\rho : G \to H$ and a standard Borel probability G-space (X, μ_X), we define the *cocycle associated to* ρ as

$$\sigma_\rho : G \times X \to H\,, \quad \sigma_\rho(g, x) \coloneqq \rho(g). \tag{2.14}$$

Even if the cocycle σ_ρ actually depends on the space X as a function, its value does not change when the variable x varies, hence we drop X in the notation.

Beyond the large family of examples that it provides, the previous definition also shows how the rich theory of representations sits inside the wider world of cocycles.

Example 2.4.3. We now move to the framework of differentiable manifolds. Let X be a compact n-manifold equipped with a measure μ_X and choose a locally compact second countable subgroup $G < \mathrm{Diffeo}(X)$ of μ_X-preserving diffeomorphisms of X. In this setting, for any point $x \in X$ and element $g \in G$ and since the tangent bundle is measurably trivial, the differential $d_x g$ lies in $\mathrm{GL}(n, \mathbb{R})$. Hence we can define the *tangent cocycle* of X as

$$\sigma_{\mathrm{tang}} : G \times X \to \mathrm{GL}(n, \mathbb{R})\,, \quad \sigma_{\mathrm{tang}}(g, x) \coloneqq d_x g$$

for almost every $x \in X$ and for every $g \in G$. The cochain rule of the differential implies that

$$\begin{aligned} \sigma_{\mathrm{tang}}(g \circ h, x) &= d_x(g \circ h) \\ &= d_{h(x)} g \circ dh_x \\ &= \sigma_{\mathrm{tang}}(g, h(x))\sigma_{\mathrm{tang}}(h, x) \end{aligned}$$

and hence the cocycle condition (2.13) is satisfied.

This example gives an interpretation of measurable cocycles as the analogous of the differential in the measurable context.

Example 2.4.4. Given a locally compact second countable group G and a closed subgroup $H < G$, we consider a measurable section $s : G/H \to G$ of the natural projection $G \to G/H$. Since for any $x \in G/H$ and $g \in G$ we have that

$$s(g \cdot x)H = \pi(s(g \cdot x)) = \pi(gs(x)) = g \cdot s(x)H\,,$$

then $s(g \cdot x)^{-1}gs(x)$ lies in H and we can define a cocycle

$$\sigma_s : G \times G/H \to H\,, \quad \sigma_s(g,x) \coloneqq s(g \cdot x)^{-1}gs(x).$$

By definition we have

$$\begin{aligned}\sigma_s(gh,x) &= s(gh \cdot x)^{-1}ghs(x)\\ &= s(gh \cdot x)^{-1}gs(h \cdot x)s(h \cdot x)^{-1}hs(x)\\ &= \sigma_s(g, h \cdot x)\sigma_s(h,x)\,,\end{aligned}$$

that is the cocycle condition (2.13).

To understand the concrete meaning of the cocycle σ_s defined above, we focus on the specific case when $G = \mathbb{S}^1 \times \mathbb{S}^1$ is the 2-dimensional torus with the group product structure (here we consider the standard additive operation on $\mathbb{S}^1$). If also $H = \mathbb{S}^1 \cong \{0\} \times \mathbb{S}^1 < G$, the geometrical interpretation of H is a meridian μ_0. Hence the projection $\pi : G \to G/H$ collapses any other meridian $\{\xi\} \times \mathbb{S}^1 \subset G$ to the point $\xi \in \mathbb{S}^1 \cong G/H$.

Fix a point $\eta_0 \in \mathbb{S}^1$ and let

$$s_{\eta_0} : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1\,, \quad s_{\eta_0}(\xi) = (\xi, \eta_0)$$

be the section that associates to any point ξ of the circle the corresponding point (ξ, η_0) on the longitude $\mathbb{S}^1 \times \{\eta_0\}$, as shown in Figure 2.2.A. Hence we claim that the cocycle $\sigma_{s_{\eta_0}}$ coincides with the cocycle σ_{π_2} induced by the projection $\pi_2 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ on the second factor (see Example 2.4.2). In fact, for every $g = (\xi, \eta) \in \mathbb{S}^1 \times \mathbb{S}^1$ and every $x \in \mathbb{S}^1$, we have that

$$\begin{aligned}\sigma_{s_{\eta_0}}(g,x) &= g + s_{\eta_0}(x) - s_{\eta_0}(g+x)\\ &= (\xi,\eta) + (x,\eta_0) - (\xi + x, \eta_0) = (0,\eta) = \sigma_{\pi_2}(g)\end{aligned}$$

(see Figure 2.2.B), which can be rewritten as

$$g + s_{\eta_0}(x) = \pi_2(g) + s_{\eta_0}(g+x)\,.$$

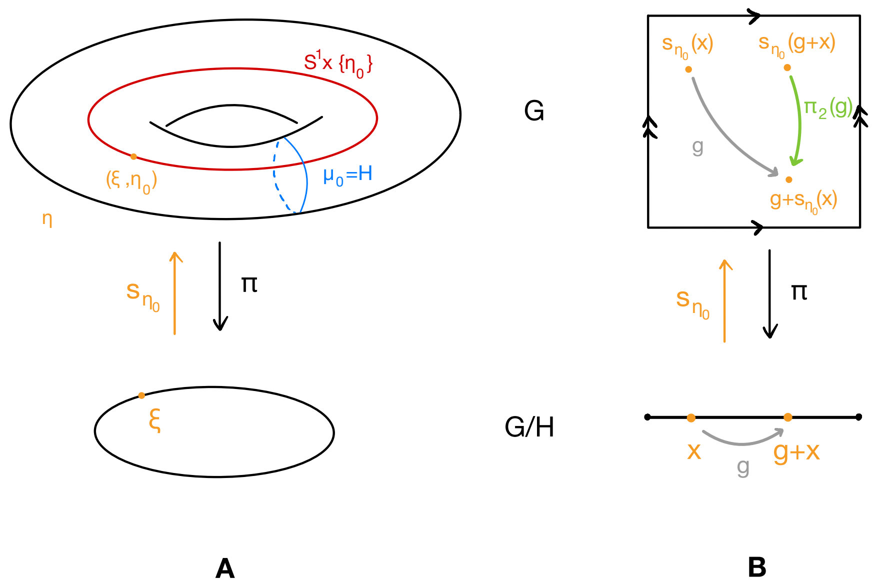

Figure 2.2: The cocycle associated to the trivial section of the 2-torus

Example 2.4.5. Our next example describes a class of cocycles that has been deeply studied so far. For instance, it is the object of Zimmer superrigidity theorem mentioned in the introduction [Zim80]. Let G and H be locally compact second countable groups and let (X, μ_X) and (Y, μ_Y) be measure spaces equipped with a measure preserving G (respectively H)-measurable action. Assume also that the H-action is free. An *orbit equivalence* between X and Y is a measurable isomorphism $\pi : X \to Y$ such that

$$\pi(Gx) = H\pi(x)$$

for every $x \in X$. Given such an equivalence we can associate to every pair $(g, x) \in G \times X$ a unique element $h_{g,x} \in H$ such that

$$\pi(gx) = h_{g,x}\pi(x).$$

Hence we define a measurable map

$$\sigma_\pi : G \times X \to H\,, \quad \sigma_\pi(g, x) := h_{g,x}$$

which turns to be a cocycle. In fact, for almost every $x \in X$ and for every $g_1, g_2 \in G$ we have

$$h_{g_1 g_2,x}\pi(x) = \pi(g_1 g_2 x) = h_{g_1,g_2 x}\pi(g_2 x) = h_{g_1,g_2 x}h_{g_2,x}\pi(x)$$

and, since H acts freely on Y, it must be $h_{g_1 g_2,x} = h_{g_1,g_2 x} h_{g_2,x}$.

When G and H are higher rank semisimple Lie groups with trivial center and no compact factors and if both the actions are free, ergodic and irreducible, Zimmer showed that the cocycle σ_π defined above is cohomologuous to the cocycle induced by an isomorphism $\rho : G \to H$ and hence the actions are conjugated [Zim80, Theorem 4.3].

Sometimes, before studying a class of mathematical objects, it is natural to consider an equivalence relation that allows to identify elements lying in the same equivalence class. In addition, as suggested by Example 2.4.2, measurable cocycles can be interpreted as a generalization of the notion of representation, and in that context it is particularly interesting to consider homomorphisms modulo conjugation. For this reason, we are going to introduce an equivalence relation between cocycles which is nothing that the natural generalization of conjugacy between representations.

Definition 2.4.6. Let $\sigma_1, \sigma_2 : G \times X \to H$ be two measurable cocycles, let $f : X \to H$ be a measurable map and denote by σ_1^f the cocycle defined as

$$\sigma_1^f(g,x) \coloneqq f(gx)^{-1}\sigma_1(g,x)f(x) \tag{2.15}$$

for almost every $g \in G$ and almost every $x \in X$. The cocycle σ_1^f is the *f-twisted cocycle associated to* σ_1. We say that σ_1 is *cohomologous* to σ_2 (writing $\sigma_1 \simeq \sigma_2$) if there exists a measurable map f such that $\sigma_2 = \sigma_1^f$.

Remark 2.4.7. If $\rho_1, \rho_2 : G \to H$ are two conjugated continuous representations and (X, μ_X) is a standard Borel probability G-space, then there exists an element $h \in H$ such that

$$\rho_2(h) = h^{-1}\rho_1(g)h$$

for every $g \in G$. Hence by taking the essentially constant function

$$f_h : X \to H\,, \quad f(x) \equiv h\,,$$

we get that

$$\sigma_{\rho_2}(g,x) = \rho_2(g) = h^{-1}\rho_1(g)h = \sigma_{\rho_1}^{f_h}$$

that is $\sigma_{\rho_2} \simeq \sigma_{\rho_1}$. This confirms that cohomology between cocycles naturally extends conjugation between representations.

We also notice that the converse does not hold in general, namely two representations inducing cohomologous cocycles are not necessarily conjugated, since the measurable map $X \to H$ which realizes the cohomology may depend on X.

The equivalence relation introduced in Definition 2.4.6 hides a cohomological interpretation of the classes of measurable cocycles, that we will briefly describe (refer to Feldman and Moore [Moo76, FM77] and to Furman [Fur10] for a detailed discussion).

Given a standard Borel space (X, μ_X), to any μ_X-invariant equivalence relation $\mathcal{R} \subset X \times X$ and Polish abelian group H one can associate a cochain complex $(C^\bullet(\mathcal{R}; H), d^\bullet)$ as follows. We consider the space

$$\mathcal{R}^{(\bullet)} \coloneqq \{(x_0, \dots, x_\bullet) \in X^{\bullet+1} \,|\, (x_i, x_i + 1) \in \mathcal{R} \,,\, \forall i = 0, \dots, \bullet - 1\}$$

endowed with the measure $\mu_X^{(\bullet)}$ defined as

$$\mu_X^{(\bullet)}(A) \coloneqq \int_X \sharp\{(x_1, \dots, x_\bullet) \,|\, (x_0, \dots, x_\bullet) \in A\} d\mu_X(x_0) \,.$$

We notice that at degree 0 we simply have $\mathcal{R}^{(0)} \equiv X$, while at degree 1 one has $\mathcal{R}^{(1)} \equiv \mathcal{R}$.

We define the set

$$C^\bullet(\mathcal{R}; H) \coloneqq \{f : \mathcal{R}^{(\bullet)} \to H \,,\, f \text{ measurable}\}/ \sim_{\mu_X^{(\bullet)}}$$

where $f \sim_{\mu_X^{(\bullet)}} g$ if they coincide $\mu_X^{(\bullet)}$-almost everywhere, and the operator

$$d^\bullet : C^\bullet(\mathcal{R}; H) \to C^{\bullet+1}(\mathcal{R}; H) \,,$$
$$d^\bullet(f)(x_0, \dots, x_{\bullet+1}) \coloneqq \prod_{i=0}^{\bullet+1} f(x_0, \dots, \widehat{x_i}, \dots, x_{\bullet+1})^{(-1)^i} \,.$$

where H is considered with a multiplicative structure.

It follows by the definition that $d^{\bullet+1} \circ d^\bullet = 0$, hence the pair $(C^\bullet(\mathcal{R}; H), d^\bullet)$ forms a cochain complex whose cohomology

$$\mathrm{H}^n(\mathcal{R}; H) \coloneqq \mathrm{H}^n(C^\bullet(\mathcal{R}; H), d^\bullet) = \mathrm{Ker}(d^\bullet)/\mathrm{Im}(d^{\bullet-1})$$

is the *cohomology of* $\mathcal{R}$ *with values in* H. We notice that the assumption that H is abelian is essential to define $\mathrm{H}^n(\mathcal{R}; H)$ only for $n > 1$. In fact, we are interested in the 1-dimensional cohomology of a specific class of equivalence relations, called *orbital equivalence relations*, with values in generic topological groups. Precisely, in the setting of Definition 2.4.1, one can define the equivalence relation $\mathcal{R}_G$ where $(x, y) \in \mathcal{R}_G \Leftrightarrow y = gx$ for some $g \in G$. In this case we define the space $Z^1(\mathcal{R}_G; H)$ as the set of functions $c : \mathcal{R}_G \to H$ satisfying the relation

$$c(x, z) = c(x, y)c(y, z) \tag{2.16}$$

for almost every $x, y, z \in \mathcal{R}_G^{(2)}$. Hence we have a natural identification between measurable cocycles $G \times X \to H$ and $Z^1(\mathcal{R}_G; H)$ realized by the following map

$$\begin{aligned}
\Theta : \{\sigma : G \times X \to H \ , \ \sigma \text{ cocycle } \} &\longrightarrow Z^1(\mathcal{R}_G; H) \\
\sigma &\longmapsto f_\sigma(x, gx) \coloneqq \sigma(g, x) \, .
\end{aligned}$$

The 1-cohomology group of $\mathcal{R}_G$ with values in H, denoted by $\mathrm{H}^1(G \curvearrowright X; H)$, is defined as the quotient $Z^1(\mathcal{R}_G; H)/\sim$ where

$$f_1 \sim f_2 \Leftrightarrow \exists\, h \in \mathrm{Meas}(X, H) \,|\, f_1(x, gx) = h(gx)^{-1} f_2(x, gx) h(x) \, , \ \widetilde{\forall} \ x \in X, g \in G$$

where the condition $f_1(x, gx) = h(gx)^{-1} f_2(x, gx) h(x)$ is exactly the one of Equation (2.15) applied to the cocycles $\Theta^{-1}(f_1)$ and $\Theta^{-1}(f_2)$. In other words, the map Θ factors through the equivalence relation of cohomology between cocycles and defines a bijection

$$\{\sigma : G \times X \to H \ , \ \sigma \text{ cocycle } \}/_{\simeq} \leftrightarrow \mathrm{H}^1(G \curvearrowright X; H) \, .$$

We conclude this part by introducing a weaker notion of equivalence between cocycles, that we will exploit in Chapter 6.

Definition 2.4.8. If $\sigma_1 : G \times X \to H_1$ and $\sigma_2 : G \times X \to H_2$ are two measurable cocycles, we say that σ_1 and σ_2 are *equivalent* (writing $\sigma_1 \sim \sigma_2$) if there exists a group isomorphism $s : H_1 \to H_2$ such that $s \circ \sigma_1 \simeq \sigma_2$.

Straightening the comparison between representations and cocycles, we now focus on the image of a cocycle. It is well widely recognized the crucial role of the image of a representation into a semi-simple algebraic groups, as one can find in the works by Burger, Iozzi and Wienhard [BIW10] and in the one by Pozzetti [Poz15]. For representations into algebraic groups the image is a subgroup in the target and hence its Zariski closure is a group as well. Even if the image of a cocycle has no algebraic properties, we can get an analogous definition using the notion of algebraic hull. This technical step is necessary since a priori the image of a cocycle is not a subgroup of the target group.

Definition 2.4.9. Let $\mathbf{G}$ be a semisimple real algebraic group and denote by $H = \mathbf{H}(\mathbb{R})$ the real points of $\mathbf{H}$, namely the real solutions of the polynomials defining $\mathbf{G}$. The *algebraic hull* of a measurable cocycle $\sigma : G \times X \to H$ is the (conjugacy class of the) smallest algebraic subgroup $\mathbf{L}$ of $\mathbf{H}$ such that $\mathbf{L}(\mathbb{R})^\circ$ contains the image of a cocycle cohomologous to σ.

We say that σ is *Zariski dense* if its algebraic hull coincides with the whole group $\mathbf{H}$.

We remind the reader to [Zim84, Proposition 9.2.1] for a proof of well-definition of algebraic hull. The argument relies on the Noetherianity of the target group.

Since we are strongly interested into Zariski dense cocycles and because of the invariance of such property with respect to cohomology, we denote by

$$\mathrm{H}^1_{\mathrm{ZD}}(G \curvearrowright X; H)$$

the subset of classes of Zariski dense cocycles in $\mathrm{H}^1(G \curvearrowright X; H)$.

To conclude this section, we describe an example of cocycle that is not Zariski dense.

Example 2.4.10. Consider the real hyperbolic space $\mathbb{H}^3_{\mathbb{R}}$ and the group of orientation preserving isometry $\mathrm{Isom}^+(\mathbb{H}^3_{\mathbb{R}}) \cong \mathrm{PSL}(2,\mathbb{C})$ endowed with its Haar measurable structure. We consider a locally compact second countable group Γ and a standard Borel probability Γ-space (X, μ_X). We claim that a measurable cocycle

$$\sigma : \Gamma \times X \to \mathrm{PSL}(2,\mathbb{C})$$

that preserves an equivariant family $\{\mathcal{Y}_x\}_{x \in X}$ of totally geodesic copies of $\mathbb{H}^2_{\mathbb{R}}$, namely such that $\sigma(\gamma, x)\mathcal{Y}_x = \mathcal{Y}_{\gamma x}$ for almost every $\gamma \in \Gamma$ and almost every $x \in X$, cannot be Zariski dense. In fact, by Proposition 2.3.7, $\mathrm{PSL}(2,\mathbb{C})$ acts transitively on totally geodesic copies of $\mathbb{H}^2_{\mathbb{R}}$ inside $\mathbb{H}^3_{\mathbb{R}}$, so that the set of such embeddings is identified with the quotient $\mathrm{PSL}(2,\mathbb{C})/\mathrm{Stab}_{\mathrm{PSL}(2,\mathbb{C})}\iota_0$, where ι_0 denotes the standard embedding. Hence, by composing the map $x \mapsto \mathcal{Y}_x$ with a measurable section $\mathrm{PSL}(2,\mathbb{C})/\mathrm{Stab}_{\mathrm{PSL}(2,\mathbb{C})}\iota_0 \to \mathrm{PSL}(2,\mathbb{C})$ one can construct a measurable map

$$g : X \to \mathrm{PSL}(2,\mathbb{C})\,, \quad x \mapsto g_x$$

such that $g_x \mathcal{Y}_x = \mathcal{Y}_{x_0}$ for some fixed $x_0 \in X$. In other words, the image of the twisted cocycle σ^g (see Definition 2.4.6) is contained into $\mathrm{Stab}_{\mathrm{PSL}(2,\mathbb{C})}(\mathcal{Y}_{x_0})$, that is a proper subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and hence σ is not Zariski dense.

2.4.2 Boundaries and boundary maps

The goal of this section is to introduce boundaries for locally compact second countable groups and then the notion of boundary maps for cocycles. We will finally describe the pullback along cocycles by showing the importance of boundary maps in this theory. Since there exist several notions of boundaries in different contexts, we spend a few lines to clarify our approach.

The notion of boundary for a group G that we use is due to Bader and Furmann [BF14b] and it is based on the notion of relative metric ergodicity (see Definition

2.1.11). Such spaces have two crucial properties: firstly they are amenable G-spaces, and this make them a suitable choice for the Burger–Monod characterization of bounded cohomology of G. Secondly, the double-ergodicity of the G-action is a useful ingredient to prove rigidity phenomena, as we will see in Chapter 4.

After the general definition of boundaries we focus on the Furstenberg–Poisson boundary [Fur63]. We give an explicit realization of those objects in the case of semi-simple Lie groups (Example 2.4.14) and for finitely generated discrete groups (Example 2.4.15).

Let G be a locally second countable group. In view of Definition 2.1.11, we give the following

Definition 2.4.11. A *G-boundary* is an amenable Lebesgue G-space B such that the projections $\mathrm{pr}_1 : B \times B \to B$ and $\mathrm{pr}_2 : B \times B \to B$ on the two factors are relatively metrically ergodic.

Remark 2.4.12. As observed in [BF14b, Remarks 2.4], a G-boundary in the sense of Definition 2.4.11 is a *strong G-boundary* in the sense of Burger and Monod [BM02]. Here by strong boundary we mean an amenable G-space such that the diagonal action on the product is ergodic. Thanks to Theorem 2.2.17 and Theorem 2.2.16, it can be exploited to compute the continuous bounded cohomology of G.

We now recall the following useful property of boundaries.

Proposition 2.4.13 ([MS04, Proposition 2.4])**.** *Let B a strong boundary for Γ and Ω be a standard Borel probability Γ-space. Then the diagonal action of Γ on $B \times X$ and $B \times B \times X$ is ergodic.*

As an example of G-boundary we consider the case of lattices into connected semi-simple Lie groups, that will come in handy in Chapter 4, and the case of finitely generated discrete groups which are the main objects of Chapter 6.

Example 2.4.14. For a lattice $\Gamma < G$ into a connected semi-simple Lie group G, the *Furstenberg–Poisson boundary* [Fur63] of Γ coincides with the quotient G/P by any minimal parabolic subgroup endowed with induced measure class and, by [BF14b, Theorem 2.5], it is a Γ-boundary in the sense of Definition 2.4.11.

For instance, in Chapters 4 and 5 we will deal with complex hyperbolic lattices, that are torsion-free lattices in the group

$$\mathrm{PU}(1,n) \coloneqq \mathrm{SU}(1,n)/\pm\lambda\mathrm{Id}\,,$$

where $\mathrm{SU}(1,n)$ is the isometry group of the complex hyperbolic $\mathbb{H}^n_{\mathbb{C}}$ space recalled in Example 2.3.3. In this case and in view of Example 2.3.10, as a minimal parabolic subgroup of $\mathrm{PU}(1,n)$ we can take $P = \mathrm{Stab}_{\mathrm{PU}(1,n)}\xi$, that is the stabilizer in $\mathrm{PU}(1,n)$ of the north pole $\xi = [1,0,\ldots,0,1]$ in the Shilov boundary $\mathcal{S}_{1,n} \cong \mathbb{S}^{2n-1}$ (see for instance [Qui, Section 2.2.2] for an explicit description of the group P). Anyhow, since $\mathrm{PU}(1,n)$ acts transitively on $\mathcal{S}_{1,n}$, we deduce that the Furstenberg–Poisson boundary of a torsion-free lattice $\Gamma < \mathrm{PU}(1,n)$ coincides with its Shilov boundary, namely

$$\mathrm{PU}(1,n)/P \cong \mathbb{S}^{2n-1}.$$

We notice that, since $\mathrm{rk}(\mathrm{PU}(1,n)) = 1$, a minimal parabolic subgroup is also maximal, and the previous identification immediately follows.

We conclude by pointing out that, in the higher rank case, there exists a natural map from the Furstenberg–Poisson boundary to the Shilov boundary. In fact, the first one coincides with the quotient G/P where $P < G$ is minimal parabolic and the second one can be identified with $\mathbf{G}/\mathbf{Q}(\mathbb{R})$ for some maximal parabolic subgroup $\mathbf{Q} < \mathbf{G}$ (see Section 2.3.1), and the inclusion $P < Q := \mathbf{Q}(\mathbb{R})$ induces a projection $G/P \to \mathbf{G}/\mathbf{Q}(\mathbb{R})$. This is the case of the group $\mathrm{SU}(p,q)$ when $1 < p \leq q$ recalled in Example 2.3.4, where minimal (respectively maximal) parabolic subgroups are stabilizers of complete flags of $Q_{p,q}$-isotropic subspaces (respectively of maximal $Q_{p,q}$-isotropic subspaces).

Example 2.4.15. When Γ is any finitely generated group endowed with the discrete topology, we recall the realization of a Poisson boundary for Γ (see [BI04, Theorem 7.1] for details). If S is a set of generators for Γ, we define a probability measure on Γ as

$$\mu_S = \frac{1}{2|S|}\sum_{s\in S}\delta_s + \delta_{s^{-1}}.$$

To realize a Poisson boundary for Γ related to the measure μ_S we start with the realization of a Poisson boundary for the free group $\mathbb{F}_S$ on the set S. Let $\mathcal{T}_S(\infty)$ be the boundary of the Cayley graph $\mathcal{T}_S$ of $\mathbb{F}_S$, namely the set of all reduced words on S of infinite length. We endow such a boundary with the $\mathbb{F}_S$-quasi invariant measure defined by

$$\overline{m}(C(x)) = \frac{1}{2r(2r-1)^{n-1}}$$

where x is any reduced word of length n, $r = |S|$ and $C(x)$ denotes the set of all reduced words of infinite length starting with x. Hence the pair $(\mathcal{T}_S(\infty), \overline{m})$ is the Poisson boundary of $\mathbb{F}_S$ related to the measure

$$m = \frac{1}{2|S|}\sum_{s\in S}\delta_s + \delta_{s^{-1}}.$$

Coming back to Γ, if $\rho : \mathbb{F}_S \to \Gamma$ is the representation of Γ realizing it as a quotient, we denote by $N = \ker\rho$ and we consider the set $\mathrm{L}^\infty(\mathcal{T}_S(\infty), \overline{m})^N$ of N-invariant essentially bounded functions on $\mathcal{T}_S(\infty)$. By Mackey realization Theorem [Mac62] there exists a standard measure space (B, ν) equipped with a measurable map $p : \mathcal{T}_S(\infty) \to B$ such that $p_*(\overline{m}) = \nu$ and the pull back via p identifies $\mathrm{L}^\infty(B, \nu)$ with $\mathrm{L}^\infty(\mathcal{T}_S(\infty), \overline{m})^N$. The fact that (B, ν) actually is a Poisson boundary for Γ follows from the fact that the pull back along ρ identifies μ-harmonic bounded functions on Γ, namely functions $f : \Gamma \to \mathbb{R}$ such that $f(\widetilde{\gamma}) = \int_\Gamma f(\gamma\widetilde{\gamma})d\mu(\gamma)$, with essentially bounded function on (B, ν). Moreover, the ergodicity of the diagonal action of $\mathbb{F}_S$ on $\mathcal{T}_S(\infty)$ implies the ergodicity of the diagonal action of Γ on $B \times B$. Finally, it can be proved that the action of Γ on B is also amenable (see [BI04, Proposition 7.1]).

As observed in Remark 2.4.12, the amenability of the G-action on a G-boundary reveals the link between boundaries and continuous bounded cohomology. On the other hand, the next definition relates boundaries and measurable cocycles.

Definition 2.4.16. Let G be a locally compact second countable group, let B be a G-boundary and let H be a locally compact group. Consider a standard Borel probability G-space (X, μ_X) and a Lebesgue H-space (Y, ν). A *boundary map* for a measurable cocycle $\sigma : G \times X \to H$ is a measurable map

$$\phi : B \times X \to Y \ ,$$

which is σ-*equivariant*, that is

$$\phi(g\xi, gx) = \sigma(g, x)\phi(\xi, x) \ , \tag{2.17}$$

for almost every $g \in G$ and almost every $b \in B, x \in X$.

Remark 2.4.17. We push further the comparison between cocycles and representation by relating boundary maps to the notions introduced in Section 2.4.1.

(i) In the setting of Definition 2.4.16, if $\sigma = \sigma_\rho$ is the cocycle induced by a representation $\rho : G \to H$ as in Example 2.4.2, then a ρ-equivariant map $\varphi : B \to Y$ naturally defines a σ_ρ-equivariant map $\phi : B \times X \to Y$ as

$$\phi(b, x) := \phi(b)$$

for every $b \in B$ and $x \in X$.

(ii) If $\phi : B \times X \to Y$ is a boundary map for a cocycle $\sigma : G \times X \to H$ and $f : X \to G$ is a measurable function, the map $\phi^f : B \times X \to Y$ defined as

$$\phi^f(b, x) := f(x)^{-1}\phi(b, x)$$

is a boundary map for the twisted cocycle σ^f introduced in Definition 2.4.6.

As we will see in the next section, boundary maps allows to write down an alternative implementation of the pullback along cocycles. On the other hand in Section 2.3 we showed how to find a representative in the resolution of essentially bounded measurable functions on the boundary for some preferred cohomology class. This approach will be very fruitful to prove rigidity results in the context of cocycles, as well it has been for representations [BI02, BI04, BIW07]. For this reason, the investigation about the existence of boundary maps is a crucial point in the theory of measurable cocycles that will be discussed in Chapter 3.

2.4.3 Pull back along measurable cocycles

In this section we exploit all the notions introduced in this chapter in order to define the main tool that we need to investigate rigidity results for cocycles, namely the *pullback along cocycles*. This construction is nothing that the generalization of the pullback along representations in the more general setting of measurable cocycles. Since for our purposes we consider cocycles $\Gamma \times X \to H$ where Γ is endowed with the discrete topology, we restrict to this specific setting. As we will see, by extending faithfully the pull back for representations one get a map from the real bounded cohomology of H into the bounded cohomology of Γ with coefficients into $\mathrm{L}^\infty(X;\mathbb{R})$. In order to remove the dependence on X and hence to land into the real bounded cohomology of Γ, we will need to integrate over X with respect to the Γ-invariant measure μ_X.

We first describe the natural extension of the pullback for representations, that was introduced by Burger and Iozzi [BI02], without using boundary maps.

The second part is devoted to describe an alternative version of the pullback in case of existence of boundary maps. This is nothing that the work done by Moraschini and Savini in [Sav22, MS20, MS21, Sav20] and it will be the right approach to prove all our main results. For an almost self-contained description of this theory we refer to the recent works by Moraschini, Savini and the author [MS20, Sav20, SS21c, SS21b].

We first consider a measurable cocycle $\sigma : \Gamma \times X \to H$ where Γ is endowed with the discrete topology. We define the map

$$\mathrm{C}^\bullet_{\mathrm{b}}(\sigma) : \mathrm{C}^\bullet_{\mathrm{cb}}(H;\mathbb{R}) \to \mathrm{C}^\bullet_{\mathrm{b}}(\Gamma;\mathrm{L}^\infty(X;\mathbb{R}))$$

as follows

$$\mathrm{C}^\bullet_{\mathrm{b}}(\sigma)(\psi)(\gamma_0,\dots,\gamma_\bullet)(x) \coloneqq \psi(\sigma(\gamma_0^{-1},x)^{-1},\dots,\sigma(\gamma_\bullet^{-1},x)^{-1}). \tag{2.18}$$

A slight modification of the argument in [Sav20, Lemma 2.7] shows that the map $\mathrm{C}^\bullet_{\mathrm{b}}(\sigma)$ induces a map at a cohomological level.

Lemma 2.4.18. *The map* $\mathrm{C}_{\mathrm{b}}^{\bullet}(\sigma)$ *is a well-defined cochain map which restricts to invariant cochains, namely*

$$\mathrm{C}_{\mathrm{b}}^{\bullet}(\sigma) : \mathrm{C}_{\mathrm{cb}}^{\bullet}(H;\mathbb{R})^{H} \to \mathrm{C}_{\mathrm{b}}^{\bullet}(\Gamma;\mathrm{L}^{\infty}(X;\mathbb{R}))^{\Gamma} ,$$

and hence it induces a map between cohomology groups

$$\mathrm{H}_{\mathrm{b}}^{\bullet}(\sigma) : \mathrm{H}_{\mathrm{cb}}^{\bullet}(H;\mathbb{R}) \to \mathrm{H}_{\mathrm{b}}^{\bullet}(\Gamma;\mathrm{L}^{\infty}(X;\mathbb{R})). \tag{2.19}$$

Proof. The fact that $\mathrm{C}_{\mathrm{b}}^{\bullet}(\sigma)$ is a cochain map is an easy computation, and it actually preserves boundedness. We prove that the image of an H-invariant cochain is Γ-invariant. Let $\psi \in \mathrm{C}_{\mathrm{cb}}^{\bullet}(H;\mathbb{R})^{H}$ and let $\gamma, \gamma_0, \dots, \gamma_\bullet$ elements of Γ. Hence we have

$$\begin{aligned}
(\gamma \cdot \mathrm{C}_{\mathrm{b}}^{\bullet}(\sigma)(\psi)(\gamma_0, \dots, \gamma_\bullet)(x) &= \mathrm{C}_{\mathrm{b}}^{\bullet}(\sigma)(\psi)(\gamma^{-1}\gamma_0, \dots, \gamma^{-1}\gamma_\bullet)(\gamma^{-1}x) \\
&= \psi(\sigma(\gamma_0^{-1}\gamma, \gamma^{-1}x)^{-1}, \dots, \sigma(\gamma_\bullet^{-1}\gamma, \gamma^{-1}x)^{-1}) \\
&= \psi(\sigma(\gamma, \gamma^{-1}x)^{-1}\sigma(\gamma_0^{-1}, x)^{-1}, \dots, \\
&\quad \psi(\sigma(\gamma_0^{-1}, x)^{-1}, \dots, \sigma(\gamma_\bullet^{-1}, x)^{-1}) \\
&= \mathrm{C}_{\mathrm{b}}^{\bullet}(\sigma)(\psi))(\gamma_0, \dots, \gamma_\bullet)(x) ,
\end{aligned}$$

where we moved from the first line to the second one using the definition, from the second line to the third one using the cocycle condition of Equation (2.13) and we concluded by exploiting the H-invariance of ψ. $\square$

Remarkably, the map induced in bounded cohomology depends only on the cohomology class of σ.

Proposition 2.4.19 ([SS21b, Proposition 2.15])**.** *Let* Γ *be a discrete group and let* (X, μ_X) *be a standard Borel probability* Γ*-space. Given a measurable cocycle* $\sigma : \Gamma \times X \to H$ *and a measurable map* $f : X \to H$*, it holds that*

$$\mathrm{H}_{\mathrm{b}}^{\bullet}(\sigma^{f}) = \mathrm{H}_{\mathrm{b}}^{\bullet}(\sigma) \ .$$

Proof. Following the line of either [Mon01, Lemma 8.7.2] or [Sav20, Lemma 2.9], we are going to prove that the pullback induced by σ^f and σ are chain homotopic. Consider a cochain $\psi \in \mathrm{C}_{\mathrm{cb}}^{\bullet}(H;\mathbb{R})^{H}$. For every $(\gamma_0, \dots, \gamma_\bullet) \in \Gamma^{\bullet+1}$ and for almost every $x \in X$, we have that

$$\begin{aligned}
\mathrm{C}_{\mathrm{b}}^{\bullet}(\sigma^f)(\psi)(\gamma_0, \dots, \gamma_\bullet)(x) &= \psi((\sigma^f(\gamma_0^{-1}, x))^{-1}, \dots, (\sigma^f(\gamma_\bullet^{-1}, x))^{-1}) \\
&= \psi(f(x)^{-1}\sigma(\gamma_0^{-1}, x)^{-1}f(\gamma_0^{-1}x), \dots) \\
&= \psi(\sigma(\gamma_0^{-1}, x)^{-1}f(\gamma_0^{-1}x), \dots) \ ,
\end{aligned}$$

where we moved from the first line to the second one using the definition of σ^f and we exploited the H-invariance of ψ to move from the second line to the third one. We want to prove that the right action by measurable maps is actually chain homotopic to the identity. In this way the claim will follow.

For $0 \leq i \leq \bullet - 1$ we now define the following map

$$s_i^\bullet(\sigma, f) : \mathrm{C}_{\mathrm{cb}}^\bullet(H, \mathbb{R}) \to \mathrm{C}_{\mathrm{cb}}^{\bullet-1}(\Gamma; \mathrm{L}^\infty(X; \mathbb{R})) \ , \quad s_i^\bullet(\sigma, f)(\psi)(\gamma_0, \ldots, \gamma_{\bullet-1})(x) :=$$

$$= \psi(\sigma(\gamma_0^{-1}, x)^{-1} f(\gamma_0^{-1} x), \ldots, \sigma(\gamma_i^{-1}, x)^{-1} f(\gamma_i^{-1} x), \sigma(\gamma_i^{-1}, x)^{-1}, \ldots, \sigma(\gamma_{\bullet-1}^{-1}, x)^{-1}) \ ,$$

and we set $s^\bullet(\sigma, f) := \sum_{i=0}^{\bullet-1} (-1)^i s_i^\bullet(\sigma, f)$. If we define for $-1 \leq i \leq \bullet$ the map

$$\rho_i^\bullet(\sigma, f) : \mathrm{C}_{\mathrm{cb}}^\bullet(H; \mathbb{R}) \to \mathrm{C}_{\mathrm{cb}}^\bullet(\Gamma; \mathrm{L}^\infty(X; \mathbb{R})) \ , \quad \rho_i^\bullet(\sigma, f)(\psi)(\gamma_0, \ldots, \gamma_\bullet)(x) :=$$

$$= \psi(\sigma(\gamma_0^{-1}, x)^{-1} f(\gamma_0^{-1} x), \ldots, \sigma(\gamma_i^{-1}, x)^{-1} f(\gamma_i^{-1} x), \sigma(\gamma_{i+1}^{-1}, x)^{-1}, \ldots, \sigma(\gamma_\bullet^{-1}, x)^{-1}) \ ,$$

we can notice that $\rho_{-1}^\bullet(\sigma, f) = \mathrm{C}_{\mathrm{b}}^\bullet(\sigma)$. As noticed in the proof of [Mon01, Lemma 8.7.2], one gets

$$s_i^{\bullet+1} \delta_j^\bullet = \begin{cases} \delta_j^{\bullet-1} s_{j-1}^\bullet \ , & j < i \\ \rho_{i-1}^\bullet \ , & j = i \\ \rho_i^\bullet \ , & j = i+1 \\ \delta_{j-1}^{\bullet-1} s_j^\bullet \ , & j > i \end{cases} \tag{2.20}$$

where we have decomposed $\delta^\bullet : \mathrm{C}_{\mathrm{cb}}^\bullet(H; \mathbb{R}) \to \mathrm{C}_{\mathrm{cb}}^{\bullet+1}(H; \mathbb{R})$ as $\delta^\bullet = \sum_{j=0}^{n} (-1)^j \delta_j^\bullet$ and $\delta_j^\bullet(\psi)(h_0, \ldots, h_{\bullet+1}) := \psi(h_0, \ldots, h_{j-1}, h_{j+1}, \ldots, h_{\bullet+1})$ (notice that, with a slight abuse of notation, we are using $\delta^\bullet$ for both the coboundary operators on bounded functions on H and on Γ). By Equation (2.20) we obtain

$$\begin{aligned} s^{\bullet+1}(\sigma, f)\delta^\bullet &= -\delta^{\bullet+1} s^\bullet(\sigma, f) + \sum_{i=0}^{\bullet} (\rho_{i-1}^\bullet(\sigma, f) - \rho_i^\bullet) = \\ &= -\delta^{\bullet+1} s^\bullet(\sigma, f) + \mathrm{C}_{\mathrm{b}}^\bullet(\sigma) - \rho_\bullet^\bullet(\sigma, f) \, . \end{aligned}$$

It is immediate to notice that on the subcomplex of the H-invariants cochains it holds that

$$\rho_\bullet^\bullet(\sigma, f) = \mathrm{C}_{\mathrm{b}}^\bullet(\sigma^f) \ ,$$

and hence on the invariant subcomplex we get that

$$s^{\bullet+1}(\sigma, f)\delta^\bullet + \delta^\bullet s^\bullet(\sigma, f) = \mathrm{C_b}^\bullet(\sigma) - \mathrm{C}_{\mathrm{b}}^\bullet(\sigma^f) \ .$$

This concludes the proof. $\square$

As pointed out above, the pullback defined by Equation (2.18) get back a class in $\mathrm{H}^\bullet_{\mathrm{b}}(\Gamma;\mathrm{L}^\infty(X;\mathbb{R}))$, whereas the standard pullback along representations provides a class in the real cohomology of Γ. To get an analogous map for cocycles, we proceed with the following construction.

We define the *integration map*

$$\begin{aligned}\mathrm{I}^\bullet_X : \mathrm{C}^\bullet_{\mathrm{b}}(\Gamma;\mathrm{L}^\infty(X;\mathbb{R}))^\Gamma \to \mathrm{C}^\bullet_{\mathrm{b}}(\Gamma;\mathbb{R})^\Gamma\,, \\ \psi \mapsto \mathrm{I}^\bullet_X(\psi)(\gamma_0,\dots,\gamma_\bullet) := \int_X \psi(\gamma_0^{-1},\dots,\gamma_\bullet^{-1})(x)d\mu_X(x)\end{aligned} \tag{2.21}$$

that is a well-defined cochain map [Sav20, Lemma 2.7] and hence, by composing with the function defined in Equation (2.18), we get the following map at a cohomological level

$$\begin{aligned}\mathrm{H}^\bullet_{\mathrm{b}}(\sigma^X) : \mathrm{H}^\bullet_{\mathrm{cb}}(H;\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(\Gamma;\mathbb{R})\ , \\ \mathrm{H}^\bullet_{\mathrm{b}}(\sigma^X)([\psi]) := [\mathrm{I}^\bullet_X \circ \mathrm{C}^\bullet_{\mathrm{b}}(\sigma)(\psi)]\,.\end{aligned} \tag{2.22}$$

We now move to the implementation of the pullback through boundary maps. In particular we suppose the existence of a σ-equivariant measurable function $\phi : B \times X \to Y$ for a cocycle $\sigma : \Gamma \times X \to H$, where B is a Γ-boundary in the sense of Definition 2.4.11 and Y is a Lebesgue H-space. In this setting we can naturally define a map at the level of cochains as

$$\begin{aligned}\mathrm{C}^\bullet(\Phi) : \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^H \to \mathrm{L}^\infty_{\mathrm{w}^*}(B^{\bullet+1};\mathrm{L}^\infty(X;\mathbb{R}))^\Gamma\,, \\ \mathrm{C}^\bullet(\Phi)(\psi)(b_0,\dots,b_\bullet)(x) := \psi(\phi(b_0,x),\dots,\phi(b_\bullet,x))\end{aligned} \tag{2.23}$$

for every $\psi \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^H$ and almost every $(b_0,\dots,b_\bullet) \in B^{\bullet+1}$ and $x \in X$. Since the above function is a well-defined chain map and it does not increase the norm [MS20, Lemma 4.2], it induces maps at the level of cohomology groups

$$\mathrm{H}^k(\Phi) : \mathrm{H}^k(\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^H) \to \mathrm{H}^k_{\mathrm{b}}(\Gamma;\mathrm{L}^\infty(X;\mathbb{R})) \tag{2.24}$$

for every $k \geq 0$. We notice that we are tacitly post-composing by the isomorphism provided by Theorem 2.2.16.

An immediate application of [BM02, Proposition 1.5.2] implies the following

Lemma 2.4.20. *The following diagram commutes*

$$\begin{tikzcd}\mathrm{H}^k(\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^H) \arrow[r, "\mathfrak{c}^k"] \arrow[d, "\mathrm{H}^k(\Phi)"] & \mathrm{H}^k_{\mathrm{cb}}(H;\mathbb{R}) \arrow[dl, "\mathrm{H}^k_{\mathrm{b}}(\sigma)"] \\ \mathrm{H}^k_{\mathrm{b}}(\Gamma;\mathrm{L}^\infty(X;\mathbb{R})) & \end{tikzcd} \tag{2.25}$$

for every $k \geq 0$.

Proof. Consider the resolutions $(\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R}),\delta^\bullet)$ and $(\mathrm{C}^\bullet_{\mathrm{cb}}(H;\mathbb{R}),\delta^\bullet)$ of Banach H-modules $\mathbb{R}$. The first one is strong by [Mon01, Lemma 7.5.5], while the second one is a strong resolution by relatively injective modules by Theorem 2.2.16. Hence, by [BM02, Proposition 1.5.2], there exists a cochain map

$$\alpha^\bullet : \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R}) \to \mathrm{C}^\bullet_{\mathrm{cb}}(H,\mathbb{R})$$

that is unique up to homotopy. Similarly, the resolutions $(\mathrm{L}^\infty_{\mathrm{w}^*}(B^{\bullet+1};\mathrm{L}^\infty(X;\mathbb{R})),\delta^\bullet)$ and $(\mathrm{C}^\bullet_{\mathrm{b}}(\Gamma,\mathrm{L}^\infty(X;\mathbb{R})),\delta^\bullet)$ are both strong resolutions by relatively injective modules, hence there exists a homotopy equivalence of complexes

$$\beta^\bullet : \mathrm{L}^\infty_{\mathrm{w}^*}(B^{\bullet+1};\mathrm{L}^\infty(X;\mathbb{R})) \to \mathrm{C}^\bullet_{\mathrm{b}}(\Gamma,\mathrm{L}^\infty(X;\mathbb{R})).$$

Hence the maps

$$\beta^\bullet \circ \mathrm{C}^\bullet(\Phi)\,,\ \mathrm{C}^\bullet_{\mathrm{b}}(\sigma)\circ\alpha^\bullet : \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R}) \to \mathrm{C}^\bullet_{\mathrm{b}}(\Gamma;\mathrm{L}^\infty(X;\mathbb{R}))\,,$$

where we tacitly consider the restriction to the H-invariant cochain for $\alpha^\bullet$ and to the Γ-invariant ones for $\beta^\bullet$, are cochain maps which extends the inclusion of coefficients $\mathbb{R}\to\mathrm{L}^\infty(X;\mathbb{R})$. Again by [BM02, Proposition 1.5.2], they must be homotopic, and in particular the induced maps in cohomology coincide, namely

$$\mathrm{H}^k(\Phi) = \mathrm{H}^k_{\mathrm{b}}(\sigma)\circ\mathfrak{c}^k$$

for every $k\geq 0$ where, as usual, we omit the isomorphisms given by Theorem 2.2.17. $\square$

In other words we have obtained an extension of the work by Burger and Iozzi in [BI02] in the context of measurable cocycles, writing down an explicit formula for the pullback along cocycles through boundary maps.

We conclude with a boundary version also for the map defined by Equation (2.22). When $\sigma:\Gamma\times X\to H$ is a measurable cocycle, B is a Γ-boundary and $\phi: B\times X\to Y$ is a σ-equivariant measurable map, we define the following map at a cohomological level

$$\begin{gathered}\mathrm{H}^k(\Phi^X):\mathrm{H}^k(\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^H)\to\mathrm{H}^k_{\mathrm{b}}(G;\mathbb{R})\\ \mathrm{H}^\bullet_{\mathrm{b}}(\Phi^X)([\psi]) := [\mathrm{I}^\bullet_X\circ\mathrm{C}^\bullet(\Phi)(\psi)]\,.\end{gathered}\tag{2.26}$$

Also for the integrated version of the pullback we lead to an analogous commutative diagram

$$\begin{tikzcd}\mathrm{H}^k(\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^H) \arrow[r, "\mathfrak{c}^k"] \arrow[d, "\mathrm{H}^k(\Phi^X)"] & \mathrm{H}^k_{\mathrm{cb}}(H;\mathbb{R}) \arrow[dl, "\mathrm{H}^k_{\mathrm{b}}(\sigma^X)"] \\ \mathrm{H}^k_{\mathrm{b}}(\Gamma;\mathbb{R}) & \end{tikzcd}\tag{2.27}$$

for every $k \geq 0$.

Remark 2.4.21. Even if we have described two different versions of the pullback that by Lemma 2.4.20 actually coincide up to compose with the canonical map of Equation (2.9), we prefer not to distinguish them with two denominations. When the notion of pullback will bring up, we will regard to rule out any possible ambiguity. In particular, in Chapter 4 and in Chapter 5 we will use the integrated version (Equation (2.22) and Equation (2.26)), while in Chapter 6 we will exploit the one defined by Equation (2.19) and Equation (2.24).

2.5 CAT(0)-spaces and measurable fields

Since one of our main goal is to investigate the existence of boundary maps for measurable cocycles, the final preliminary section is devoted to introduce some tools that we will need in Chapter 3, in particular to prove Theorem 1. Precisely, we first introduce the basics of *CAT(0)-spaces*, the notion of *telescopic dimension* and the *Euclidean De Rham decomposition*. Then we define *measurable fields* and some results as a measurable versions of the decomposition into Euclidean and non-Euclidean factors and the measurable Adam–Ballmann theorem. For details about the first part we refer to the book of Bridson and Haefliger [BH99, Part II] or to the paper of Caprace and Lytchak [CL09], while for the second part we refer to the work of Anderegg and Henry [AH14] and of Bader, Duchesne and Lecureux [Duc12, BDL16].

2.5.1 CAT(0)-spaces

We first recall basic definitions and known facts about CAT(0)-space.

Definition 2.5.1. A metric space $(\mathcal{X}, d)$ is a *CAT(0)-space* if it is geodesic and for every triple of distinct points $x, y, z \in \mathcal{X}$, given a point m in the geodesic segment between y and z, the following inequality holds

$$d(x,m)^2 \leq \frac{1}{2}(d(x,y)^2 + d(x,z)^2) - \frac{1}{4}d(y,z)^2 .$$

A complete CAT(0)-space is also called a *Hadamard space*.

Since embedded flats into CAT(0)-spaces play an important role in the study of their geometry, we recall the following decomposition into Euclidean and non-Euclidean factors. Precisely, the *Euclidean De Rham decomposition* of a CAT(0)-space $\mathcal{X}$ is its canonical isometric splitting into an Hilbert space H and a factor Z which cannot be further decomposed as a product of a non-trivial Euclidean factor

[BH99, Theorem 6.15]. Moreover, for every point $x \in \mathcal{X}$ the spaces H (respectively Z) identifies with a unique closed convex subspaces of $\mathcal{X}$ containing x.

Given a subset $\mathcal{Y} \subset \mathcal{X}$ of a metric space, its *diameter* is defined as

$$\operatorname{diam}(\mathcal{Y}) \coloneqq \sup_{x,y \in \mathcal{Y}} d(x,y),$$

and $\mathcal{Y}$ is said to be *bounded* if it has finite diameter. A convex bounded set $\mathcal{Y}$ has some preferred points called *circumcenters*, which are the centers of balls of minimal radius containing $\mathcal{Y}$. Notice that, without the assumption of convexity, one can still give the notion of circumcenter but such points may not belong to $\mathcal{Y}$. An equivalent definition can be given in terms of actions of isometries. Precisely, the circumcenters of a bounded subset $\mathcal{Y} \subset \mathcal{X}$ of a generic metric space are the points fixed by any isometry stabilizing $\mathcal{Y}$. A peculiarity of CAT(0)-spaces is that every bounded subset has a unique circumcenter, which we call *center*. This fact follows from a more general property of CAT(κ)-spaces, see [BH99, Proposition 2.7] for details.

Before introducing the notion of telescopic dimension, we need the one of *geometric dimension*. This concept was first introduced by Kleiner [Kle99] in terms of the space of directions at each point, and then has been reformulated by Caprace and Lytchack [CL09, Theorem 1.3] in the following way. If $\mathcal{X}$ is a CAT(0)-space, then its geometric dimension is $\leq n$ if for each subset $\mathcal{Y}$ of finite diameter the following inequality holds

$$\operatorname{rad}(\mathcal{Y}) \leq \sqrt{\frac{n}{2(n+1)}} \operatorname{diam}(\mathcal{Y}),$$

where $\operatorname{rad}(\mathcal{Y})$ is the *circumradius* of $\mathcal{Y}$, namely the infimum of all positive numbers r such that $\mathcal{Y}$ is contained in some closed ball of radius r. The result by Caprace and Lytchack leads to a characterization of telescopic dimension, originally given by [Kle99], that we assume here as a definition (refer to [CL09] for more details).

Definition 2.5.2. A CAT(0)-space $\mathcal{X}$ has *telescopic dimension* $\leq n$ if for any $\delta > 0$ there exists some constant $D > 0$ such that for every bounded set of $\mathcal{Y}$ of diameter $> D$, we have

$$\operatorname{rad}(\mathcal{Y}) \leq \left(\delta + \sqrt{\frac{n}{2(n+1)}} \right) \operatorname{diam}(\mathcal{Y}).$$

Remark 2.5.3. The Hermitian symmetric spaces of the form $\mathcal{X}(p,q)$ with $p \leq q$ introduced in Section 2.3, are CAT(0)-spaces of telescopic dimension p [Duc12, Corollary 1.4]. In particular, this implies that the visual boundary $\partial\mathcal{X}(p,q)$ has geometric dimension $p-1$ [CL09, Proposition 2.1].

For a complete CAT(0)-space $\mathcal{X}$ with finite telescopic dimension, Caprace and Lytchak proved that every filtering family of closed convex subspaces of $\mathcal{X}$ either intersects at $\mathcal{X}$ or at $\partial\mathcal{X}$ [CL09, Theorem 1.1]. Notice that this is equivalent to quasi-compactness of the space $\overline{\mathcal{X}} = \mathcal{X} \cup \partial\mathcal{X}$ endowed with the topology defined by Monod in [Mon06, Section 3.7]. The following technical result is an example of application of [CL09, Theorem 1.1], and it turned out to be the useful in the proof of [BDL16, Theorem 1.1] and [DLP21, Theorem 1.9]. It will be exploited to prove Theorem 1.

Proposition 2.5.4 ([BDL16, Proposition 2.1])**.** *Let E be an Euclidean space and $f : E \to \mathbb{R}$ be a convex function. If we denote by $m = \inf\{f(x) \,|\, x \in E\}$, then we have the following four possible cases:*

(i) If m is not attained, then $\bigcap_{\epsilon>0} \partial E_\epsilon \neq \emptyset$ where $E_\epsilon \coloneqq f^{-1}((m, m+\epsilon))$ is not empty and has a center.

If m is attained, we denote by $E_m = f^{-1}(m)$ and by $E_m = F \times T$ its Euclidean De Rham decomposition. Then one of the following holds

(ii) E_m is bounded and thus it has a center;

(iii) T is bounded and $\partial E_m = \partial F$ is a sphere;

(iv) T is not bounded and $\partial T \subset \partial E$ has radius less then $\frac{\pi}{2}$.

Notice that, as mentioned in point *(iii)*, boundaries of flats are Euclidean spheres, that can be also interpreted as CAT(1)-spaces. In particular, boundaries of maximal flats are subcomplexes called apartments of the building structure of the visual boundary. Since we will not directly use such construction, we refer to [AB08] for the general theory of such building. We only notice that the existence of circumcenters for bounded subsets [BH99, Proposition 2.7] holds also in this case. More precisely, every subset of radius at most $\frac{\pi}{2}$ in a sphere has a center, and this property will be used in the proof of Theorem 1.

2.5.2 Measurable fields of CAT(0)-spaces and the Adam–Ballmann dichotomy

In this section we introduce measurable fields of CAT(0)-spaces and some results that we will exploit in the next section to prove the existence of boundary maps for cocycles.

Definition 2.5.5. Given a standard probability space (Ω, μ_Ω), a *measurable field of CAT(0)-spaces* is a collection of CAT(0)-spaces $\mathbf{X} = \{X_\omega\}_{\omega\in\Omega}$ together with a countable family $\mathcal{F} \subset \prod_{\omega\in\Omega} X_\omega$ such that

- for all $x, y \in \mathcal{F}$ the map $\omega \mapsto d_\omega(x_\omega, y_\omega)$ is measurable;
- for almost every $\omega \in \Omega$, the set $\{f_\omega \,|\, f \in \mathcal{F}\}$ is dense in X_ω.

A *section* $\mathbf{X}$ is an element $x \in \prod_{\omega\in\Omega} X_\omega$ such that, for every $y \in \mathcal{F}$ the map $\omega \mapsto d_\omega(x_\omega, y_\omega)$ is measurable.
A *subfield* $\mathbf{Y}$ of $\mathbf{X}$ is a collection of non-empty closed convex subsets $Y_\omega \subset X_\omega$ such that, for every section x of $\mathbf{X}$ the map $\omega \mapsto d_\omega(x_\omega, Y_\omega)$ is measurable.

If G is a locally compact group and Ω is a G-space, a G-action on $\mathbf{X}$ is the datum of a collection $\{\sigma(g,\omega)\}_{g\in G,\omega\in\Omega}$ where

- for every $g \in G$ and almost every $\omega \in \Omega$, we have $\sigma(g,\omega) \in \mathrm{Isom}(X_\omega, X_{g\omega})$;
- for every $g, h \in G$ and almost every $\omega \in \Omega$, the following equality holds
$$\sigma(gh,\omega) = \sigma(g, h\omega)\sigma(h,\omega); \tag{2.28}$$
- for every $x, y \in \mathcal{F}$, the map $(g,\omega) \mapsto d(x_\omega, \sigma(g, g^{-1}\omega)y_{g^{-1}\omega})$ is measurable.

Remark 2.5.6. The Equation (2.28) might remind the reader to the cocycle condition of Definition 2.4.1. In fact, a cocycle $\sigma : \Gamma \times X \to \mathcal{X}$ into a CAT(0)-space $\mathcal{X}$ naturally defines a Γ-action for example on the constant field $\mathbf{Y}$ where $\Omega = X$ and $Y_x = \mathcal{X}$ for every $x \in X$.

A G-action $\{\sigma(g,\omega)\}_{g\in G,\omega\in\Omega}$ on a measurable field $\mathbf{X}$ induces a natural G-action on every subfield $\mathbf{Y}$ by $g\mathbf{Y} = \{\sigma(g, g^{-1}\omega)Y_{g^{-1}\omega}\}_\omega$. Similarly, if $\partial\mathbf{X}$ denotes the boundary field of $\mathbf{X}$, namely the field consisting of the boundaries of each X_ω, a G-action on $\mathbf{X}$ induces an action on the set of sections of $\partial\mathbf{X}$ defined as $(g\xi)_\omega = \{\sigma(g, g^{-1}\omega)\xi_{g^{-1}\omega}\}$.

As proved by Caprace and Lytchak [CL09, Proposition 1.8], any isometric action of a locally compact group on a complete CAT(0)-space of finite telescopic dimension either has a fixed at infinity or admits an invariant non-empty closed convex subset which is minimal, namely it does not contain a proper subset with the same properties. This allows to reduce the investigation of existence of boundary maps to minimal actions, since the boundary of a closed convex subset naturally embeds into $\mathcal{X}$ (see [Duc12, Theorem 1.7] and [BDL16, Theorem 1.1]). The following result can be see as the generalization of [CL09, Proposition 1.8] to measurable fields and will be our starting point in the proof of Theorem 2.

Proposition 2.5.7 ([Duc12, Proposition 8.11]). *Suppose* $\mathbf{X}$ *is a measurable field of CAT(0)-spaces of finite telescopic dimension,* G *acts on* $\mathbf{X}$ *and* Ω *is* G*-ergodic. Then there exists a minimal invariant subfield of* $\mathbf{X}$ *or there exists an invariant section of* $\partial\mathbf{X}$*.*

A second construction that we will use is the extension of the Euclidean De Rham decomposition for measurable fields of CAT(0)-spaces.

Proposition 2.5.8 ([Duc12, Proposition 9.2]). *Let* x *be a section of a measurable field* $\mathbf{X}$*. There exists* $n \in \mathbb{N}$ *and two subfields* $\mathbf{E}$ *and* $\mathbf{Y}$ *of* $\mathbf{X}$ *containing* x *such that* $\mathbf{X} = \mathbf{E} \times \mathbf{Y}$ *and* $E_\omega \cong \mathbb{R}^n$ *for almost every* $\omega \in \Omega$*. Moreover,* $\mathbf{E}$ *is maximal for those properties.*

If y *is an other section of* $\mathbf{X}$ *and* $\mathbf{X} = \mathbf{E'} \times \mathbf{Y'}$ *is another such decomposition associated to* y *then for almost every* $\omega \in \Omega$*, the projections* $\pi_{E_\omega|E'_\omega}$ *and* $\pi_{Y_\omega|Y'_\omega}$ *are isometries.*

In particular, the G*-action* $\{\sigma(g,\omega)\}_{g\in G,\omega\in\Omega}$ *on* $\mathbf{X}$ *splits as*

$$\sigma(g,\omega) = \sigma_{\mathbf{E}}(g,\omega) \times \sigma_{\mathbf{Y}}(g,\omega)$$

where $\{\sigma_{\mathbf{E}}(g,\omega)\}_{g\in G,\omega\in\Omega}$ *and* $\{\sigma_{\mathbf{Y}}(g,\omega)\}_{g\in G,\omega\in\Omega}$ *are respectively actions on* $\mathbf{E}$ *and* $\mathbf{Y}$*.*

The following measurable version of the Adam–Ballmann dichotomy [AB98] is a crucial result to construct boundary maps.

Theorem 2.5.9 ([Duc12, Theorem 1.8]). *Let* G *be a locally compact second countable group and* Ω *an ergodic and amenable* G*-space. Let* X *be a measurable field of complete CAT(0)-spaces of finite telescopic dimension. If* G *acts on* $\mathbf{X}$ *then there is an invariant section of the boundary field* $\partial\mathbf{X}$ *or there exists an invariant Euclidean subfield of* $\mathbf{X}$*.*

We conclude by showing how the previous result relates measurable fields and boundaries (see Definition 2.4.11). Precisely, the way to organize the components of a measurable field into a standard Borel space with a fiberwise isometric Γ-action is explained by the following

Lemma 2.5.10 ([DLP21, Lemma 4.11]). *Let* Γ *be a countable group and let* $\mathbf{X}$ *be a measurable field over a Lebesgue* Γ*-space* Ω*. Then there exists a full-measure subset* $\Omega_0 \subset \Omega$*, a standard Borel structure on* $X := \bigsqcup_{\omega\in\Omega_0} X_\omega$ *and a Borel map* $p : X \to \Omega_0$ *that admits a fiberwise isometric* Γ*-action. Moreover,* $p^{-1}(\omega)$ *is* X_ω *with the metric* d_ω*.*

In virtue of Lemma 2.5.10, an invariant section of the boundary field $\partial\mathbf{X}$ of a measurable field $\mathbf{X}$ on the space $B \times X$ is a boundary map in the sense of Definition 2.4.16, and in fact this will be our fundamental tool in the proof of Theorem 1.

Chapter 3

Boundary maps

It is clear from Section 2.4.3 the relevance of boundary maps in the theory of numerical invariants, since they provide a useful implementation of the pullback along cocycles. This chapter is devoted to prove the existence of a boundary map for certain families of cocycles.

In the first part we prove Theorem 1, namely the existence of a boundary map for non-elementary cocycles into CAT(0)-spaces of finite telescopic dimension. To this end, we exploit the geometric properties of CAT(0)-spaces introduced in Section 2.5.1 and the measurable fields defined in Section 2.5.2. For the original version of this result refer to [SS21a].

In the second part we deal with cocycles into algebraic groups. Precisely we consider Zariski dense cocycles into simple Lie groups of non-compact type and, by adapting the argument used in [BF14a, Theorem 5.3], we prove Theorem 2. This part is contained in [SS21c].

Finally we prove some properties of boundary maps. First we show that for Zariski dense cocycles into $\mathrm{PU}(p,q)$ boundary maps must have Zariski dense slices. Then we prove that, for cocycles into groups of Hermitian type, almost every pair of points in the image of a boundary map into the Shilov boundary consist of opposite points.

3.1 Boundary maps for cocycles into CAT(0)-spaces

We consider cocycles $\sigma : \Gamma \times X \to H$ where Γ is a countable group, (X, μ_X) is an ergodic standard Borel probability Γ-space and H is the isometry group of a CAT(0)-space of finite telescopic dimension. Moreover, we assume that the σ-

action on $\mathcal{X}$ does not preserve a family of flats, which is the natural extension of non-elementarity for representations.

Definition 3.1.1. Let $\sigma : \Gamma \times X \to H$ where H is the isometry group of a complete CAT(0) space $\mathcal{X}$. We say that σ is *non-elementary* if there exists no σ-equivariant family of Euclidean subspaces of $\mathcal{X}$ and no σ-equivariant family of points in $\partial\mathcal{X}$.

We are now ready to prove

Theorem 1. *Let Γ be a locally compact second countable group, (X, μ_X) be an ergodic standard Borel probability Γ-space and B a Γ-boundary. For every non-elementary cocycle $\sigma : \Gamma \times X \to H$ into the isometry group of a CAT(0)-space $\mathcal{X}$ of finite telescopic dimension there exists a boundary map $\phi : B \times X \to \partial\mathcal{X}$.*

Proof. We consider the constant field $\mathbf{X} = \{\mathcal{X}_x\}_{x \in X}$ endowed with the Γ-action defined by σ. We first notice that by Proposition 2.5.7 either we have a minimal subfield $\mathbf{Y} \subset \mathbf{X}$ or there exists a section of $\partial\mathbf{X}$. Since the last one is ruled out by non-elementary, we can assume the existence of a minimal subfield.

According to Proposition 2.5.8, we consider the Euclidean De Rham decomposition $\mathbf{Y} = \mathbf{E} \times \mathbf{Z}$ and we denote by $\sigma_{\mathbf{Z}}$ and $\sigma_{\mathbf{Y}}$ the Γ-actions induced respectively on $\mathbf{Z}$ and $\mathbf{Y}$. By ergodicity we have that one of the following options is verified for almost every $x \in X$: either $\operatorname{diam}(Z_x) \leq \frac{\pi}{2}$ or not. In the first case we can denote by z_x the center of Z_x, whose existence is ensured by [BL05, Proposition 1.4] and by considering the σ-equivariant family $\{E_x \times \{z_x\}\}_{x \in X}$ we get a contradiction to the hypothesis of non-elementarity. Hence we assume that almost all the Z_x's have diameter greater than $\frac{\pi}{2}$.

We claim that the Γ-action $\sigma_{\mathbf{Z}}$ on $\mathbf{Z}$ is minimal and non-elementary. Before proving the claim, notice that this implies that it is sufficient to find an invariant section of $\mathbf{Z}$, since the boundaries ∂Z_x's are contained in the ∂Y_x's and hence in $\partial\mathcal{X}$.

Assume that $\mathbf{Z}$ is not $\sigma_{\mathbf{Z}}$-minimal. Hence by Proposition 2.5.7 there exists a minimal invariant subfield $\mathbf{W} \subset \mathbf{Z}$ whose product with $\mathbf{E}$ is a strict subfield of $\mathbf{E} \times \mathbf{Z} = \mathbf{Y}$, contradicting the minimality of $\mathbf{Y}$. Similarly, a $\sigma_{\mathbf{Z}}$-equivariant family of flats $\{F_x\}$ would produce a σ-invariant family of flats in $\mathcal{X}$, which is ruled out by non-elementarity of σ.

Hence it remains to prove the existence of an invariant section of the boundary field $\partial\mathbf{Z}$, where $\mathbf{Z}$ has trivial Euclidean factor and it is endowed with a minimal action $\{\sigma_{\mathbf{Z}}(\gamma, x)\}$.

We consider the measurable field $\mathbf{U} = \{U_{\xi,x}\}_{\xi,x \in B \times X}$ where $U_{\xi,x} = Z_x$ for every pair $(\xi, x) \in B \times X$. Recall that by Proposition 2.4.13 the spaces $B \times X$ and $B \times B \times X$ are ergodic Γ-spaces. By [Zim84, Proposition 4.3.4] $B \times X$ is also Γ-amenable. In this context we apply [Duc12, Theorem 2.5.9] and we have two possible cases: either there exists a section of $\partial\mathbf{U}$ or there exists an invariant Euclidean subfield $\mathbf{E} \subset \mathbf{U}$. Since in the first case we are done, we need to rule out the second one.

We consider the distance map

$$d : B \times B \times X \to \mathbb{R}, \quad (\xi_1, \xi_2, x) \mapsto d(E_{\xi_1,x}, E_{\xi_2,x}) := \inf_{y \in E_{\xi_1,x}} d(y, E_{\xi_2,x})$$

where the $E_{\xi,x}$'s are the sheets of the Euclidean subfield $\mathbf{E} = \{E_{\xi,x}\}_{(\xi,x)\in B\times X}$. Following [BDL16], we have four possible cases, and by ergodicity one of them must occur almost surely. Moreover, again by ergodicity, the distance map is essentially equal to some value, say d_0, for almost every $x \in X$ and $\xi_1, \xi_2 \in B$.

Case (i): Suppose that d_0 it is not attained for almost every $x \in X$ and $\xi_1, \xi_2 \in B$. Hence for almost every $x \in X$ and $\xi_1 \in B$ we can define the subspaces

$$E^n_{\xi_1,\xi_2,x} := \left\{ y \in E_{\xi_1,x} \,|\, d(y, E_{\xi_2,x}) < d_0 + \frac{1}{n} \right\}$$

which are nested subspaces of $E_{\xi_1,x}$. By [Duc12, Proposition 8.10] we have a σ-equivariant map

$$\psi : B \times B \times X \to \bigcup \partial E_{\xi,x} \subset \partial\mathbf{E},$$

where we are considering the measurable field $\{E'_{\xi_1,\xi_2,x}\}_{(\xi_1,\xi_2,x)\in B\times B\times X}$ such that $E'_{\xi_1,\xi_2,x} = E_{\xi_1,x}$ for every $x \in X$ and $\xi_1, \xi_2 \in B$. It follows directly from Lemma 2.5.10 that the projection p of (a full-measure subset of) $\bigcup \partial E'_{\xi_1,\xi_2,x}$ on $B \times X$ has a Γ-fiberwise isometric action, so that we can apply relative metric ergodicity to the following diagram

$$\begin{tikzcd}
B \times B \arrow[r, "\Psi"] \arrow[d, "\pi_1"] & \mathrm{Meas}(X, \partial\mathbf{E}) \arrow[d, "p_X"] \\
B \arrow[r, "j"] & \mathrm{Meas}(X, B \times X).
\end{tikzcd}$$

Here Ψ and j are induced respectively by ψ (namely $\Psi(\xi_1, \xi_2)(x) := \psi(x, \xi_1, \xi_2)$) and by the inclusion of constants (namely $j(\xi)(x) := (\xi, x)$), while π_1 is the projection on the first factor and p_X is defined as $p_X(f)(x) := p(f(x))$. The Γ-action on $\mathrm{Meas}(X, \partial\mathbf{E})$ is the one induced by σ, precisely

$$(\gamma f)(\cdot) := \sigma(\gamma, \gamma^{-1}\cdot) f(\gamma^{-1}\cdot) \tag{3.1}$$

for any $\gamma \in \Gamma$ and $f \in \mathrm{Meas}(X, \partial\mathbf{E})$. By applying the cocycle condition of Equation (2.13) we have that for almost every $\gamma_1, \gamma_2 \in \Gamma$

$$\begin{aligned}
(\gamma_1\gamma_2 f)(\cdot) &= \sigma(\gamma_1\gamma_2, \gamma_2^{-1}\gamma_1^{-1}\cdot) f(\gamma_2^{-1}\gamma_1^{-1}\cdot) \\
&= \sigma(\gamma_1, \gamma_2\gamma_2^{-1}\gamma_1^{-1}\cdot)\sigma(\gamma_2, \gamma_2^{-1}\gamma_1^{-1}\cdot) f(\gamma_2^{-1}\gamma_1^{-1}\cdot) \\
&= \sigma(\gamma_1, \gamma_1^{-1}\cdot)\sigma(\gamma_2, \gamma_2^{-1}\gamma_1^{-1}\cdot) f(\gamma_2^{-1}\gamma_1^{-1}\cdot) = \gamma_1(\gamma_2 f)(\cdot)\,.
\end{aligned}$$

Moreover, since ψ is σ-equivariant, we have that

$$\begin{aligned}\Psi(\gamma\xi_1, \xi_2)(\cdot) &= \psi(\gamma\xi_1, \gamma\xi_2, \cdot)\\ &= \psi(\gamma\xi_1, \gamma\xi_2, \gamma\gamma^{-1}\cdot)\\ &= \sigma(\gamma, \gamma^{-1}\cdot)\psi(\xi_1, \xi_2, \gamma^{-1}\cdot)\\ &= \sigma(\gamma, \gamma^{-1}\cdot)\Psi(\xi_1, \xi_2)(\gamma^{-1}\cdot)\end{aligned}$$

and hence Ψ is Γ-equivariant.

Concerning the projection p_X, we can equip it with a fiberwise isometric Γ-action as follows. For any $x \in X$, we denote by $d_{h(x)}$ the metric on the leaf $\partial E_{h(x)}$ of the field $\partial\mathbf{E}$ and we define a metric on p_X as

$$d(f, g) \coloneqq \int_X \frac{d_{h(x)}(f(x), g(x))}{1 + d_{h(x)}(f(x), g(x))} d\mu_X(x)$$

for every $f, g \in p_X^{-1}(h) \subset \mathrm{Meas}(X, \partial\mathbf{E})$. Hence, for every $\gamma \in \Gamma$ we have

$$\begin{aligned}d(\gamma f, \gamma g) &= \int_X \frac{d_{\gamma h(x)}((\gamma f)(x), (\gamma g)(x))}{1 + d_{\gamma h}((\gamma f)(x), (\gamma g)(x))} d\mu_X(x)\\ &= \int_X \frac{d_{\gamma h(x)}(\sigma(\gamma, \gamma^{-1}x)f(\gamma^{-1}x), \sigma(\gamma, \gamma^{-1}x)g(\gamma^{-1}x))}{1 + d_{\gamma h}(\sigma(\gamma, \gamma^{-1}x)f(\gamma^{-1}x), \sigma(\gamma, \gamma^{-1}x)g(\gamma^{-1}x))} d\mu_X(x)\\ &= \int_X \frac{d_h(f(\gamma^{-1}x), g(\gamma^{-1}x))}{1 + d_h(f(\gamma^{-1}x), g(\gamma^{-1}x))} d\mu_X(x)\\ &= \int_X \frac{d_h(f(x), g(x))}{1 + d_h(f(x), g(x))} d\mu_X(x) = d(f, g)\ ,\end{aligned}$$

where we used definition of the metrics on the fibers and the Γ-invariance of the metric d to move from the second line to the third one and we concluded exploiting the Γ-invariance of μ_X.

By relative metric ergodicity we have a lifting $B \to \mathrm{Meas}(X, \partial\mathbf{E})$, thus Ψ does not depend on the second factor. Hence we have a σ-invariant map $B \times X \to \partial\mathbf{E} \subset \partial\mathbf{U}$, whose existence is ruled out by the dichotomy of Theorem 2.5.9.

We can suppose that the distance $d_{\xi_1,\xi_2,x}$ is attained almost surely and we define the non-empty subsets

$$W_{\xi_1,\xi_2,x} \coloneqq \{w \in E_{\xi_1,x} \,|\, d(w, E_{\xi_2,x}) = d_0\} \subset E_{\xi_1,x}.$$

Case (ii): The $W_{\xi_1,\xi_2,x}$'s are bounded. We can associate to any such subset its circumcenter $c_{\xi_1,\xi_2,x}$. The map

$$\psi : B \times B \times X \to \mathbf{E}\,, \quad \psi(\xi_1, \xi_2, x) \coloneqq c_{\xi_1,\xi_2,x}$$

is σ-equivariant and relative metric ergodicity applied to the same diagram of case (i)

$$\begin{array}{ccc} B \times B & \xrightarrow{\Psi} & \mathrm{Meas}(X, \mathbf{E}) \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle p_X} \\ B & \xrightarrow{j} & \mathrm{Meas}(X, B \times X) \end{array}$$

implies that Ψ does not depend on the second factor. If we replace π_1 in the above diagram with the projection $\pi_2 : B \times B \to B$ on the second factor, the same argument shows that Ψ is also independent on the second factor. In other words, we obtain a map $\psi : X \to \mathbf{E}$ such that

$$\psi(\gamma x) = \sigma(\gamma, x)\psi(x).$$

Since points are 0-dimensional flats, this contradicts non-elementarity.

Thus the $W_{\xi_1,\xi_2,x}$'s are not bounded, and we can consider their Euclidean De Rham decomposition

$$W_{\xi_1,\xi_2,x} = F_{\xi_1,\xi_2,x} \times T_{\xi_1,\xi_2,x},$$

where the $F_{\xi_1,\xi_2,x}$'s are maximal Euclidean factors.

Case (iii): If the $T_{\xi_1,\xi_2,x}$'s are not bounded, as in case (i) we realize a map

$$\psi : B \times B \times X \to \partial\mathbf{T}\,, \quad \psi(\xi_1, \xi_2, x) \coloneqq c_{\xi_1,\xi_2,x}$$

where $c_{\xi_1,\xi_2,x}$ is the center of $\partial T_{\xi_1,\xi_2,x}$ and $\mathbf{T}$ denotes the measurable field given by the $T_{\xi_1,\xi_2,x}$'s. Notice that $c_{\xi_1,\xi_2,x}$ can be defined thanks to [BDL16, Proposition 2.1]. Using the same arguments of case (i), we get a contradiction.

Case (iv): Finally, if the $T_{\xi_1,\xi_2,x}$'s are bounded we consider a subfield **E'** of **E** whose sheets are defined by

$$E'_{\xi_1,\xi_2,x} \coloneqq F_{\xi_1,\xi_2,x} \times \{t_{\xi_1,\xi_2,x}\}$$

for every $x \in X$ and $\xi_1, \xi_2 \in B$, where $t_{\xi_1,\xi_2,x}$ is the circumcenter of $T_{\xi_1,\xi_2,x}$. The same argument used in [BDL16] shows that in fact $E_{\xi_1,\xi_2,x} = E_{\xi_1,x}$ for almost every $x \in X$ and $\xi_1, \xi_2 \in B$. Moreover, $E_{\xi,x}$ and $E_{\xi',x}$ are parallel for almost every $x \in X$ and almost every $\xi, \xi' \in B$, which means that $d(E_{\xi,x}, E_{\xi',x}) = 0$ and is denoted as

$$E_{\xi,x} /\!/ E_{\xi',x}\,.$$

By Fubini's theorem there exists an element $\xi_0 \in B$ and a full-measure subset $\Omega \subset B \times X$ such that

$$E_{\xi,x} /\!/ E_{\xi_0,x}$$

for almost every $(\xi, x) \in \Omega$. Let $\Omega^\Gamma \coloneqq \bigcap\limits_{\gamma \in \Gamma} \gamma\Omega$, which is still of full-measure since Γ is countable. We consider the set

$$C_x \coloneqq \text{ convex hull } (\{E_{\xi,x}\}_{(\xi,x)\in\Omega^\Gamma}),$$

which can be decomposed into Euclidean De Rham factors $E_x \times T_x$ such that

$$E_x /\!/ E_{\xi,x} /\!/ E_{\xi_0,x} \tag{3.2}$$

for every $(\xi, x) \in \Omega^\Gamma$.

Moreover, for almost every $x \in X$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}\sigma_{\mathbf{Z}}(\gamma, x)C_x = & \text{ convex hull } (\sigma_{\mathbf{Z}}(\gamma, x)E_{\xi,x})_{(\xi,x)\in\Omega^\Gamma} \\ = & \text{ convex hull } (E_{\gamma\xi,\gamma x})_{(\xi,x)\in\Omega^\Gamma} \\ = & \text{ convex hull } (E_{\xi,\gamma x})_{(\xi,\gamma x)\in\Omega^\Gamma} = C_{\gamma x}\,,\end{aligned}$$

where to pass from the first line to the second one we used the fact that $\mathbf{E}$ is a subfield of $\mathbf{U}$ and to pass from the second line to the third one we exploited the action on Ω^Γ.

Now, by the minimality of $\mathbf{Z}$ we must have $C_x = Z_x$ for almost every $x \in X$ and since Z_x has trivial Euclidean factor, by Equation (3.2) we have

$$\dim(E_{\xi,x}) = 0$$

for every $(\xi, x) \in \Omega^\Gamma$. Hence we have a section $B \times X \to \mathbf{U}$ and, by the same argument used in *case (ii)*, we have a contradiction. □

3.2 Boundary maps for cocycles into algebraic groups

In this section we consider a Zariski dense measurable cocycle $\sigma : \Gamma \times X \to H$, where Γ is a locally compact second countable group, (X, μ_X) is a standard Borel probability Γ-space and H is a simple Lie group of non-compact type. Moreover, we denote by B a Γ-boundary in the sense of Definition 2.4.11.

It is worth noticing that we do not know how to adapt the approach adopted by Burger and Iozzi to prove the existence of boundary maps for representations in the analogous setting [BI04, Proposition 7.2]. The crucial, and for us inaccessible, point is in the notion of mean proximality, which we are not able to adapt in our context (see also [Zim80, Mar91, BI04] for details about this approach). We chose instead the point of view introduced by Bader and Furman in [BF14b]. Here the authors exploited the notion of relative metric ergodicity to show the existence of boundary maps for Zariski dense representations of locally compact second countable groups into connected simple Lie groups [BF14b, Theorem 3.4].

We recall that in the above setting, by [BF14a, Theorem 5.3] there exists an algebraic subgroup $L < H$ and a Γ-equivariant universal map $\phi : X \to H/L$

such that, for any algebraic H-space V and for any Γ-equivariant measurable map $\psi : X \to V$, there exists a Γ-equivariant measurable map $\pi : L/H \to V$ that makes the following diagram commutative

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & H/L \arrow[dl, "\pi"] \\ & V. & \end{tikzcd}$$

This universal property is the fundamental ingredient in the proof of

Theorem 2. *Let Γ be a locally compact and second countable group and let H be a simple Lie group of non-compact type. Let (X, μ_X) be an ergodic standard Borel probability Γ-space and let $\sigma : \Gamma \times X \to H$ be a Zariski dense measurable cocycle. Then, for any Γ-boundary B there exists a σ-equivariant map $\phi : B \times X \to H/P$ where $P < H$ is a minimal parabolic subgroup.*

Proof. Since B is a strong boundary by Remark 2.4.12, by Proposition 2.4.13 both $B \times X$ and $B \times B \times X$ are ergodic Γ-space. Thus we denote by L, L_0 the algebraic subgroups of H and by $\phi : B \times X \to H/L$, $\phi_0 : B \times B \times X \to H/L_0$ the Γ-equivariant universal maps associated respectively to $B \times X$ and to $B \times B \times X$.

Since $B \times X$ is amenable by Proposition 2.1.18, then there exists a σ-equivariant map $\nu : B \times X \to \mathcal{M}^1(H/P)$ where $P < H$ is a minimal parabolic subgroup and $\mathcal{M}^1(H/P)$ is the space of probability measures on H/P.

By ergodicity of Γ on $B \times X$ and by the smooth action of H on $\mathcal{M}^1(H/P)$ [Zim84, Corollary 3.2.23], it follows that the induced map

$$\bar{\nu} : B \times X \to \mathcal{M}^1(H/P)/H$$

is essentially constant. Equivalently, ν has image essentially contained in a single H-orbit, namely we get a map $B \times X \to H/\mathrm{Stab}_H(\mu_0)$ where $\mathrm{Stab}_H(\mu_0)$ is the stabilizer in H of some probability measure $\mu_0 \in \mathcal{M}^1(H/P)$. By [Zim84, Corollary 3.2.23] we have that $\mathrm{Stab}_H(\mu_0)$ is algebraic and amenable. Hence we can exploit the universal property of ϕ, in order to get a Γ-equivariant map $H/L \to H/\mathrm{Stab}_H(\mu_0)$. Thus, up to conjugacy, $L < \mathrm{Stab}_H(\mu_0)$ and moreover, by amenability of $\mathrm{Stab}_H(\mu_0)$, it follows that L is amenable.

Consider now the map $\phi \circ \pi_2$ where $\pi_2 : B \times B \times X \to B \times X$ is the projection on the last two factors. By the universal property of ϕ_0, we get a Γ-equivariant map $\pi : H/L_0 \to H/L$ such that the following diagram commutes

$$\begin{tikzcd} B \times B \times X \arrow[rr, "\phi_0"] \arrow[dr, "\phi\circ\pi_2"'] & & H/L_0 \arrow[dl, "\pi"] \\ & H/L. & \end{tikzcd}$$

Again, up to conjugation, we can assume that $L_0 < L$ and, denoting by

$$R := \mathrm{Rad}_u(L)$$

the unipotent radical of L, we get the chain of inclusion $L_0 < L_0R < L$ and the induced chain of projections $H/L_0 \xrightarrow{p_1} H/L_0R \xrightarrow{p_2} H/L$.

Define now the maps

$$\Phi : B \to \mathrm{Meas}(X, H/L), \quad \Phi(\xi)(\cdot) := \phi(\xi, \cdot)$$

and

$$\Phi_0 : B \times B \to \mathrm{Meas}(X, H/L_0), \quad \Phi_0(\xi_1, \xi_2)(\cdot) := \phi_0(\xi_1, \xi_2, \cdot).$$

Hence, by the same arguments used in the proof of Theorem 1, we have that Φ is Γ-equivariant (where the Γ on $\mathrm{Meas}(X, H/L)$ is the one defined in Equation (3.1). Similarly, one can check the Γ-equivariance of Φ_0.

Consider now the following commutative diagram

$$\begin{tikzcd}
B \times B \arrow[r, "\Phi_0"] \arrow[d, "\mathrm{pr}_2"] & \mathrm{Meas}(X, H/L_0R) \arrow[d, "p_2^X"] \\
B \arrow[r, "\Phi"] \arrow[ur, dotted, "\Psi"] & \mathrm{Meas}(X, H/L)
\end{tikzcd}$$

where $p_2^X(f)(\cdot) := p_2(f(\cdot))$, and $\mathrm{pr}_2 : B \times B \to B$ is the projection on the second factor. We remark that the existence of the map Ψ follows from the fact that p_2^X is fiberwise Γ-isometric and from the relative metric ergodicity of pr_2. In fact, a metric on $\mathrm{Meas}(X, H/L_0R)$ compatible with the Γ-action defined in (3.1), can be found as follows. Let $\mathbf{e} \in \mathrm{Meas}(X, H/L)$ be the constant function $\mathbf{e}(x) := L$. If we denote by d the L/R-invariant metric on L/L_0R cited in the proof of [BF14a, Theorem 3.4], we can set the metric d_0 on the special fiber $(p_2^X)^{-1}(\mathbf{e}) \cong \mathrm{Meas}(X, L/L_0R)$ as

$$d_0(f, g) := \int_X \frac{d(f(x), g(x))}{1 + d(f(x), g(x))} d\mu_X(x)$$

for every $f, g \in \mathrm{Meas}(X, L/L_0R)$. Since the group $\mathrm{Meas}(X, H)$ acts transitively on $\mathrm{Meas}(X, H/L_0R)$, we can move the metric d_0 on the whole $\mathrm{Meas}(X, H/L_0R)$. To show the compatibility of the collection of metrics on the fibers, let $h \in$

$\mathrm{Meas}(X, H/L)$ and let $f, g \in (p_2^X)^{-1}(h)$. Then

$$\begin{aligned} d_{\gamma.h}(\gamma f, \gamma g) = & \int_X \frac{d_{\gamma.h}(\sigma(\gamma,\gamma^{-1}x)f(\gamma^{-1}x), \sigma(\gamma,\gamma^{-1}x)g(\gamma^{-1}x))}{1 + d_{\gamma.h}(\sigma(\gamma,\gamma^{-1}x)f(\gamma^{-1}x), \sigma(\gamma,\gamma^{-1}x)g(\gamma^{-1}x))} d\mu_X(x) = \\ = & \int_X \frac{d_h(f(\gamma^{-1}x), g(\gamma^{-1}x))}{1 + d_h(f(\gamma^{-1}x), g(\gamma^{-1}x))} d\mu_X(x) = \\ = & \int_X \frac{d_h(f(x), g(x))}{1 + d_h(f(x), g(x))} d\mu_X(x) = d_h(f, g) \ , \end{aligned}$$

where we used the transitivity of $\mathrm{Meas}(X, H)$ and the definition of the metrics on the fibers to move from the first line to the second one and we concluded exploiting the Γ-invariance of μ_X.

Define the Γ-equivariant map $\psi : B \times X \to H/L_0R$ as $\psi(\xi, x) := \Psi(\xi)(x)$ for almost every $\xi \in B$ and almost every $x \in X$. By the universal property of ϕ, there exists $q : H/L \to H/L_0R$ which is in fact a isomorphism, and hence, up to conjugation, we can assume that $L_0R = L$.

By defining

$$\phi : B \times B \times X \to H/L \times H/L, \quad (\phi \times \phi)(\xi_1, \xi_2, x) := (\phi(\xi_1, x), \phi(\xi_2, x)) \ ,$$

we know by the universal property of ϕ_0 that we have the following commutative diagram

$$\begin{tikzcd} B \times B \times X \arrow[rr, "\phi_0"] \arrow[dr, "\phi\times\phi"'] & & H/L_0 \arrow[dl] \\ & H/L \times H/L. & \end{tikzcd}$$

Additionally, notice that given $\gamma_1, \gamma_2 \in \Gamma$ we have

$$(\phi \times \phi)(\gamma_1\xi_1, \gamma_2\xi_2, x) = (\sigma(\gamma_1, \gamma_1^{-1}x)\phi(\xi_1, \gamma_1^{-1}x), \sigma(\gamma_2, \gamma_2^{-1}x)\phi(\xi_2, \gamma_2^{-1}x)) \ ,$$

for almost every $\xi_1, \xi_2 \in B, x \in X$ by the σ-equivariance of ϕ. As a consequence of the Zariski density of σ, the essential image of $\phi \times \phi$ is Zariski dense in $H/L \times H/L$. Thus H/L_0 is Zariski dense in $H/L \times H/L$ or, equivalently, RL_0R is Zariski dense in H. Thus, by [BF14a, Lemma 3.5] H is parabolic and, being amenable, is also minimal. This concludes the proof.

□

We can now prove the existence of a boundary map in the setting of Theorem 3.

Corollary 3.2.1. *Let $n \geq 2$ and let $\Gamma < \mathrm{PU}(1,n)$ a torsion-free lattice. Consider an ergodic standard Borel probability Γ-space. If $\sigma : \Gamma \times X \to \mathrm{SU}(p,q)$ is a Zariski dense measurable cocycle, then there exists a boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{S}_{p,q}$.*

Proof. By [BF14a, Theorem 2.3] the Furstenberg boundary of Γ, which coincides with the visual boundary $\partial\mathbb{H}^p_{\mathbb{C}}$, is actually a Γ-boundary. Hence, applying Theorem 2 we get a map into the Furstenberg boundary of $\mathrm{SU}(p,q)$ and we compose with the projection on the Shilov boundary (induced by the inclusion of a minimal parabolic subgroup into a maximal one). □

Remark 3.2.2. In the setting of Corollary 3.2.1, since Zariski density implies non-elementarity, Theorem 1 provides a boundary map $\partial\mathbb{H}^n_{\mathbb{C}} \times X \to \partial\mathcal{X}(p,q)$. Moreover, by ergodicity we have that it takes values in the set of isotropic k-dimensional subspace in the boundary $\partial\mathcal{X}(p,q)$ for some $k \leq p$. To see this, for each pair $(\xi, x) \in \partial\mathbb{H}^n_{\mathbb{C}} \times X$ one can take the smallest cell in the spherical building of $\partial\mathcal{X}(p,q)$ which contains $\phi(\xi,x)$, that corresponds to a totally isotropic flag of $\mathbb{C}^{p,q}$ (see [Duc12]). By ergodicity the type of this flag must be the same for almost every pair in $\partial\mathbb{H}^n_{\mathbb{C}} \times X$ and by taking the maximal isotropic spaces of any flag we get the desired map. If we assume that σ is Zariski dense, the same argument in [DLP21, Theorem 1.7] show that $k = p$, which means that the target is the Shilov boundary of $\mathcal{X}(p,q)$. This gives an alternative proof of Corollary 3.2.1 which relies on the geometry of the symmetric space $\mathcal{X}(p,q)$.

3.3 Properties

Given a measurable map $\phi : B \to \mathcal{S}_G$ from a Γ-boundary B into the Shilov boundary of a semisimple Lie group of Hermitian type, its *essential Zariski closure* is the minimal Zariski closed subset V of $\mathcal{S}_G$ such that $\mu(\phi^{-1}(V)) = 1$. Such a set exists since the intersection of finitely many closed subset of full measure has full measure and $\mathcal{S}_G$ is an algebraic variety, in particular it is Noetherian. We say that such a measurable boundary map is *Zariski dense* if its essential Zariski closure is the whole $\mathcal{S}_G$.

Since the Zariski density of the slices of a boundary map will be needed in the proof of the main theorem, we are going to prove the next

Proposition 3.3.1. *Let $\Gamma < \mathrm{PU}(1,n)$ be a complex hyperbolic lattice with $n \geq 2$ and consider a Zariski dense cocycle $\sigma : \Gamma \times X \to \mathrm{SU}(p,q)$ together with a boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{S}_{p,q}$. Then for almost every $x \in X$ the slice $\phi_x \coloneqq \phi(\cdot, x)$ is Zariski dense.*

Proof. Before starting the proof, recall that the Shilov boundary $\mathcal{S}_{p,q}$ corresponds to the real points $\mathcal{S}_{p,q} = (\mathbf{H}/\mathbf{Q})(\mathbb{R})$ of the quotient of the complexification $\mathbf{H}$ of $\mathrm{SU}(p,q)$ (which is $\mathrm{SL}(p+q,\mathbb{C})$) modulo a maximal parabolic subgroup $\mathbf{Q}$ stabilizing a maximal isotropic subspace of $\mathbb{C}^{p+q}$. For almost every $x \in X$, we are going to denote by $\mathbf{V}_x \subset \mathbf{H}/\mathbf{Q}$ the smallest Zariski closed set such that $V_x := \mathbf{V}_x(\mathbb{R}) \subset \mathcal{S}_{p,q}$ and $\phi_x^{-1}(V_x)$ has full measure in $\partial\mathbb{H}^n_{\mathbb{C}}$. As noticed in [Poz15] those sets exist by the Noetherian property.

By embedding suitably $\mathbf{H}/\mathbf{Q}$ in some complex projective space $\mathbb{P}^N(\mathbb{C})$, we can define a map,

$$\mathfrak{v} : X \to \mathrm{Var}(\mathbb{P}^N(\mathbb{C})) \ , \quad \mathfrak{v}(x) := \mathbf{V}_x \ .$$

Here $\mathrm{Var}(\mathbb{P}^N(\mathbb{C}))$ is the set of all the possible closed varieties inside $\mathbb{P}^N(\mathbb{C})$ with the measurable structure coming from the Hausdorff metric (Zariski closed sets are closed in the Euclidean topology and this makes sense). The map $\mathfrak{v}$ is measurable since the slice ϕ_x varies measurably with respect to $x \in X$ as a consequence of [Mar91, Chapter VII, Lemma 1.3], by the measurability of ϕ. Moreover, for almost every $\gamma \in \Gamma$ and almost every $x \in X$, by exploiting the fact that σ acts by isometries on $\mathcal{S}_{p,q}$ we have

$$\phi(\gamma x) = \sigma(\gamma, x)\phi(x) \in \sigma(\gamma, x)V_x$$

and hence, by minimality of $V_{\gamma x}$, it follows that $\sigma(\gamma, x)V_x \subset V_{\gamma x}$. Similarly, exploiting minimality of V_x, one can prove that $\sigma(\gamma, x)^{-1}V_{\gamma x} \subset V_x$ and this implies the following equality

$$\mathbf{V}_{\gamma x} = \sigma(\gamma, x)\mathbf{V}_x \, . \tag{3.3}$$

In other words we proved that the map $\mathfrak{v}$ is σ-equivariant, namely

$$\mathfrak{v}(\gamma x) = \sigma(\gamma, x)\mathfrak{v}(x)$$

for almost every $\gamma \in \Gamma$ and almost every $x \in X$.

On $\mathrm{Var}(\mathbb{P}^N(\mathbb{C}))$ the group $\mathrm{GL}(N+1,\mathbb{C})$ acts naturally on the left. As noticed in the proof of [Zim84, Proposition 3.3.2], the set $\mathrm{Var}(\mathbb{P}^N(\mathbb{C}))$ decomposes as a countable union of varieties and the action of $\mathrm{GL}(N+1,\mathbb{C})$ on those varieties is algebraic and hence smooth. Seeing $\mathrm{SU}(p,q)$ as a subgroup of $\mathrm{GL}(N+1,\mathbb{C})$, we argue that the quotient $\Sigma := \mathrm{Var}((\mathbb{P}^N(\mathbb{C}))/\mathrm{SU}(p,q)$ is countably separated and $\mathfrak{v}$ induces a map

$$\overline{\mathfrak{v}} : X \to \Sigma \ , \quad \overline{\mathfrak{v}}(x) := \mathrm{SU}(p,q) \cdot \mathbf{V}_x \ ,$$

which is Γ-invariant, since $\mathfrak{v}$ was σ-equivariant by Equation (3.3). By the ergodicity of Γ on X, the above map must be essentially constant. Equivalently $\mathfrak{v}$ must take values into a unique orbit $\mathrm{SU}(p,q) \cdot \mathbf{V}_{x_0}$, for some $x_0 \in X$. Now, by composing the homeomorphism $\mathrm{SU}(p,q)\cdot\mathbf{V}_{x_0}\cong\mathrm{SU}(p,q)/\mathrm{Stab}_{\mathrm{SU}(p,q)}(\mathbf{V}_{x_0})$ with a measurable section

$$\mathrm{SU}(p,q)/\mathrm{Stab}_{\mathrm{SU}(p,q)}(\mathbf{V}_{x_0})\to\mathrm{SU}(p,q)$$

(see [Zim84, Corollary A.8]), we get a measurable map $g:X\to\mathrm{SU}(p,q)$ such that

$$\mathbf{V}_x=g(x)\mathbf{V}_{x_0}\,.$$

This implies that σ is cohomologous to a cocycle preserving $\mathbf{V}_{x_0}$ and the latter must coincide with $\mathbf{H}/\mathbf{Q}$ by the Zariski density assumption on σ. Hence for almost every $x\in X$, we have $\mathbf{V}_x=\mathbf{H}/\mathbf{Q}$ and hence $V_x=\mathrm{SU}(p,q)$, which means that almost every slice is essentially Zariski dense. $\square$

Our last result study the image of boundary maps of Zariski dense cocycles and involves the notion of opposite points introduced in Section 2.3.1.

Proposition 3.3.2. *Let $G=\boldsymbol{G}(\mathbb{R})$ where $\boldsymbol{G}$ is connected component of the isometry group of an Hermitian symmetric space. Let $\sigma:\Gamma\times X\to G$ be Zariski dense and let $\phi:B\times X\to G/Q$ be a boundary map where $Q=\boldsymbol{Q}(\mathbb{R})<\boldsymbol{G}(\mathbb{R})=G$ is some maximal parabolic subgroup such that $\mathcal{S}_G$ is identified with G/Q. Then for almost every $x\in X$ and $b_1,b_2\in B$, $\phi(b_1,x),\phi(b_2,x)$ are opposite in G/Q.*

Proof. Following the proof of [BI04, Proposition 7.2], we consider the map

$$B\times B\times X\to\mathbb{N},\quad (b_1,b_2,x)\mapsto\dim(\phi(b_1,x)\cap\phi(b_2,x)).$$

Such a map is Γ-invariant and, by the ergodicity of Γ on the product $B\times B\times X$ (see Proposition 2.4.13), it is essentially equal to a constant d. Assume by contradiction that d is positive. Then, if we denote by $V_x:=\mathrm{EssIm}(\phi_x)$ the essential image of ϕ_x, then one can adapt the argument in [BI04, Proposition 7.2] as follows: for any $b\in B$ and $x\in X$ we consider the set $\mathbf{nt}(\phi(b,x))$ of points in G/Q that are non-opposite to $\phi(b,x)$. Fix an $x\in X$ with such that

$$\dim(\phi(b_1,x)\cap\phi(b_2,x))=d\ ,$$

for almost every $b_1,b_2\in B$. As a consequence, if we suitably fix $b_2\in B$, it holds

$$\phi(b_1,x)\in\mathbf{nt}(\phi(b_2,x))\ ,$$

for almost every $b_1\in B$. Since $\mathbf{nt}(\phi(b_2,x))$ is a proper Zariski closed set of G/Q, this implies that the Zariski closure of the essential image $\overline{V_x}^Z$ must be a proper Zariski closed set. By Proposition 3.3.1 almost every slice of a boundary map of a Zariski dense cocycle has Zariski dense essential image, thus we get a contradiction and the statement is proved. $\square$

Now, combining Theorem 2 with Proposition 3.3.2, we obtain the following

Corollary 3.3.3. *Let Γ be a finitely generated group, let (X, μ_X) be an ergodic standard Borel probability Γ-space and consider a Zariski dense measurable cocycle $\sigma : \Gamma \times X \to G$ into an Hermitian Lie group. Then there exists a boundary map $\phi : B \times X \to G/Q$ where $Q < G$ is some maximal parabolic subgroup. Moreover, for almost every $x \in X$ and $b_1, b_2 \in B$, it holds $(\phi(b_1, x), \phi(b_2, x)) \in (G/Q)^{(2)}$.*

Chapter 4

Toledo invariant and superrigidity

In this chapter we apply the machinery of pullback along cocycles described in Section 2.4.3 in the context of cocycles $\sigma : \Gamma \times X \to \mathrm{SU}(p,q)$ where $\Gamma < \mathrm{PU}(1,n)$ is a torsion free lattice and Γ acts ergodically on X (for the definition of both $\mathrm{PU}(1,n)$ and $\mathrm{SU}(p,q)$ see Examples 2.3.3 and 2.3.4). In particular we define the *Toledo invariant* associated to a measurable cocycle, extending the notion of Toledo invariant for representations (see for instance Burger, Iozzi and Wienhard [BIW10] and Pozzetti [Poz15]). Then we provide a useful formula involving this numerical invariant by exploiting the version of the pullback through boundary maps. Such a formula, that is the natural adaption of the one obtained by Burger and Iozzi for representations [BI09], will lead to characterize a family of cocycles, called *maximal cocycles* that are the main object of this part.

We define, once and for the whole chapter, the following

Setup 4.0.1. *Fix integers $n \geq 2$ and $1 \leq p \leq q$ and consider the following objects:*

- $\Gamma < \mathrm{PU}(1,n)$ *is a torsion-free lattice;*
- (X, μ_X) *is an ergodic standard Borel probability* Γ*-space;*
- $\sigma : \Gamma \times X \to \mathrm{SU}(p,q)$ *is a measurable cocycle;*
- $\phi : \partial \mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{S}_{p,q}$ *is a boundary map for* σ*.*

We notice that, when σ is Zariski dense, the existence of the boundary map ϕ is ensured by Corollary 3.2.1.

4.1 The Toledo invariant of a measurable cocycle

In the setting of Setup 4.0.1, we notice how the target of the pullback defined by Equation (2.22) is the second bounded cohomology of Γ. In order to obtain a map into the second continuous bounded cohomology of $\mathrm{PU}(1,n)$, we need to define the *transfer map*, that is the function

$$\mathrm{T}_b^\bullet : \mathrm{H}_\mathrm{b}^\bullet(\Gamma;\mathbb{R}) \to \mathrm{H}_\mathrm{cb}^\bullet(\mathrm{PU}(1,n);\mathbb{R}) \ , \tag{4.1}$$

induced in cohomology by

$$\widetilde{\mathrm{T}_b^\bullet} : \mathrm{C}_\mathrm{b}^\bullet(\Gamma;\mathbb{R})^\Gamma \to \mathrm{C}_\mathrm{cb}^\bullet(\mathrm{PU}(1,n);\mathbb{R})^{\mathrm{PU}(1,n)} \ ,$$

$$(\widetilde{\mathrm{T}_b^\bullet}\psi)(\gamma_0,\ldots,\gamma_\bullet) \coloneqq \int_{\Gamma\backslash \mathrm{PU}(1,n)} \psi(\overline{g}\gamma_0,\ldots,\overline{g}\gamma_\bullet)d\mu(\overline{g}) \, .$$

Here μ is the probability measure induced on the quotient by the Haar measure on $\mathrm{PU}(1,n)$ (see [BBI13, MS20, MS21] for more details about the transfer map).

We recall that $\mathrm{H}^2_\mathrm{cb}(\mathrm{PU}(1,n);\mathbb{R}) \cong \mathbb{R}$ and that it is generated by the Cartan class $[c_n]$ (see Remark 2.3.14). Hence the machinery developed in Section 2.4.3 leads to the following

Definition 4.1.1. In the situation of Setup 4.0.1, the *Toledo invariant associated to* σ is the real number $\mathrm{t}_b(\sigma)$ satisfying

$$\mathrm{T}_b^2(\mathrm{H}_\mathrm{b}^2(\sigma^X)(k^b_{\mathrm{SU}(p,q)})) = \mathrm{t}_b(\sigma)[c_n] \ , \tag{4.2}$$

where $k^b_{\mathrm{SU}(p,q)}$ is the Kähler class of $\mathrm{SU}(p,q)$.

We recall that the transfer map is also induced by the map

$$\widehat{\mathrm{T}_b^\bullet} : \mathrm{L}^\infty((\partial\mathbb{H}^n_\mathbb{C})^{\bullet+1});\mathbb{R})^\Gamma \to \mathrm{L}^\infty((\partial\mathbb{H}^n_\mathbb{C})^{\bullet+1});\mathbb{R})^{\mathrm{PU}(1,n)}$$

$$(\widehat{\mathrm{T}_b^\bullet}\psi)(\xi_0,\ldots,\xi_\bullet) \coloneqq \int_{\Gamma\backslash \mathrm{PU}(1,n)} \psi(\overline{g}\xi,\ldots,\overline{g}\xi_\bullet)d\mu(\overline{g})$$

defined at the level of resolutions on boundaries.

This, together with the implementation of the pullback defined through boundary maps and to the Diagram (2.27), shows that the composition of the map defined by Equation (2.26) with T_b^2 applied to the class $[\beta_{\mathrm{SU}(p,q)}]$ (see Definition 2.3.12), defines a class $\mathrm{T}_b^2(\mathrm{H}^2(\Phi^X)([\beta_{\mathrm{SU}(p,q)}]))$ which satisfies

$$\mathrm{T}_b^2(\mathrm{H}^2(\Phi^X)([\beta_{\mathrm{SU}(p,q)}])) = \mathrm{t}_b(\sigma)[c_n]$$

(here, as usual, we are tacitly composing with the canonical map of Equation(2.9)). Hence we obtain the following formula

$$\int_{\Gamma\backslash \mathrm{PU}(1,n)} \int_X \beta_{\mathrm{SU}(p,q)}(\phi(\overline{g}\xi_0,x),\phi(\overline{g}\xi_1,x),\phi(\overline{g}\xi_2,x))d\mu_X(x)d\mu(\overline{g}) = \qquad (4.3)$$

$$= \mathrm{t}_b(\sigma)c_n(\xi_0,\xi_1,\xi_2).$$

Moreover, as proved for instance in [BBI13, Poz15, BBI18], Equation (4.3) holds for every triple (ξ_0,ξ_1,ξ_2) of pairwise distinct points in $\partial\mathbb{H}^n_{\mathbb{C}}$.

Remark 4.1.2. It is worth noticing that Equation (4.3) is a suitable adaptation of [MS21, Proposition 1.1] to this particular context. The absence of coboundary terms is due to the doubly ergodic action of Γ on the boundary $\partial\mathbb{H}^n_{\mathbb{C}}$ and to the fact that all the considered cochains are alternating. Additionally, the Toledo invariant $\mathrm{t}_b(\sigma)$ is the multiplicative constant $\lambda_{\beta_{\mathrm{SU}(p,q)},c_n}(\sigma)$ associated to $\sigma, \beta_{\mathrm{SU}(p,q)}, c_n$, namely

$$\mathrm{t}_b(\sigma) = \lambda_{\beta_{\mathrm{SU}(p,q)},c_n}(\sigma) \ ,$$

according to [MS21, Definition 3.21].

We immediately show the basic properties of the numerical invariant just defined, that are a almost direct consequence of Proposition 2.3.13. Before we need the following definition, that characterizes boundary maps which preserves chains. This particular class of equivariant maps plays an important role in the proof of superrigidity results, both for representations (see Pozzetti [Poz15, Theorem 4.1]) and in our context (Theorem 3).

Definition 4.1.3. A measurable map $\varphi : \partial\mathbb{H}^n_{\mathbb{C}} \to \mathcal{S}_{p,q}$ is *chain-preserving* if, for almost every $\xi_1,\xi_2 \in (\partial\mathbb{H}^n_{\mathbb{C}})^{(2)}$ and for almost every η lying on the chain from ξ_1 and ξ_1, we have $\phi(\eta) \in \langle \phi(\xi_1),\phi(\xi_2)\rangle$.

Hence we have the following

Proposition 4.1.4. *In the situation of Setup 4.0.1, the Toledo invariant* $\mathrm{t}_b(\sigma)$ *satisfies:*

(1) $|\mathrm{t}_b(\sigma)| \leq \mathrm{rk}(\mathcal{X}(p,q)) = \min(p,q)$*;*

(2) $|\mathrm{t}_b(\sigma)| = \mathrm{rk}(\mathcal{X}(p,q)) = \min(p,q)$ *if and only if the slice* $\phi_x \coloneqq \phi(\ \cdot\ ,x)$ *is chain-preserving for almost every* $x \in X$*.*

Proof. Ad 1. By Proposition 2.3.13 we know that $||c_n||_\infty \leq 1$ and also that $||\beta_{\mathrm{SU}(p,q)}||_\infty \leq \mathrm{rk}(\mathcal{X}(p,q))$. Hence we obtain

$$|\mathrm{t}_b(\sigma)| = \|\mathrm{t}_b(\sigma)c_n\|_\infty = \|\widehat{\mathrm{T}^2_b}(\mathrm{C}^2(\Phi^X)(\beta_{\mathrm{SU}(p,q)}))\|_\infty \leq \mathrm{rk}(\mathcal{X}(p,q)) \ ,$$

since both the transfer map $\widehat{\mathrm{T}^2_b}$ and the pullback map $\mathrm{C}^2(\Phi^X)$ are norm non-increasing.

Ad 2. Assume that the slice ϕ_x is chain preserving for almost every $x \in X$. Fixed a point $x \in X$, if ϕ_x is chain preserving and the triple (ξ_0, ξ_1, ξ_2) lies on a chain, then the triple $(\phi_x(\overline{g}\xi_0), \phi_x(\overline{g}\xi_1), \phi_x(\overline{g}\xi_2))$ lies on a m-chain for almost every $\overline{g} \in \Gamma\backslash\mathrm{PU}(1,n)$. Hence, if we fix a triple $(\xi_0, \xi_1, \xi_2) \in (\partial\mathbb{H}^n_{\mathbb{C}})^{(3)}$ of positive points on a chain, it holds $c_n(\xi_0, \xi_1, \xi_2) = 1$ and by hypothesis it follows

$$\beta_{\mathrm{SU}(p,q)}(\phi_x(\overline{g}\xi_0), \phi_x(\overline{g}\xi_1), \phi_x(\overline{g}\xi_2)) = \mathrm{rk}(\mathcal{X}(p,q))$$

for almost every $\overline{g} \in \Gamma\backslash\mathrm{PU}(1,n), x \in X$. In this way we obtain

$$\begin{aligned} \mathrm{t}_b(\sigma) &= \int_{\Gamma\backslash\mathrm{PU}(1,n)} \left(\int_X \beta(\phi_x(\overline{g}\xi_0), \phi_x(\overline{g}\xi_1), \phi_x(\overline{g}\xi_2)) d\mu_X(x) \right) d\mu(\overline{g}) = \\ &= \int_{\Gamma\backslash\mathrm{PU}(1,n)} \left(\int_X \mathrm{rk}(\mathcal{X}(p,q)) d\mu_X(x) \right) d\mu(\overline{g}) = \mathrm{rk}(\mathcal{X}(p,q)) \ , \end{aligned}$$

as claimed.

For the converse assume $\mathrm{t}_b(\sigma) = \mathrm{rk}(\mathcal{X}(p,q))$. Fixing a positive triple $(\xi_0, \xi_1, \xi_2) \in (\partial\mathbb{H}^n_{\mathbb{C}})^{(3)}$ on a chain, it follows by Equation (4.3) that,

$$\beta(\phi_x(\overline{g}\xi_0), \phi_x(\overline{g}\xi_1), \phi_x(\overline{g}\xi_2)) = \mathrm{rk}(\mathcal{X}(p,q))$$

for almost every $\overline{g} \in \Gamma \setminus \mathrm{PU}(1,n)$ and $x \in X$. By the σ-equivariance of ϕ we argue that

$$\beta(\phi_x(g\xi_0), \phi_x(g\xi_1), \phi_x(g\xi_2)) = \mathrm{rk}(\mathcal{X}(p,q)) \ ,$$

for almost every $g \in \mathrm{PU}(1,n)$ and $x \in X$. By the transitivity of the $\mathrm{PU}(1,n)$-action on chains, the map ϕ_x is chain preserving, as desired.

The same arguments can be used for the negative case. □

By Proposition 4.1.4 it follows naturally the next

Definition 4.1.5. In the situation of Setup 4.0.1, a cocycle σ is *maximal* if $\mathrm{t}_b(\sigma) = \mathrm{rk}(\mathcal{X}(p,q)) = \min(p,q)$.

It is worth mentioning that the notion of maximal measurable cocycles is a substantial extension of that one of maximal representations. Indeed, given any maximal $\rho : \Gamma \to \mathrm{SU}(p,q)$ in the sense of Pozzetti [Poz15] and any measurable function $f : X \to \mathrm{SU}(p,q)$, it is easy to check that the twisted cocycle σ_ρ^f is actually maximal. Moreover, if ρ is Zariski dense then it admits an essentially Zariski dense boundary map $\varphi : \partial\mathbb{H}^n_{\mathbb{C}} \to \mathcal{S}_{p,q}$ [Poz15, Proposition 2.9]. Hence the induced boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{S}_{p,q}$ defined as in Remark 2.4.17 has in fact essentially Zariski dense slices. In particular it satisfies the hypothesis of Theorem 3, which can be seen as the converse of what noticed above.

We conclude this section with a characterization of boundary maps with Zariski dense slices associated to maximal cocycles.

Lemma 4.1.6. *In the situation of Setup 4.0.1, if σ is maximal and the slice ϕ_x has essentially Zariski dense image for almost every $x \in X$, then ϕ_x is rational for almost every $x \in X$.*

Proof. It follows by [Poz15, Theorem 1.6] since ϕ_x is essentially Zariski dense for almost $x \in X$ and it is chain preserving by Proposition 4.1.4. □

4.2 Superrigidity for maximal Zariski dense cocycles

The aim of this section is to prove Theorem 3. As anticipated, the proof follows the line of that in [Zim80, Theorem 4.1] and is based on both Lemma 4.1.6 and on the following result about invariant measures on quotients of algebraic groups, which is an immediate consequence of [BDL17, Theorem 3.9].

Lemma 4.2.1. *Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{R}$-group and let $\mathbf{G}_0$ be a $\mathbb{R}$-subgroup. Denote by $G = \mathbf{G}(\mathbb{R})$ and $G_0 = \mathbf{G}_0(\mathbb{R})$ the associated real points, respectively. Consider a lattice Γ in G. Then, any measure on G/G_0 which is invariant by left translations in Γ, it is also a G-invariant measure.*

Proof. By [BDL17, Theorem 3.9] the stabilizer of a Γ-invariant measure is an almost algebraic subgroup of G that must coincide with the whole group by Borel Density Theorem [Zim84, Theorem 3.2.5]. □

We are now able to give the proof of

Theorem 3. *Consider $n \geq 2$ and $1 \leq p \leq q$. Let $\Gamma < \mathrm{PU}(1,n)$ be a torsion-free lattice and let (X, μ_X) be an ergodic standard Borel probability Γ-space. If $\sigma : \Gamma \times X \to \mathrm{SU}(p,q)$ is a maximal Zariski dense measurable cocycle, then it is cohomologous to the restriction of a cocycle associated to a representation $\rho : \mathrm{PU}(1,n) \to \mathrm{SU}(p,q)$.*

Proof. Assuming the same algebraic structures on $\partial\mathbb{H}^n_{\mathbb{C}}$ and $\mathcal{S}_{p,q}$ as those ones described in Example 2.3.10, we denote the set of rational maps between boundaries by

$$\mathcal{Q} := \mathrm{Rat}(\partial\mathbb{H}^n_{\mathbb{C}}, \mathcal{S}_{p,q}) \ .$$

As described in Example 2.1.8, there exists a natural action of $\mathrm{PU}(1,n)\times\mathrm{SU}(p,q)$ on it defined as follows: for each $h\in\mathrm{PU}(1,n)$, $g\in\mathrm{SU}(p,q), \xi\in\partial\mathbb{H}^n_{\mathbb{C}}$ and $f\in\mathcal{Q}$,

$$((h,g)\cdot f))(\xi) \coloneqq g\cdot f(h^{-1}\xi)\ .$$

Since σ is Zariski dense, by Corollary 3.2.1 we know that there exists a boundary map $\phi:\partial\mathbb{H}^n_{\mathbb{C}}\times X\to\mathcal{S}_{p,q}$. Being σ also maximal, by Lemma 4.1.6 we can define the function

$$\Phi: X\to\mathcal{Q},\quad x\mapsto\phi_x$$

and by composing it with the projection $\mathcal{Q}\to\mathcal{Q}/\mathrm{SU}(p,q)$ we obtain

$$\widehat{\Phi}: X\to\widehat{\mathcal{Q}}\coloneqq\mathcal{Q}/\mathrm{SU}(p,q),\quad x\mapsto\mathrm{SU}(p,q)\cdot\phi_x\ .$$

Since ϕ is a boundary map for σ, its equivariance implies

$$\begin{aligned}\Phi(\gamma x) &= \phi_{\gamma x}(\ \cdot\ ) = \\ &= \phi(\cdot,\gamma x) = \\ &= \phi(\gamma\gamma^{-1}\cdot,\gamma x) = \\ &= \sigma(\gamma,x)\phi(\gamma^{-1}\cdot,x) = \\ &= \sigma(\gamma,x)(\gamma\Phi(x)).\end{aligned} \tag{4.4}$$

In the equation above, notice that $\gamma\in\Gamma$ acts on the quotient $\widehat{\mathcal{Q}}$ via

$$\gamma\cdot(\mathrm{SU}(p,q)\cdot\psi) := \mathrm{SU}(p,q)\cdot(\gamma\cdot\psi)\,,$$

where $\gamma\cdot\psi$ is the rational map $(\gamma\cdot\psi)(\xi)=\psi(\gamma^{-1}\xi)$, for $\xi\in\partial\mathbb{H}^n_{\mathbb{C}}$. As a consequence of Equation (2.17) we get

$$\Phi(\gamma x)\in\mathrm{SU}(p,q)\cdot\gamma\cdot\Phi(x)\ ,$$

and hence it holds

$$\widehat{\Phi}(\gamma x)=\gamma\cdot\widehat{\Phi}(x)\ ,$$

from which we deduce that $\widehat{\Phi}$ is a Γ-equivariant map on the quotient. It follows that the induced map

$$\widehat{\widehat{\Phi}}: X\to\widehat{\widehat{\mathcal{Q}}} := \mathcal{Q}/\mathrm{PU}(1,n)\times\mathrm{SU}(p,q),\quad x\mapsto\mathrm{PU}(1,n)\times\mathrm{SU}(p,q)\cdot\phi_x.$$

is Γ-invariant and, since Γ acts ergodically on X, it is essentially constant or, equivalently, $\widehat{\Phi}$ takes values in a single $\mathrm{PU}(1,n)$-orbit. Notice that to conclude that $\widehat{\widehat{\Phi}}$ is essentially constant, we exploited the fact that $\widehat{\widehat{\mathcal{Q}}}$ is countably separated because of the smoothness of the joint action of both $\mathrm{PU}(1,n)$ and $\mathrm{SU}(p,q)$ on $\mathcal{Q}$ (see Example 2.1.8).

Let $x_0 \in X$ be a point such that $\widehat{\Phi}$ takes value in the orbit $\mathrm{PU}(1,n) \cdot \widehat{\Phi}(x_0)$ and set $G_0 \coloneqq \mathrm{Stab}_{\mathrm{PU}(1,n)}(\widehat{\Phi}(x_0))$. The latter is an algebraic $\mathbb{R}$-subgroup of $\mathrm{PU}(1,n)$ by [Zim84, Proposition 3.3.2]. Since the orbit $\mathrm{PU}(1,n) \cdot \widehat{\Phi}(x_0)$ may be identified with $\mathrm{PU}(1,n)/G_0$ by the smoothness of the action [Zim84, Theorem 2.1.14], we can compose the map

$$\widehat{\Phi} : X \to \mathrm{PU}(1,n) \cdot \widehat{\Phi}(x_0) \cong \mathrm{PU}(1,n)/G_0 \ ,$$

with a measurable section

$$s : \mathrm{PU}(1,n)/G_0 \to \mathrm{PU}(1,n) \ ,$$

which exists by [Zim84, Corollary A.8]. The previous composition gives us a map

$$g : X \to \mathrm{PU}(1,n)$$

which is measurable (being the composition of measurable maps) and such that

$$\widehat{\Phi}(x) = g(x)\widehat{\Phi}(x_0)$$

for almost every $x \in X$. By definition, we have

$$\widehat{\Phi}(\gamma x) = g(\gamma x)\widehat{\Phi}(x_0)$$

for every $\gamma \in \Gamma$ and almost every $x \in X$. On the other hand, by equivariance we get

$$\widehat{\Phi}(\gamma x) = \gamma(\widehat{\Phi}(x))$$

and thus

$$(\gamma g(x))^{-1} g(\gamma x) \in G_0.$$

The induced map

$$\bar{g} : X \to \mathrm{PU}(1,n)/G_0$$

is Γ-equivariant and this ensures the existence of a Γ-invariant finite measure (by push-forward) on $\mathrm{PU}(1,n)/G_0$. By Lemma 4.2.1, such a measure is in fact $\mathrm{PU}(1,n)$-invariant and, since G_0 is a closed subgroup, it coincides with $\mathrm{PU}(1,n)$ again by the Borel Density Theorem [Zim84, Theorem 3.2.5]. Hence $\widehat{\Phi}$ is essentially constant or, equivalently, Φ takes values in a single $\mathrm{SU}(p,q)$-orbit. Denote again by ϕ_0 an element in the orbit and choose a map

$$f : X \to \mathrm{SU}(p,q)$$

satisfying

$$\Phi(x) = f(x)\phi_0.$$

The measurability of f follows by the same argument we used to prove the measurability of g. By rewriting Equation (4.4) using f we obtain

$$f(\gamma x)\phi_0 = \sigma(\gamma, x)f(x)\gamma\phi_0 \tag{4.5}$$

and then

$$\gamma^{-1}\phi_0 = f(\gamma x)^{-1}\sigma(\gamma, x)f(x)\phi_0. \tag{4.6}$$

We define

$$\beta : \Gamma \times X \to \mathrm{SU}(p, q), \quad \beta(\gamma, x) := f(\gamma x)^{-1}\sigma(\gamma, x)f(x)$$

and, by Equation (4.6), we get

$$\phi_0(\xi) = \beta(\gamma, x_1)^{-1}\beta(\gamma, x_2)\phi_0(\xi)$$

for all $\gamma \in \Gamma$ and for almost all $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}, x_1, x_2 \in X$. Hence $\beta(\gamma, x_1)^{-1}\beta(\gamma, x_2)$ lies in the stabilizer of the image of ϕ_0. Since the latter is essentially Zariski dense, the product $\beta(\gamma, x_1)^{-1}\beta(\gamma, x_2)$ actually stabilizes $\mathcal{S}_{p,q}$. Since the pointwise stabilizer of $\mathcal{S}$ (that is the kernel of the action of $\mathrm{SU}(p, q)$ on $\mathcal{S}_{p,q}$) is trivial, it follows that β does not depend on X and hence it is the cocycle associated to a representation

$$\rho : \Gamma \to \mathrm{SU}(p, q) \ .$$

Moreover, by Equation (4.6), the map ϕ_0 is a boundary map for ρ, it is rational and has essentially Zariski dense image in $\mathrm{SU}(p, q)$. It follows by [Poz15, Theorem 1.1] that ρ is the restriction of a representation

$$\widetilde{\rho} : \mathrm{PU}(1, n) \to \mathrm{SU}(p, q)$$

and, finally, σ is cohomologous to the restriction to Γ of the induced cocycle $\sigma_{\tilde{\rho}}$, as desired. □

We can now prove that, except when either $p = 1$ or $p = q$, there are no maximal Zariski dense cocycle as in the statement of Theorem 3. Precisely, we have the following

Proposition 4. *Consider $n \geq 2$. Let $\Gamma < \mathrm{PU}(1, n)$ be a torsion-free lattice and let (X, μ_X) be an ergodic standard Borel probability Γ-space. Assuming $1 < p < q$, there is no maximal Zariski dense measurable cocycle $\sigma : \Gamma \times X \to \mathrm{SU}(p, q)$.*

Proof. Following the proof of Theorem 3, given such a maximal cocycle, there exists a maximal representation $\rho : \Gamma \to \mathrm{SU}(p,q)$. By [Poz15, Corollary 1.2], if $p \neq q$, such a representation cannot exist. $\square$

Since maximal measurable cocycles into $\mathrm{SU}(p,q)$ cannot be Zariski dense when $1 < p < q$, it is quite natural to ask which could be their algebraic hull. The following result provides an answer to this question and it can be seen as the analogous to the characterization given in [Poz15, Theorem 1.3].

Proposition 5. *Fix positive integers $n \geq 2$ and $1 < p < q$. Let $\Gamma < \mathrm{PU}(1,n)$ be a complex hyperbolic lattice, (X, μ_X) be an ergodic standard Borel probability Γ-space and consider a maximal cocycle $\sigma : \Gamma \times X \to \mathrm{SU}(p,q)$. Denoting by $\mathbf{H}$ the algebraic hull of σ and by $H = \mathbf{H}(\mathbb{R})^\circ$, then H splits as the product $K \times L_{nt} \times L_t$, where K is a compact subgroup of $\mathrm{SU}(p,q)$, L_t is a Hermitian Lie group of tube-type and L_{nt} is a Hermitian Lie group not of tube-type that splits again as a product of several copies of $\mathrm{SU}(1,n)$.*

Proof. Being maximal, σ is *tight*, that is the pull back along σ preserves the norm of the Kähler class $k^b_{\mathrm{SU}(p,q)}$. By [Sav20] the group H is reductive and hence it splits as the product of a compact factor $L_c = K$ and a non compact factor L_{nc}. By splitting L_{nc} in simple factors $L_1, \ldots, L_k$, we notice that the composition of σ with any projection $\pi_i : L_1 \times \ldots \times L_k \to L_i$ is a Zariski dense maximal measurable cocycle from a complex hyperbolic lattice to L_i. It follows by [MS21, Theorem 1.5] that none of the L_i's can be isomorphic to $\mathrm{SU}(1,1)$. Hence the inclusion $L_{nc} \to \mathrm{SU}(p,q)$ satisfies the hypothesis of [Poz15, Proposition 2.5], which states that each factor L_i is either of tube-type or isomorphic to some $\mathrm{SU}(p_i,q_i)$ where $1 \leq p_i \leq q_i$. We denote by L_t the tube-type part and we focus on the non-tube-type factors. Again by [MS21, Theorem 1.5], if one of $\mathrm{SU}(p_i,q_i)$'s is actually of the form $\mathrm{SU}(1,s)$ (that is $p_i = 1$), we must have that s is equal to q by Zariski density. By Proposition 4 the Zariski density of an ergodic cocycle taking values into $\mathrm{SU}(p_i,q_i)$ implies necessarily that $p_i = 1$ and we are done. $\square$

Chapter 5

Infinite dimension and reducibility

In Chapter 4 we have studied cocycles from complex hyperbolic lattices into the Hermitian group $\mathrm{SU}(p, q)$. In particular we focused on maximal and Zariski dense cocycles, where the first properties refers to a multiplicative constant that we called Toledo invariant (Definition 4.1.1), and the second one is related to the notion of algebraic hull (Definition 2.4.9). In this chapter we study cocycles $\Gamma \times X \to \mathrm{PU}(p, \infty)$ where Γ is a complex hyperbolic lattice and the target is the isometry group of the Hermitian symmetric space $\mathcal{X}(p, \infty)$ introduced in Example 2.3.5. The approach adopted is different from the one of Chapter 4, and this is motivated by the fact that the absence of algebraicity does not allow to speak about algebraic hull, and hence about Zariski density (see also Example 2.3.5). The idea, inspired also by the work of Duchesne, Lecureux and Pozzetti in [DLP21], is to introduce a notion of *finite dimensional algebraic subgroup* of $\mathrm{PU}(p, \infty)$ and hence the one of *finitely reducible cocycles*, that refers to cocycles admitting a representative in its cohomology class with image contained into a finite dimensional algebraic subgroup of $\mathrm{PU}(p, \infty)$. This notion is strictly related to the embedding between Hermitian symmetric spaces of the form $\mathcal{X}(p, q)$ introduced by Definition 2.3.6. In particular the standard embedding and Proposition 2.3.7 will be an important ingredient to prove that maximal cocycles are finitely reducible, that is the content of Theorem 6.

After a brief introduction about the Toledo invariant, that is a mere adaption of Section 4.1, we will introduce finite dimensional algebraic subgroups of $\mathrm{PU}(p, \infty)$ and the definition of finite reducibility. Then we will pass to the proof of Theorem 6. Finally, as a consequence of Theorem 1 and of Theorem 6 we will prove Theorem 7, that is version of Mostow rigidity for infinite dimensional cocycles.

For the original version of the results discussed in this chapter refer to [SS21a].

From now on we will consider a measurable cocycle $\sigma : \Gamma \times X \to \mathrm{PU}(p,\infty)$ and a boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{I}_p$ where Γ is a complex hyperbolic lattice, (X, μ_X) is an ergodic standard Borel probability Γ-space and $\mathcal{I}_p = \mathcal{I}_p(p,\infty)$ denotes set of p-chains in the visual boundary $\partial\mathcal{X}(p,\infty)$, according to Definition 2.3.11. Since $\mathcal{I}_p$ is a subset of the space of p-dimensional subspaces of an Hilbert space $\mathcal{H}$, we endow it with the measurable structure coming from the Grassmannian $\mathrm{Gr}_p(\mathcal{H})$.

A crucial difference between the finite case and the infinite one in the context of symmetric spaces is that $\mathrm{PU}(p,q)$ is locally compact for $q < \infty$ whereas $\mathrm{PU}(p,\infty)$ is not. To overcome this problem we will deal with the bounded cohomology groups $\mathrm{H}^\bullet_{\mathrm{b}}(\mathrm{PU}(p,\infty);\mathbb{R})$, namely its continuous bounded cohomology if we endow $\mathrm{PU}(p,\infty)$ with the discrete topology.

We recall that, by Example 2.3.5, the space $\mathcal{X}(p,\infty)$ is an Hermitian symmetric space and hence we can exploit the material introduced in Section 2.3. Precisely, as in the finite dimensional setting, we have a cohomology class $k^b_{\mathrm{PU}(p,\infty)} \in \mathrm{H}^2_{\mathrm{b}}(\mathrm{PU}(p,\infty);\mathbb{R})$ called *bounded Kähler class* of $\mathrm{PU}(p,\infty)$, that satisfies

$$||k^b_{\mathrm{PU}(p,\infty)}||_\infty = \mathrm{rk}\mathcal{X}(p,\infty) = p. \tag{5.1}$$

Exploiting both the pullback defined by Equation (2.22) and the transfer map defined by Equation (4.1) and since $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(1,n);\mathbb{R}) \cong \mathbb{R}$, we have that

$$\mathrm{T}^2_b \circ \mathrm{H}^2_{\mathrm{b}}(\sigma^X)(k^b_{\mathrm{PU}(p,\infty)}) = \mathrm{t}_b(\sigma)[c_n] \tag{5.2}$$

for some real number $\mathrm{t}_b(\sigma)$ that we call *Toledo invariant* associated to σ (see also Definition 4.1.1). As in the finite dimensional case (Definition 4.1.5), since both T^2_b and $\mathrm{H}^2_{\mathrm{b}}(\sigma^X)$ are norm non-increasing, then $|\mathrm{t}_b(\sigma)| \leq p$ we can define *maximal cocycles* as those with Toledo invariant equal to p.

The analogous of Equation 4.3 in this context can be obtained by rewriting Equation (5.2) as follows

$$\mathrm{T}^2_b \circ \mathrm{H}^2(\Phi^X)([\beta]) = \mathrm{t}_b(\sigma)[c_n]$$

and hence getting the analogous of Equation (4.3)

$$\begin{aligned}\int_{\Gamma\backslash\mathrm{PU}(1,n)} \left(\int_X \beta(\phi(\bar{g}\xi_0,x),\phi(\bar{g}\xi_0,x),\phi(\bar{g}\xi_0,x))d\mu_X(x) \right) d\mu_{\Gamma\backslash\mathrm{PU}(1,n)}(\bar{g}) = \\ = \mathrm{t}_b(\sigma)\cdot c_n(\xi_0,\xi_1,\xi_2)\end{aligned} \tag{5.3}$$

that holds for every triple of distinct points (ξ_0,ξ_1,ξ_2) in $\partial\mathbb{H}^n_{\mathbb{C}}$ ([Poz15, SS21c]).

5.1 Algebraic subgroups of $\mathrm{GL}(\mathcal{H})$.

In this section we describe a notion of algebraicity for subgroups of the group of bounded linear operators of an infinite dimensional Hilbert space. This allows to recover a notion of algebraic subgroups of $\mathrm{PU}(p,\infty)$ which a priori is not defined. We first introduce the notion of polynomial map.

Definition 5.1.1. A map $f : L(\mathcal{H}) \to \mathbb{R}$ is a *polynomial map* if it is a finite sum of maps $f_1, \dots, f_k$ where for each $i = 1, \dots, k$ there exists $h_i \in L^{n_i}(L(\mathcal{H}), \mathbb{R})$ such that $f_i(g) = h_i(g, \dots, g)$ for every $g \in L(\mathcal{H})$. The *degree* of f is the maximum of the n_i's.

Now, in parallel to the finite dimensional case, we define an algebraic subgroup as the set of the zero locus of some family of polynomial maps. More precisely,

Definition 5.1.2. A subgroup G of $\mathrm{GL}(\mathcal{H})$ is *algebraic* if there exists a positive integer n and family $\mathcal{P}$ of polynomial maps of degrees at most n such that

$$G = \{g \in \mathrm{GL}(\mathcal{H}) \mid P(g, g^{-1}) = 0 \ , \ \forall P \in \mathcal{P}\}.$$

A *strict algebraic subgroup* is a proper algebraic subgroup of $\mathrm{GL}(\mathcal{H})$.

To define a linear algebraic subgroup of $\mathrm{GL}(n, \mathbb{R})$ we consider polynomial equations in matrix coefficients. The generalization to infinite dimension of this notion is the content of the following definition (see [DLP21, Definition 3.4]).

Definition 5.1.3. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and choose an orthogonormal basis $(e_n)_{n\in\mathbb{N}}$. A homogeneous polynomial map $P : L(\mathcal{H}) \times L(\mathcal{H}) \to \mathbb{R}$ is *standard* of degree d if there exist two naturals ℓ, m such that $\ell + m = d$ and a family of real coefficients $(\lambda_i)_{i\in\mathbb{N}^{2\ell}}$ and $(\mu_j)_{j\in\mathbb{N}^{2m}}$ such that for any $(M, N) \in L(\mathcal{H}) \times L(\mathcal{H})$ we have that P can be expressed as the absolute convergent series

$$P(M, N) = \sum_{i\in\mathbb{N}^{2\ell}, j\in\mathbb{N}^{2m}} \lambda_i \mu_j P_i(M) P_j(N)$$

where $P_i(M) = \prod_{k=0}^{\ell-1} < Me_{i_{2k}}, e_{i_{2k+1}} >$ and $P_j(N) = \prod_{k=0}^{m-1} < Me_{i_{2k}}, e_{i_{2k+1}} >$.

A *standard polynomial map* is a finite sum of standard homogeneous polynomial maps.

An algebraic subgroup of $L(\mathcal{H})$ is *standard* if it is defined by a family of standard polynomial maps.

Hence we have the following interesting property, that shows how proper standard algebraic subgroups are closely related to finite dimensional subspace of $\mathcal{H}$.

Lemma 5.1.4 ([DLP21, Lemma 3.6])**.** *If H is a strict standard algebraic group, then there exists a finite dimensional subspace E of $\mathcal{H}$ such that the the group $H_E := \{g \in H \mid g(E) = E \ , \ g_{|E^\perp} = Id\}$ is a strict algebraic subgroup of* $\mathrm{GL}(E)$*.*

We call the subspace E *support* of the strict algebraic subgroup H and the group H_E the E*-part* of H. We are now ready to give the following

Definition 5.1.5. A *finite dimensional algebraic subgroup* is a standard algebraic subgroup of $\mathrm{GL}(\mathcal{H})$ of the form H_E.

Hence, it follows by Lemma 5.1.4 the following characterization of finite dimensional algebraic subgroups.

Lemma 5.1.6. *If E is a finite dimensional subspace of $\mathcal{H}$ and H is a subgroup of* $\mathrm{GL}(\mathcal{H})$ *contained in* $\mathrm{GL}(E)$*, then H is algebraic in* $\mathrm{GL}(E)$ *if and only if it is finite dimensional algebraicin* $\mathrm{GL}(\mathcal{H})$*.*

Proof. If H is finite dimensional algebraicin $\mathrm{GL}(\mathcal{H})$ then $H = H_E$ and by Lemma 5.1.4 it is algebraic in $\mathrm{GL}(E)$. Conversely, if H is algebraic in $\mathrm{GL}(E)$, it is also an algebraic subgroup in $\mathrm{GL}(\mathcal{H})$. Moreover, any polynomial which defines H on $\mathrm{GL}(E)$ can be turned into a polynomial on the entries of the matrices. Hence the same polynomials, seen as standard polynomial maps in the sense of Definition 5.1.3, define a standard algebraic subgroup in $\mathrm{GL}(\mathcal{H})$. Since it fixes $E^\perp$ then it coincides with its E-part and we are done. $\square$

We come back to the groups $\mathrm{U}(p,q)$. It is well know that the group $\mathrm{U}(p,\infty)$ is algebraic subgroup of $\mathrm{GL}(\mathcal{H})$. Indeed, if $V_0 := \mathrm{Span}\{e_0, \dots e_p\}$, we have that

$$\mathrm{U}(p,\infty) = \{g \in \mathrm{GL}(\mathcal{H}) \mid g^* \mathrm{Id}_{p,\infty} g = \mathrm{Id}_{p,\infty}\}$$

where $\mathrm{Id}_{p,\infty}$ is the linear map $\mathrm{Id}_{V_0} \oplus -\mathrm{Id}_{V_0^\perp}$. Since the map $(A,B) \mapsto A^*\mathrm{Id}_{p,\infty}B - \mathrm{Id}_{p,\infty}$ is bilinear on $L(\mathcal{H}) \times L(\mathcal{H})$ then $\mathrm{U}(p,\infty)$ is algebraic in $\mathrm{GL}(\mathcal{H}_\mathbb{R})$ and hence in $\mathrm{GL}(\mathcal{H})$ (see [DLP21] for more details and for the proof that $\mathrm{GL}(\mathcal{H})$ is actually standard). By Proposition 5.1.6 we can say immediately that the groups $\mathrm{U}(p,q)$ with $q < \infty$, seen as subgroups of $\mathrm{U}(p,\infty)$ inside $\mathrm{GL}(\mathcal{H})$, are actually finite dimensional algebraicsince they stabilize the embedding of $\mathcal{X}(p,q)$ inside $\mathcal{X}(p,\infty)$.

Since we work with the quotients $\mathrm{PU}(p,q)$ instead of the groups $\mathrm{U}(p,q)$, we call *finite algebraic* a subgroup of $\mathrm{PU}(p,\infty)$ if its preimage under the projection $\mathrm{U}(p,\infty) \to \mathrm{PU}(p,\infty)$ is finite dimensional algebraicin $\mathrm{GL}(\mathcal{H})$ in the sense of Definition 5.1.5.

5.2 Finite reducibility of a cocycle

The final aim of this section is to prove Theorem 6. Before passing to the proof, we need to introduce the notion of reducibility of a cocycle. In fact, given cocycle $\sigma : \Gamma \times X \to \mathrm{PU}(p, \infty)$ one can asks when the image of σ is contained in some suitable subgroup of $\mathrm{PU}(p, \infty)$. More precisely, definitions and results given in Section 5.1 allow us to define a class of cocycles for which some nice algebraic properties of the image are recovered.

Definition 5.2.1. A cocycle $\sigma : \Gamma \times X \to \mathrm{PU}(1, n)$ is *finitely reducible* if it admits a cohomologous cocycle with image contained in a finite dimensional algebraic subgroup of $\mathrm{PU}(p, \infty)$.

Before proving the main theorem, we recall by [DLP21] the following

Definition 5.2.2. A measurable map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \to \mathcal{I}_p$ *almost surely maps chains to chains* if for almost every chain $\mathcal{C} \subset \partial\mathbb{H}^n_{\mathbb{C}}$ there is a p-chain $\mathcal{T} \subset \mathcal{I}_p$ such that for almost every point $\xi \in \mathcal{C}$, $\phi(\xi) \in \mathcal{T}$.

An equivalent condition to the one above, which corresponds to the one given in Definition 4.1.3 in the finite dimensional case, is to check that for almost every pair $(\xi_0, \xi_1) \in \partial\mathbb{H}^n_{\mathbb{C}} \times \partial\mathbb{H}^n_{\mathbb{C}}$ the points $\phi(\xi_0)$ and $\phi(\xi_1)$ are opposite and, for almost every $\eta \in \mathcal{C}_{\xi_0,\xi_1}$, the subspace $\phi(\eta)$ is contained in $\langle \phi(\xi_0), \phi(\xi_1) \rangle$ [Poz15, Lemma 4.2].

Before passing to the proof of Theorem 6, we need the following result about maps that almost surely maps chains to chains, which is a slight refinement of [DLP21, Proposition 6.2]. Since there is a natural embedding $\partial\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{P}^n\mathbb{C}$, we can say that a set of $k \leq n+1$ points in $\partial\mathbb{H}^n_{\mathbb{C}}$ is *generic* if, for every $1 < h \leq k$, any subset of h points does not span a $(h-2)$-dimensional subspace.

Lemma 5.2.3. *Let $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \to \mathcal{I}_p$ be a measurable map that almost surely maps chains to chains. Then there exists a unique minimal totally geodesic embedded copy of $\mathcal{X}(p, q) \subset \mathcal{X}(p, \infty)$ that contains the image of almost every $(n+1)$-tuple of generic points in $\partial\mathbb{H}^n_{\mathbb{C}}$. Moreover, $p \leq q \leq np$.*

Proof. We argue by induction on n. The case $n = 1$ is clear, since there is only one chain $\mathcal{C}$ in $\partial\mathbb{H}^1_{\mathbb{C}}$ and for almost every $\eta_1, \eta_2 \in \mathcal{C}$ the subspace $\langle \phi(\eta_1), \xi(\eta_2) \rangle \subset \mathcal{H}$ defines a copy of $\mathcal{X}(p, p) \subset \mathcal{X}(p, \infty)$. The fact that ϕ almost surely maps chains to chains implies that for almost every ξ in $\partial\mathbb{H}^1_{\mathbb{C}}$ we have $\phi(\xi) < \langle \phi(\eta_1), \xi(\eta_2) \rangle$.

Assume that the statement holds for $n - 1$. Thanks to the construction in [DLP21], we can define a full-measure subset $\mathcal{G}$ of the set of $(n+1)$-tuple of points in general position of $\partial\mathbb{H}^n_{\mathbb{C}}$ such that for every $(\xi_0, \ldots, \xi_n) \in \mathcal{G}$ the following conditions hold:

- $\phi_{|\langle \xi_0,\dots,\xi_{n-1}\rangle}$ almost surely maps chains to chains;
- for almost every $\eta \in \langle \xi_0,\dots,\xi_{n-1}\rangle$ then $\langle \phi(\eta),\phi(\xi_{n-1})\rangle$ is a $2p$-dimensional subspace on which the restriction of Q has signature (p,p);
- for almost every $\eta \in \langle \xi_{n-1},\xi_n\rangle$ then $\langle \phi(\eta),\phi(\xi_{n-1})\rangle$ is a $2p$-dimensional subspace on which the restriction of Q has signature (p,p);
- for almost every $\eta_1 \in \langle \xi_{n-1},\xi_n\rangle$ and $\eta_2 \in \langle \xi_0,\dots,\xi_{n-1}\rangle$ the space $\langle \phi(\eta_1),\phi(\eta_2)\rangle$ has dimension $2p$ and the restriction of Q has signature (p,p).

As proved in [DLP21, Proposition 6.2], for almost every $(\xi_0,\dots,\xi_n) \in \mathcal{G}$ the space

$$V_{\xi_0,\dots,\xi_n} \coloneqq \langle \phi(\xi_0),\dots,\phi(\xi_n)\rangle$$

contains $\phi(\eta)$ for almost every $\eta \in \partial\mathbb{H}^n_{\mathbb{C}}$. Furthermore, the restriction of Q to $V_{\xi_0,\dots,\xi_n}$ is non-degenerate of signature (p,q) with $p \le q \le np$.

We now prove that almost every pair of tuple $((\xi_0,\dots,\xi_n),(\eta_0,\dots,\eta_n)) \in \mathcal{G}^2$ give the same subspace. We first note that, since $V_{\xi_0,\dots,\xi_n}$ contains the image of almost every point in $\partial\mathbb{H}^n_{\mathbb{C}}$, it clearly contains $\phi(\eta_0),\dots,\phi(\eta_n)$, and hence $\langle\phi(\eta_0),\dots,\phi(\eta_n)\rangle$, for almost every $(\eta_0,\dots,\eta_n) \in \mathcal{G}$. Hence there exists a full-measure subset $\mathcal{Q} \subset \mathcal{G}\times\mathcal{G}$ such that

$$V_{\xi_0,\dots,\xi_n} > V_{\eta_0,\dots,\eta_n}$$

for almost every $((\xi_0,\dots,\xi_n),(\eta_0,\dots,\eta_n)) \in \mathcal{Q}$. By taking the measure-preserving idempotent function of $\mathcal{G}\times\mathcal{G}$ which swap the tuple, one gets a second full-measure subsets $\overline{\mathcal{Q}}$. Hence the intersection $\mathcal{Q}\cap\overline{\mathcal{Q}}$ is a full-measure subset of $\mathcal{G}\times\mathcal{G}$ of pairs $(\xi_0,\dots,\xi_n),(\eta_0,\dots,\eta_n)$ such that

$$V_{\xi_0,\dots,\xi_n} = V_{\eta_0,\dots,\eta_n}\,,$$

which implies the uniqueness.

A similar argument can be used to prove minimality, namely that every linear subspace $W < \mathcal{H}$ containing the image of a full-measure subset of $\partial\mathbb{H}^n_{\mathbb{C}}$ must contain the spaces constructed above. □

Remark 5.2.4. It seems natural to investigate the effective dimension of the copy of $\partial\mathcal{X}(p,q)$ which contains the essential image of ϕ provided by Lemma 5.2.3. For instance, given a chain preserving map $\psi : \partial\mathbb{H}^n_{\mathbb{C}} \to \partial\mathbb{H}^p_{\mathbb{C}}$, Burger and Iozzi [BI07] proved the following dichotomy: if the image of almost every triple (ξ_0,ξ_1,ξ_2) of generic points is generic as well, then ψ coincides almost everywhere with the map induced on boundaries by an isometric holomorphic embedding $\mathbb{H}^n_{\mathbb{C}} \to \mathbb{H}^p_{\mathbb{C}}$. If not, then the image is essentially contained into a chain in $\partial\mathbb{H}^p_{\mathbb{C}}$.

In our more general context, we do not know if such a dichotomy holds. However, in our setting, the two cases described above can be interpreted as the limit cases as follows. In fact, if $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \to \mathcal{I}_p$ as in Lemma 5.2.3 sends almost every $(n+1)$-tuple of generic points of $\partial\mathbb{H}^n_{\mathbb{C}}$ to $(n+1)$ generic points of $\mathcal{I}_p$, then we have that the essential image of ϕ is contained in $\partial\mathcal{X}(p, np)$. On the other hand, by the same argument used in [BI07, Proposition 2.2], if there is a positive measure subset of triple in $(\partial\mathbb{H}^n_{\mathbb{C}})^3$ not on a chain whose image lies on a chain, then the image of ϕ is essentially contained into one copy of $\partial\mathcal{X}(p, p)$. We point out that this two cases do not produce a dichotomy, but a characterization of the cases when $q = p$ and $q = np$ in the notation of Lemma 5.2.3.

Now we are ready to give the proof of

Theorem 6. *Let $\Gamma < \mathrm{PU}(1, n)$ be a complex hyperbolic lattice with $n \geq 1$ and let (X, μ_X) be an ergodic standard Borel probability Γ-space. Consider a measurable cocycle $\sigma : \Gamma \times X \to \mathrm{PU}(p, \infty)$ with $p \geq 1$ and suppose there exists a boundary map $\phi : \partial\mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{I}_p$. If σ is maximal, then it is finitely reducible.*

Proof. By Equation (5.3) and using [DLP21, Corollary 6.1], it follows that almost every slice ϕ_x almost maps chains to chains. Hence, by Lemma 5.2.3, for almost every $x \in X$ there exists a unique minimal totally geodesic embedding $\mathcal{X}_x(p, q_x) \subset \mathcal{X}(p, \infty)$ such that $\mathrm{EssIm}(\phi_x) \subset \partial\mathcal{X}_x(p, q_x)$ for some $p \leq q_x \leq np$. Moreover, the equivariance of ϕ implies that

$$\sigma(\gamma, x)\mathcal{X}_x(p, q_x) = \mathcal{X}_{\gamma x}(p, q_{\gamma x})$$

for almost every $\gamma \in \Gamma$ and $x \in X$ and, by ergodicity, we have that the dimension of the $\mathcal{X}_x(p, q_x)$'s is essentially constant, namely $q_x = q$ for almost every $x \in X$. The fact that the function $x \mapsto \dim(\mathcal{X}_x(p, q_x))$ is measurable follows from the measurability of ϕ and from an application of Fubini's theorem in the construction of the $\mathcal{X}_x(p, q_x)$'s described in Lemma 5.2.3. In fact, one can find a $(n+1)$-tuple $(\xi_0, \ldots, \xi_n)$ of points in $\partial\mathbb{H}^n_{\mathbb{C}}$ such that, for almost every $x \in X$, the space $\langle \phi_x(\xi_0), \ldots, \phi_x(\xi_n) \rangle$ contains $\phi_x(\eta)$ for almost every $\eta \in \partial\mathbb{H}^n_{\mathbb{C}}$.

If we denote by ι_x the isometric linear map that induces the embedding $\mathcal{X}_x(p, q) \subset \mathcal{X}(p, \infty)$, the uniqueness of $\mathcal{X}_x(p, q)$, together with the σ-equivariance of ϕ, implies that the map

$$X \to \mathrm{PU}(p, \infty)/\mathrm{Stab}_{\mathrm{PU}(p,\infty)}(V_0)\,, \quad x \mapsto \mathcal{X}_x(p, q) \tag{5.4}$$

is measurable (with respect to the measurable structure discussed in Remark 2.3.8) and σ-equivariant. Here $\mathrm{Stab}_{\mathrm{PU}(p,\infty)} V_0$ is the subgroup of $\mathrm{PU}(p, \infty)$ preserving the subspace V_0. Now, thanks to the differentiable structure of the group $\mathrm{PU}(p,\infty)$, we can compose the function in Equation (5.4) with a measurable section

$$\mathrm{PU}(p,\infty)/\mathrm{Stab}_{\mathrm{PU}(p,\infty)}(V_0) \to \mathrm{PU}(p,\infty)$$

in order to obtain a measurable map

$$f : X \to \mathrm{PU}(p,\infty), \quad f(x) = g_x^{-1}.$$

By construction, $f(x)$ sends $\mathcal{X}_x(p,q)$ to the standard embedded copy $\mathcal{X}_0(p,q) \subset \mathcal{X}(p,\infty)$.

According to the notation of the Definition 2.4.6, we consider the twisted cocycle $\sigma^f : \Gamma \times X \to \mathrm{PU}(p,\infty)$ defined as

$$\sigma^f(\gamma, x) \coloneqq f(\gamma x)^{-1}\sigma(\gamma, x) f(x)$$

and the associated twisted boundary map $\phi^f : \partial\mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{I}_p$ which is defined as follows

$$\phi^f(\xi, x) \coloneqq f(x)^{-1}\phi(\xi, x)$$

for almost every $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ and $x \in X$. Now, by definition of f, for almost every $x \in X$ the image of almost every slice ϕ_x is contained in the boundary of a fixed $\mathcal{X}(p,q)$.

For almost every $x \in X$, denote by E_x the full measure set of points ξ in $\partial\mathbb{H}^n_{\mathbb{C}}$ such that $\phi^f_x(\xi) \in \partial\mathcal{X}(p,q)$. Consider now the set $E = \bigcup_{x \in X} E_x \times \{x\}$ (that is of full measure in $\partial\mathbb{H}^n_{\mathbb{C}} \times X$, by Fubini's theorem) and the diagonal action of Γ given by

$$\gamma(\xi, x) = (\gamma\xi, \gamma x).$$

Since Γ is countable, we find an invariant full measure subset $\overline{E}$ such that $\phi^f(\overline{E}) \subset \partial\mathcal{X}(p,q)$. More precisely, we set

$$\overline{E} = \bigcap_{\gamma \in \Gamma} \gamma E \ ,$$

where γ acts diagonally. Being the intersection of full measure sets, it is clear that $\overline{E}$ has full measure. Now, since the image of a full measure set under ϕ^f is contained in the boundary of $\mathcal{X}_0(p,q)$, it follows that the image of the twisted cocycle σ^f is contained in $\mathrm{Stab}_{\mathrm{PU}(p,\infty)}V_0$, which is finite dimensional algebraic as desired. $\square$

Remark 5.2.5. The descending chain condition that holds for Noetherian spaces (as algebraic groups are), allows to define the algebraic hull for cocycles into algebraic groups. This can not be adapted for $\mathrm{PU}(p,\infty)$, namely there exits no well-defined minimal strict algebraic group containing the image of a twisted cocycle. Nevertheless, by Theorem 6, any maximal cocycles has a representative in its cohomology class whose image is contained into the embedding of $\mathrm{PU}(p,q)$ in $\mathrm{PU}(p,\infty)$, which is algebraic. For such particular measurable cocycles, our result recover a sense of algebraicity.

5.3 Consequences of finite reducibility

The aim of this last section is to relate Theorem 1 and Theorem 6.

We consider the setting of Theorem 6, namely Γ is a complex hyperbolic lattice, (X,μ_X) is an ergodic standard Borel probability Γ-space and $\sigma:\Gamma\times X\to\mathrm{PU}(p,\infty)$ is a maximal cocycle. If we assume that σ is non-elementary, Theorem 1 provides a boundary map $\phi:\partial\mathbb{H}^n_{\mathbb{C}}\times X\to\partial\mathcal{X}(p,\infty)$. Moreover, by Remark 3.2.2 such a map takes values into $\mathcal{I}_k(p,\infty)$ for some $k\leq p$. Unfortunately, this is not sufficient to prove reducibility as in Theorem 6, since such k might be strictly less then p.

However, for cocycles $\sigma:\Gamma\times X\to\mathrm{PU}(1,\infty)$ one can exploit the geometry of $\mathcal{X}(1,\infty)=\mathbb{H}^\infty_{\mathbb{C}}$ and of its boundary to prove

Theorem 7. *Let $\Gamma<\mathrm{PU}(1,n)$ be a complex hyperbolic lattice with $n\geq 1$ and let (X,μ_X) be an ergodic standard Borel probability Γ-space. Any maximal cocycle $\sigma:\Gamma\times X\to\mathrm{PU}(1,\infty)$ is cohomologous to a cocycle preserving a copy of $\mathbb{H}^n_{\mathbb{C}}\subset\mathbb{H}^\infty_{\mathbb{C}}$ and acting on it via the standard lattice embedding.*

Proof. We first prove that maximal cocycles cannot be elementary. In fact, by ergodicity, a σ-equivariant family of flats can be made of points or lines. In both cases one can twist σ into a cocycle whose image is contained either in the stabilizer of a point or a geodesic, which are both amenable. Since amenable groups have trivial bounded cohomology, we have a contradiction to maximality.

Since σ is not elementary, Theorem 1 provides a boundary map $\partial\mathbb{H}\times X\to\partial\mathbb{H}^\infty_{\mathbb{C}}$ and then we can apply Theorem 6. Hence we have that σ is cohomologous to a cocycle $\widetilde{\sigma}$ whose image is in the stabilizer of an embedded copy of $\mathbb{H}^n_{\mathbb{C}}$ in $\mathbb{H}^\infty_{\mathbb{C}}$. The stabilizer $\mathrm{Stab}_{\mathrm{PU}(p,\infty)}(\mathbb{H}^n_{\mathbb{C}})$ is an almost direct product with one factor isomorphic to $\mathrm{PU}(1,n)$. By composing with the projection on such factor we get a maximal cocycle. Hence we can apply [MS21, Theorem 1.5] and we are done. □

In view of Remark 3.2.2 and of Theorem 7, it is natural to ask whether Theorem 1 provides a boundary map in the general setting of Theorem 6. However, since we do not have a complete answer, we postpone the discussion to Chapter 7 where we summarize some questions that remain unsolved.

Chapter 6

Parametrized Kähler class

The main objects of this chapter are Zariski dense measurable cocycles $\Gamma \times X \to G$ where Γ is a finitely generated group, (X, μ_X) is an ergodic standard Borel probability Γ-space and $G = \mathrm{Isom}(\mathcal{X})^\circ$ for an irreducible Hermitian Lie symmetric space $\mathcal{X}$ not of tube-type. Our aim is to characterize such cocycles in terms of the pullback of the bounded Kähler class of G defined by the Bergmann cocycle (see Definition 2.3.12). As we pointed out in Section 2.4.1, equivalence classes of measurable cocycles is nothing but the first cohomology group $\mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X; G)$ of the orbital equivalence relation given by the Γ-action on X with coefficient in G. We will refer to such a group as the *Zariski dense Eilenberg-MacLane cohomology* of Γ, since it can also be identified with the group $\mathrm{H}^1(\Gamma; \mathrm{Meas}(X, G))$. With this notation, our characterization (Theorem 8) defines an inclusion

$$\mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X; G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathrm{L}^\infty(X; \mathbb{R}))\ , \quad [\sigma] \mapsto \mathrm{H}^2_{\mathrm{b}}(\sigma)(k^b_G). \tag{6.1}$$

Theorem 8 is the natural generalization of [BIW10, Theorem 1] to measurable cocycles. Since [BIW10, Theorem 1] follows from the more general [BIW10, Theorem 2], the same thing will happen in our case, precisely we will deduce Theorem 8 from a more general result (see Theorem 6.1.4). In particular, we will exploit the boundary map provided by Theorem 2 and the characterization of non-tube-type domain given in terms of the complex Hermitian triple product in Section 2.3.3.

For the original version of the results contained in this chapter refer to [SS21b].

6.1 Zariski dense Eilenberg-MacLane cohomology

Before starting, we need to introduce some notions. First of all, we define the parametrized Kähler class of a cocycle.

Definition 6.1.1. In the setting of Theorem 8, the *parametrized Kähler class* associated to σ is the class $\mathrm{H}^2_{\mathrm{b}}(\sigma)(k^b_G) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathrm{L}^\infty(X;\mathbb{R}))$.

Remark 6.1.2. The name *parametrized Kähler class* recall the dependence of the pull back of k^b_G along σ on a parameter space, namely X. Such a denomination should recall the work of Löh and Pagliantini [LP16], where the authors describe the theory of parametrized simplicial volume and introduce the notion parametrized fundamental class. We warn the reader that we are not claiming that our theory is dual to the one due to Löh and Pagliantini in the context of simplicial volume.

For our purposes we need to study more carefully the cohomology class in degree two.

Lemma 6.1.3. *Let Γ a finitely generated group and let (X, μ_X) be a standard Borel probability space. If B is a Γ-boundary, then*

$$\mathrm{H}^2_b(\Gamma; \mathrm{L}^\infty(X;\mathbb{R})) \cong \mathcal{Z}\mathrm{L}^\infty_{\mathrm{w}^*,\mathrm{alt}}(B^3; \mathrm{L}^\infty(X;\mathbb{R}))^\Gamma \ ,$$

where the letter $\mathcal{Z}$ denotes the set of cocycles.

Proof. For every $k \in \mathbb{N}$ we have the following

$$\mathrm{L}^\infty_{\mathrm{w}^*}(B^k; \mathrm{L}^\infty(X;\mathbb{R}))^\Gamma \cong \mathrm{L}^\infty(B^k \times X;\mathbb{R})^\Gamma \ ,$$

where Γ acts on $B^k \times X$ diagonally [Mon01, Corollary 2.3.3]. Recalling that a Γ-boundary is also a strong boundary in the sense of Burger and Monod [BF14b, Remarks 2.4], every essentially bounded weak-* measurable function on $B \times B \times X$ which is Γ-invariant must be essentially constant [MS04, Proposition 2.4]. Since an alternating function that is constant vanishes, we have that

$$\mathrm{L}^\infty_{\mathrm{w}^*,\mathrm{alt}}(B^2; \mathrm{L}^\infty(X;\mathbb{R}))^\Gamma = 0 \ .$$

This shows that there are no coboundaries in dimension two and so we get the result. □

We are now ready to prove the following result, that is an adaption of the arguments used in [BIW07, Theorem 4] and which implies Theorem 8.

Theorem 6.1.4. *Let Γ be a finitely generated discrete group and let (X, μ_X) be an ergodic standard Borel probability Γ-space. Let $\{\sigma_i : \Gamma \times X \to G_i\}, i = 1, \ldots, n$ be a family of Zariski dense measurable cocycles into Lie groups $G_i = Isom(\mathcal{X}_i)^\circ$ where the $\mathcal{X}_i$'s are irreducible Hermitian symmetric spaces not of tube-type. If the cocycles are pairwise inequivalent, then the subset*

$$\{\mathrm{H}^2_{\mathrm{b}}(\sigma_i)(k^b_{G_i}), 1 \leq i \leq n\} \subset \mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathrm{L}^\infty(X;\mathbb{R}))$$

is linearly independent over $\mathrm{L}^\infty(X;\mathbb{Z})$.

Notice that we are using the notion of equivalence introduced by Definition 2.4.8.

Proof of Theorem 6.1.4. Suppose the existence of coefficients $m_i \in \mathrm{L}^\infty(X;\mathbb{Z}), i = 1,\dots,n$ such that

$$\sum_{i=1}^n m_i \mathrm{H}^2_{\mathrm{b}}(\sigma_i)(k^b_{G_i}) = 0.$$

As a consequence of Corollary 3.3.3, we get a boundary map $\phi_i : B \times X \to \mathcal{S}_{G_i}$ from a Γ-boundary B into the Shilov boundary $\mathcal{S}_{G_i}$ of the group G_i. By the commutativity of the Diagram (2.25), the cocycles $\mathrm{C}^2(\Phi_i)(\beta_{G_i})$ represent canonically the pullback along each σ_i. Additionally, being alternating, by Lemma 6.1.3 there are no coboundary in degree two. Hence we get the following equation

$$\sum_{i=1}^n m_i(x)\beta_{G_i}(\phi_i(b_1,x),\phi_i(b_2,x),\phi_i(b_3,x)) = 0 \tag{6.2}$$

that holds for almost every triple $(b_1,b_2,b_3) \in B^3$ and for almost every $x \in X$. As a direct consequence of Equation (2.11) it follows that

$$\prod_{i=1}^n \langle\langle \phi_i(b_1,x),\phi_i(b_2,x),\phi_i(b_3,x)\rangle\rangle_{\mathbb{C}}^{m_i(x)} = 1 \tag{6.3}$$

for almost every triple $(b_1,b_2,b_3) \in B^3$ and for almost every $x \in X$.

For any i, Corollary 3.3.3 allows to choose ϕ_i in such a way that the subset of points $(x,b_1,b_2) \in X \times B \times B$ with $(\phi_i(b_1,x),\phi_i(b_2,x)) \in \mathcal{S}^{(2)}_{G_i}$ is of full measure. Hence, since a finite intersection of full measure sets is still of full measure, we can fix a point $x_0 \in X$ and a pair $(b_1,b_2) \in B^2$ such that $(\phi_i(b_1,x_0),\phi_i(b_2,x_0)) \in \mathcal{S}^{(2)}_{G_i}$ for every $i = 1,\dots,n$.

Exploiting the transitivity of G_i on pairs in $\mathcal{S}^{(2)}_{G_i}$, we can identify $\mathcal{S}^{(2)}_{G_i}$ with the quotient $G_i/\mathrm{Stab}_{G_i}(\phi_i(b_1,x_0),\phi_i(b_2,x_0))$ by the stabilizer in G_i of the pair $(\phi_i(b_1,x_0),\phi_i(b_2,x_0)) \in \mathcal{S}^{(2)}_{G_i}$. Furthermore, the map $X \to \mathcal{S}^{(2)}_{G_i}$ that takes x into the pair $(\phi_i(b_1,x),\phi_i(b_2,x))$ is measurable by the measurability of ϕ. Hence the composition

$$X \to \mathcal{S}^{(2)}_{G_i} \to G_i/\mathrm{Stab}_{G_i}(\phi_i(b_1,x_0),\phi_i(b_2,x_0))$$

is measurable as well and, composing again with the measurable section

$$G_i/\mathrm{Stab}_{G_i}(\phi_i(b_1,x_0),\phi_i(b_2,x_0)) \to G_i$$

given by [Zim84, Corollary A.8], we get a family of measurable functions

$$g_i : X \to G_i$$

such that, setting $\phi_i^{g_i}(b,x) \coloneqq g_i(x)^{-1}\phi_i(b,x)$, we have

- $\phi_i^{g_i}(b_1,x)=\phi_i(b_1,x_0)$ for almost every $x\in X$;
- $\phi_i^{g_i}(b_2,x)=\phi_i(b_2,x_0)$ for almost every $x\in X$.

Moreover, Equation (6.3) implies that

$$\prod_{i=1}^{n}\langle\langle\phi_i^{g_i}(b_1,x),\phi_i^{g_i}(b_2,x),\phi_i^{g_i}(b_3,x)\rangle\rangle_{\mathbb{C}}^{m_i(x)}=1 \tag{6.4}$$

holds for almost every $b_3\in B$ and for almost every $x\in X$. Rewriting (6.4) we get that

$$\prod_{i=1}^{n}\langle\langle\phi_i(b_1,x_0),\phi_i(b_2,x_0),\phi_i^{g_i}(b_3,x)\rangle\rangle_{\mathbb{C}}^{m_i(x)}=1 \tag{6.5}$$

holds for almost every $b_3\in B$ and for almost every $x\in X$.

We define the cocycle

$$\sigma:\Gamma\times X\to\prod_{i=1}^{n}G_i,\quad (\gamma,x)\mapsto(\sigma_i^{g_i}(\gamma,x))_i$$

and its boundary map

$$\phi:B\times X\to\prod_{i=1}^{n}\mathcal{S}_{G_i},\quad (b,x)\mapsto(\phi_i^{g_i}(b,x))_i.$$

and we denote by $\mathbf{L}$ the algebraic hull of σ.

Now, following Lemma 2.3.17, we denote by $\mathcal{O}_i:=\mathcal{O}_{\phi_i(b_1,x_0),\phi_i(b_2,x_0)}\subset\mathcal{S}_{G_i}$ and by $P_i=P_{\phi_i(b_1,x_0),\phi_i(b_2,x_0)}$. Then we have

$$\overline{\mathrm{EssIm}(\phi_x)}^Z\cap\prod_{i=1}^{n}\mathcal{O}_i\subset\left\{(\eta_1,\ldots,\eta_n)\in\prod_{i=1}^{n}\mathcal{O}_i\ ,\ \prod_{i=1}^{n}P_i^{m_i(x)}(\eta_i)=1\right\},$$

where ϕ_x denotes the slice $\phi(\cdot,x)$. Applying Lemma 2.3.17 to almost every $x\in X$, it follows that $\overline{\mathrm{EssIm}(\phi_x)}^Z$ is a proper Zariski closed subset of $\prod_{i=1}^{n}\mathcal{O}_i$. Additionally the family $\{V_x\}$ is σ-equivariant. By a slight modification of the argument in Proposition 3.3.1 this must implies that σ cannot be Zariski dense, otherwise the slices ϕ_x would have been Zariski dense. Thus we conclude that $\mathbf{L}$ must be a proper subgroup of $\prod_{i=1}^{n}\mathbf{G}_i$.

Now, since every σ_i is Zariski dense, also every $\sigma_i^{g_i}$ is, and the projection π_i of $\mathbf{L}$ on $\mathbf{G}_i$ is onto for every i. Moreover the kernel of such projection is a normal subgroup of the product $\prod_{j\neq i} \mathbf{G}_j$ and, since $\mathbf{L} \lneq \prod_{j=1}^{n} \mathbf{G}_j$, then $\mathrm{Ker}(\pi_i)$ is also a proper subgroup of $\prod_{j\neq i} \mathbf{G}_j$. Following [BIW07], the fact that $\mathbf{G}_i$'s are simple non Abelian implies the existence of at least one isomorphism $s : G_i \to G_j$ for some $i \neq j$ such that $s \circ \sigma_i \simeq \sigma_j$, which contradicts the pairwise inequivalence of the σ_i's. □

As anticipated, Theorem 6.1.4 implies

Theorem 8. *Let Γ be a finitely generated group, let (X, μ_X) be an ergodic standard Borel probability Γ-space and consider a Zariski dense measurable cocycle $\sigma : \Gamma \times X \to G$ where $G = \mathit{Isom}(\mathcal{X})^\circ$ and $\mathcal{X}$ is an irreducible Hermitian symmetric space not of tube-type. Then the class $\mathrm{H}^2_{\mathrm{b}}(\sigma)(k^b_G)$ in $\mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathrm{L}^\infty(X; \mathbb{R}))$ is non-zero and it is a complete invariant of the cohomology class of σ.*

Proof. The non-vanishing of $\mathrm{H}^2_{\mathrm{b}}(\sigma)(k^b_G)$ is a direct consequence of Theorem 6.1.4. It remains to prove that two cocycles $\sigma_1, \sigma_2 : \Gamma \times X \to G = \mathrm{Isom}(\mathcal{X})^\circ$ have the same parametrized Kähler class if and only if they are cohomologuous. One direction follows immediately by Proposition 2.4.19. We now prove the other implication. Assuming that $\mathrm{H}^2_{\mathrm{b}}(\sigma_1)(k^b_G) = \mathrm{H}^2_{\mathrm{b}}(\sigma_2)(k^b_G)$, Theorem 6.1.4 provides an automorphism $s : G \to G$ such that $s \circ \sigma_1 \simeq \sigma_2$, that is $s \circ \sigma_1 = \sigma_2^f$ for some measurable function $f : X \to G$. Computing the pull back of the bounded Kähler class of G and exploiting the G-invariance, we obtain that

$$\begin{aligned}
\mathrm{H}^2_{\mathrm{b}}(\sigma_2)(k^b_G) &= \mathrm{H}^2_{\mathrm{b}}(\sigma_2^f)(k^b_G) \\
&= \mathrm{H}^2_{\mathrm{b}}(s \circ \sigma_1)(k^b_G) \\
&= \epsilon(s)\mathrm{H}^2_{\mathrm{b}}(\sigma_1)(k^b_G) \\
&= \epsilon(s)\mathrm{H}^2_{\mathrm{b}}(\sigma_1)(k^b_G)
\end{aligned}$$

where the $\epsilon(s)$ is the sign of the isometry s, according to the fact that s is either holomorphic or antiholomorphic. Since $\mathrm{H}^2_{\mathrm{b}}(\sigma_2)(k^b_G) \neq 0$, then $\epsilon(s) = 1$ and $s \in \mathrm{Isom}(\mathcal{X})^\circ = G$ and hence

$$s\sigma_1 s^{-1} = \sigma_2^f \,.$$

The thesis follows by setting

$$\tilde{f} : X \to G \,, \quad \tilde{f}(x) \coloneqq f(x)s$$

and by the fact that

$$\sigma_1 = \sigma_2^{\tilde{f}} \simeq \sigma_2.$$

□

Following [BIW07], in the setting of Theorem 8 we can denote by $\mathrm{Rep}_{\mathrm{ZD}}(\Gamma;G)$ the set of Zariski dense representations of Γ in G modulo conjugation. By [BIW07, Theorem 3] the map

$$K : \mathrm{Rep}_{\mathrm{ZD}}(\Gamma;G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma;\mathbb{R}) \ , \quad [\rho] \mapsto \mathrm{H}^2_b(\rho)(k^b_G)$$

is injective. Moreover, the inclusion

$$\left\{ \begin{matrix} \text{Zariski dense representations} \\ \Gamma \to G \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{Zariski dense cocycles} \\ \Gamma \times X \to G \end{matrix} \right\} ,$$
$$\rho \mapsto \sigma_\rho .$$

induces a map

$$\mathrm{Rep}_{\mathrm{ZD}}(\Gamma;G) \to \mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X;G).$$

Finally we denote by

$$K_X : \mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X;G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma;\mathrm{L}^\infty(X;\mathbb{R}))\ , \quad [\sigma] \mapsto \mathrm{H}^2_{\mathrm{b}}(\sigma)(k^b_G)$$

the map that associates to every cohomology class of a cocycle $\sigma : \Gamma \times X \to G$ its parametrized Kähler class. Putting together the above maps and the map induced in cohomology by the inclusion of coefficients $\mathbb{R} \to \mathrm{L}^\infty(X;\mathbb{R})$, we get the following

Corollary 6.1.5. *In the setting of Theorem 8, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Rep}_{\mathrm{ZD}}(\Gamma;G) \arrow[r, "K"] \arrow[d] & \mathrm{H}^2_{\mathrm{b}}(\Gamma;\mathbb{R}) \arrow[d] \\
\mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X;G) \arrow[r, "K_X"] & \mathrm{H}^2_{\mathrm{b}}(\Gamma;\mathrm{L}^\infty(X;\mathbb{R})).
\end{tikzcd}$$

6.2 Consequences of the main theorem

The aim of this last section is to present some consequences of Theorem 8 when Γ belongs to specific families of finitely generated groups. We notice that Savini has recently studied the elementarity properties of cocycles with values into the homeomorphisms of the circle when Γ is either a higher rank lattice [Sav21, Theorem 4] or an irreducible subgroup of a product [Sav21, Theorem 3]. Here we want to follow the same line.

We start with the higher rank case. Let $\Gamma < H = \mathbf{H}(\mathbb{R})$ be a lattice where $\mathbf{H}$ is a connected, simply connected, almost simple $\mathbb{R}$-group of rank at least two. In this context Zimmer's superrigidity [Zim80] applies, hence any Zariski dense cocycle $\sigma : \Gamma \times X \to G$ is induced by a representation $\rho : \Gamma \to G$, namely we have an isomorphism $\mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X; G) \cong \mathrm{Rep}_{\mathrm{ZD}}(\Gamma, G)$. Hence by applying [BIW07, Corollary 6] the image of K and K_X coincide and

$$\left|\mathrm{H}^1_{\mathrm{ZD}}(\Gamma \curvearrowright X; G)\right| \leq \dim \mathrm{H}^2(\Gamma; \mathbb{R}).$$

In this way we have a bound on the number of Zariski dense cohomology classes.

We move now to the case of products. In the setting of Theorem 8, suppose that

$$\Gamma < H = \prod_{i=1}^{n} H_i \ ,$$

with $n \geq 2$, where each factor H_i is a locally compact and second countable group with $\mathrm{H}^2_{\mathrm{cb}}(H_i; \mathbb{R}) = 0$. We suppose that Γ is *irreducible* in the sense of Burger and Monod, namely we ask that each projection of Γ in H_i is dense in H_i. Additionally, we set

$$H_i' = \prod_{j \neq i} H_i$$

for $i = 1, \ldots, n$ and we assume that each H_i' acts ergodically on X (that is H acts on X irreducibly in the sense of [BM02]). By [BM02, Corollary 15], we get that the map

$$\mathrm{H}^2_{\mathrm{cb}}(\Gamma; \mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(\Gamma; \mathrm{L}^\infty(X; \mathbb{R}))$$

induced by the inclusion of coefficients $\mathbb{R} \hookrightarrow \mathrm{L}^\infty(X; \mathbb{R})$ defined in (2.7) is an isomorphism. Combining with the inclusion in Equation (6.1), we get an inclusion

$$\mathrm{H}^1_{\mathrm{ZD}}(\Gamma; \mathrm{Meas}(X, G)) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathbb{R}).$$

Moreover, by [BM02, Theorem 16], we get a decomposition

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathbb{R}) \cong \bigoplus_{i=1}^{n} \mathrm{H}^2_{\mathrm{cb}}(H_i; \mathbb{R}^{H_i'}) \cong \bigoplus_{i=1}^{n} \mathrm{H}^2_{\mathrm{cb}}(H_i; \mathbb{R}) \tag{6.6}$$

where the equality on the right holds thanks the irreducibility of G on X. Hence we get the following result that should be compared with [Sav21, Theorem 3].

Proposition 9. *Let $n \geq 2$. Consider an irreducible lattice $\Gamma < \prod_{i=1}^{n} H_i$ into a product of locally compact second countable groups with $\mathrm{H}^2_{\mathrm{cb}}(H_i; \mathbb{R}) = 0$ for $i = 1, \ldots, n$. Let (X, μ_X) be an irreducible standard Borel H-space and assume that the Γ-action is ergodic. Then there is no Zariski dense cocycle $\sigma : \Gamma \times X \to G$ where $G = \mathrm{Isom}(\mathcal{X})^\circ$ and $\mathcal{X}$ is any Hermitian symmetric space not of tube-type.*

Chapter 7

Open questions and further directions

We close our dissertation by summarizing some questions that remain unsolved and by suggesting some possible approaches.

Boundary maps in the set of p-chains. In the general setting of Theorem 6, as pointed out in Remark 3.2.2, Theorem 6 provides a boundary map into some $\mathcal{I}_k(p,\infty)$. In [DLP21] the authors exploited Proposition 2.5.4 to rule out the case $k<p$ for Zariski dense representations. In the tentative to adapt such argument in the context of cocycles, we stuck in the final part. Precisely, following the proof of [DLP21, Theorem 1.7], one can construct a σ-equivariant family $\{W_x\}_{x\in X}$ of non-trivial subspaces of $\Lambda^d\mathcal{H}$ for some d. Since the stabilizer of such spaces are standard algebraic subgroups, it would be enough to twist the cocycle in order to get a cocycle with image contained in one of this stabilizers. However, the action of $\mathrm{PU}(p,\infty)$ on the subspaces (a priori of infinite dimension) of $\Lambda^d\mathcal{H}$ seems to us quite mysterious. Even before, one should clarifies the measurable structures involved. To conclude as in the proof of Theorem 6 or [SS21c, Theorem 2] one should identify the $\mathrm{PU}(p,\infty)$-orbit of some W_x with the quotient $\mathrm{PU}(p,\infty)/\mathrm{Stab}_{\mathrm{PU}(p,\infty)}W_x$, for instance by proving that the action is smooth, which is also not clear to us.

In conclusion, it is plausible that a natural notion of Zariski density for cocycles in $\mathrm{PU}(p,\infty)$ that extends the one given in [DLP21] exists, and maybe it can be the suitable assumption in order to prove that a boundary map takes values in the set of maximal chains. However, we do not know either how to formalize such concept and how to approach this problem.

Bounds for the number of Zariski dense cocycles. As a consequence of the injection $\mathrm{H}^1(\Gamma \curvearrowright X; H) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathrm{L}^\infty(X; H))$ provided by Theorem 8, in Section 5.3 we provided a bound for the number of Zariski dense cocycles $\Gamma \times X \to H$ when H is not of tube-type (see Proposition 9). However, this is only a partial generalization of the bounds given by Burger–Iozzi [BI04], since they proved that, the number of Zariski dense representations from a finite generated group Γ with $\mathrm{H}^2_{\mathrm{b}}(\Gamma; \mathbb{R}) < +\infty$ into $\mathrm{SU}(p, q)$ with $p \neq q$ is finite. The previous result is based on [BI04, Proposition 9.1], which asserts that the image of a continuous path in the space $\mathrm{Rep}_{\mathrm{ZD}}(\Gamma, \mathrm{SU}(p, q))$ with $p \neq q$ contains an uncountable subset which is independent over $\mathbb{R}$. As observed by Burger Iozzi and Wienhard [BIW07], this can be straightforwardly adapted to the case of Zariski dense representations in a generic Hermitian group G not of tube-type, so that one can deduce the analogous bound for the cardinality of $\mathrm{Rep}_{\mathrm{ZD}}(\Gamma, G)$. However, we do not know how to adapt these arguments in the context of cocycles. A similar result to [BI04, Proposition 9.1] for cocycles would imply a generalization of [BIW07, Corollary 5], hence would provide an estimate for the maximal number of Zariski dense cocycles into Hermitian groups not of tube-type.

Superrigidity of maximal cocycles. In the context of representations of complex hyperbolic lattices $\Gamma < \mathrm{PU}(1, n)$ with $n \geq 2$ into Hermitian Lie groups, it is conjectured that maximality implies rigidity, namely that, modulo G-conjugation, the only maximal representations are restrictions of representations of the ambient group. The parallelisms between representations and cocycles that we deeply studied in this thesis would seem to suggests the following more general

Conjecture 7.0.1. *Fix $n \geq 2$, let $\Gamma < \mathrm{PU}(1, n)$ be a complex hyperbolic lattice, let (X, μ_X) be an ergodic standard Borel probability Γ-space and consider a group G of Hermitian type. Then any maximal cocycles $\Gamma \times X \to G$ is cohomologous to the cocycle induced by a representation $\mathrm{PU}(1, n) \to G$.*

The strategy that we adopted in this thesis (previously exploited in [BI07, Poz15, MS20, SS21c]) is essentially based on boundary maps. Hence, the first difficulty to overcome to prove (parts of) Conjecture 7.0.1 with the same approach would be proving the existence of boundary map, that we gave in Theorem 2 and Theorem 1 only for certain families of cocycles.

Furthermore, in absence of Zariski density one loses all the rigid geometric properties of boundary maps (see for instance [BI07, Theorem 1] and [Poz15, Theorem 1.6]) that allows to promote equivariant maps to rational maps. Hence the arguments based on smooth actions on the variety of rational functions used by Zimmer [Zim80] and in Chapter 4 are no more adaptable.

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